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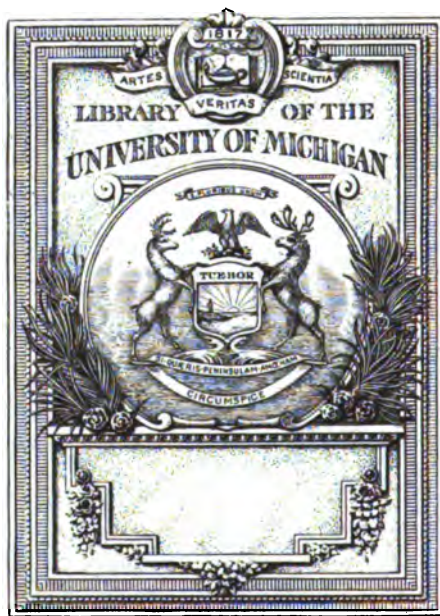
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1, 12, 21, 18, 31, 14, 41, 15, 51, 16, 61, 23, 32, 24, 42, 25, 26, 62, 33, 34, 43, 35, 53, 36, 63, 44, 15, 174, 46, 64, 55, 56, 65

36 Follows on two dice

35 to one apt. Following 2 on 2 dice

35 to 2 apt.	3
12 to 1 apt.	4
6 to 1 apt.	5
36 to 5 apt.	6
6 to 1 -	7
36 to 5 -	8
9 to 7 -	9
12 to 1 -	10
18 to 7 -	11
35 to 7 -	apt 12

On 3 dice

1, 1, 1; 12, 1, 112, 211, 122, 221, 212, 12, 3, 132, 321, 224, 142, 412, 131, 313, 311, 114, 141, 411
~~232, 223, 322, 444, 412, 124, 142, 133, 313, 331, 141, 141, 1~~
 151, 115, 511, 161, 116, 611, 171, 171, 711, 118, 181, 811, 119, 191
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 236, 263, 632, 24

112

R^d Chapman

THE
DOCTRINE
OF
PERMUTATIONS AND COMBINATIONS,
BEING
AN ESSENTIAL AND FUNDAMENTAL PART
OF THE
DOCTRINE OF CHANCES;

As it is delivered by Mr. JAMES BERNOULLI, in his excellent Treatise on the Doctrine of Chances, intituled, *Ars Conjectandi*, and by the celebrated Dr. JOHN WALLIS, of Oxford, in a Tract intituled from the Subject, and published at the end of his Treatise on Algebra :

In the former of which Tracts is contained,
A Demonstration of Sir ISAAC NEWTON's famous BINOMIAL THEOREM,
in the Cases of Integral Powers, and of the Reciprocals of Integral Powers.

TOGETHER WITH
SOME OTHER USEFUL MATHEMATICAL TRACTS.

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Mathew Bond

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E R R A T A.

IN THE PREFACE.

In page xv, line 14 from the bottom, instead of *place*, read *place*.

IN THE BOOK.

In page 5, line 20, instead of *inducunt*, read *inducunt*.

In page 7, line 13 from the bottom, instead of *Auctorum*, read *Aborum*.

In page 21, line 6, instead of $3)2(2:2$, read $3)3(2:2$.

And in the same page 21, line 11, instead of $3)2(4:3$, read $3)4(4:3$.

In page 23, line 6 from the bottom, instead of *tabulâ*, read *tabulâ*.

In page 25, line 16, instead of $\frac{n-3.e}{r}$, read $\frac{n-3.c}{r}$.

In page 27, line 5, after the word "primam" dele the figure of 1.

And in the same page 27, line the 3d from the bottom, instead of

$$\frac{n.n-1}{1.2}, \text{ read } \frac{n.n-1}{1.2}.$$

In page 28, line 10, after the word "subquintuplum," insert a comma.

And in the same page 28, line 15, instead of $\frac{n.n-1}{1.2}$, read $\frac{n.n-1}{1.2}$.

In page 30, the bottom line, instead of $-\frac{1}{2}nn$, read $+\frac{1}{2}nn$.

In page 32, in the last line but one of the lines that are parallel to the side of the page, instead of $-\frac{1}{12}nn$, read $-\frac{3}{20}nn$.

In page 50, line 2 from the bottom, insert the mark " after the word "Mathematicka."

In page 69, line 21, instead of *of rows*, read *or rows*.

In page 72, in the note at bottom, instead of *Alterations*, read *Alterations*.

In page 73, line 6, instead of 252, 462, read 210, 330.

In page 87, dele the figure of 1 at the end of the first line.

In page 101, line 3, instead of 71, read 74.

In page 102, line 12, instead of 71, read 74.

In page 103, line 12, instead of $\frac{n \times f + e + d + e + b + a}{r}$, read

$$\frac{n \times f + e + d + e + b + a}{r}.$$

In

In page 112, line 16, instead of $\frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \times 3 \times 4}$, read

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \times 3 \times 4},$$

And in the same page 112, line 17, instead of

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{2 \times 3 \times 4 \times 5}, \text{ read } \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{2 \times 3 \times 4 \times 5}.$$

In page 117, line 11 from the bottom, instead of $Ba^{1-2}b^1$, read $Ba^{1-2}b^2$.

In page 144, line 7 from the bottom, instead of $-56x$, read $-56x^5$.

In page 151, in the bottom line, instead of $-\frac{n}{1}Ax$, read $-\frac{n}{1}Ax$.

In page 158, line 3, instead of $21x^5 + 36x^6$, read $+21x^5 + 28x^6 + 36x^7$.

In page 183, lines 6, 7, and 8, the figure of 2 is not clear in the powers of 12 in the numerators of the fractions $\frac{12|7}{2}$, $\frac{12 \times 12|7}{8}$, and $\frac{7 \times 12|9}{2}$.

In page 184, line 5 from the bottom, instead of 116, 12, read 116, 122.

In page 196, line 8 from the bottom, instead of $\frac{11}{11}$, read $\frac{11}{11}$.

In page 197, the top line, instead of $\frac{5n}{6}$, read $\frac{5n}{66}$.

In page 218, line 3, instead of

$$+nBx^{n-1}d^2 + n \times \frac{n-1}{2} Bx^{n-2}d^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} Bx^{n-3}d^4, \&c$$

$$\text{read } +nBx^{n-1}d + n \times \frac{n-1}{2} Bx^{n-2}d^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}$$

$$Bx^{n-3}d^3, \&c.$$

And in the same page 218, line 4, instead of

$$+n-1 \times Cx^{n-2}d^3 + n-1 \times \frac{n-2}{2} Cx^{n-3}d^4, \&c, \text{ read}$$

$$+n-1 \times Cx^{n-2}d + n-1 \times \frac{n-2}{2} \times Cx^{n-3}d^2, \&c.$$

And again in the same page 218, line 5, instead of $+n-2Dx^{n-3}d^4, \&c$, read $+n-2 \times Dx^{n-3}d, \&c$.

In page 221, line 5 from the bottom, instead of $\frac{n \times n-1 \times n-2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, read

$$\frac{n \times n-1 \times n-2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}.$$

And

And in the same page 221, line 2 from the bottom, instead of

$$\frac{n \times n-1 \times n-2 \times n^{n-3}}{2.3.4.5.6}, \text{ read } \frac{n \times n-1 \times n-2 \times n^{n-3}}{2.3.4.5.6}.$$

And again in the same page 221, the bottom line, instead of

$$\frac{n \times n-1 \times n-2}{2.3.4.5.6}, \text{ read } \frac{n \times n-1 \times n-2}{2.3.4.5.6}.$$

In page 239, line 11, instead of a_m , read a^m .

In page 264, line 6 from the bottom, instead of $\overline{1+1}$, read $\overline{1+1}^n$.

In page 298, line 6, instead of bc , read bbc .

In page 302, line 10 from the bottom, the first figure after 10, which should be a 3, is not clear.

In page 340, line 16 from the bottom, instead of 391, read 191.

In page 341, line 15 from the bottom, instead of 3, 67, read 31, 67.

In page 350, line 11, instead of 92×76 , read 29×67 .

In page 369, line 9 from the bottom, column 17, instead of p, read 7.

In page 371, line 6 from the bottom, column 14, instead of 19, read 37.

In page 374, line 11 from the bottom, column 10, instead of 73, read 107.

In page 435, line 9 from the bottom, instead of 3, read 2.

In page 446, line 13 from the bottom, instead of $\frac{4m^2n^3}{mm+nn}^2$, read

$$\frac{4m^2n^2}{mm+nn}^2.$$

In pages 470, 471, 472, and 473, the title at the top of the pages is wrong. It should be, *Of the Extraction of the Cube-root by Mr. de Lagny's Method of Approximation.*

In page 472, lines 8 and 9 from the bottom, instead of 3.264, read 0.3264.

In page 488, line 4, after the letter a , insert the first mark of a parenthesis, to wit (.

In page 495, line 7, instead of $\frac{a^3 - c \times a}{c + 2a^3}$, read $\frac{a^3 - c \times a}{c + 2a^3}$.

In page 511, line 4 from the bottom, instead of $-Ca^{m-2}x^2$, read $+Ca^{m-2}x^2$.

And in the same page 511, line 2 from the bottom, instead of $+Ca^{m-2}$, read $+Ca^{m-2}x^2$.

In page 540, lines 10 and 15, instead of $\sqrt[3]{}$, read $\sqrt[3]{}$.

In

In page 542, line 9 from the bottom, instead of $\frac{2a \times a^m - n}{m-1 \times n + m+1 \times a^m}$,

read $\frac{2a \times a^m - n}{m-1 \times n + m+1 \times a^m}$.

In page 561, in the Note, line 19 from the bottom, instead of $-b^2x^2$,
read $-b^2x^3$.

T H E
P R E F A C E.

IT is well known to persons acquainted with the history of the Mathematicks, that Sir Isaac Newton's celebrated Theorem concerning the powers of a binomial quantity, such as $a + b$, was communicated by him to the world, in the latter part of the last century, without a demonstration. And many other writers on Algebra since Sir Isaac's death, and, amongst the rest, the famous Professor *Saunderson*, of the University of Cambridge, have followed his example in taking this important Theorem for granted, and delivering it to their readers without attempting to demonstrate it. For the chapter on this subject in the second volume of the Professor's *Elements of Algebra*, in two volumes, quarto, contains nothing more than a full and clear description of the Theorem, with an application of it to a good number of well-chosen examples, by way of illustration. This neglect of demonstrating so important a proposition has always appeared to me very strange; as the great merit and glory of the mathematical Sciences consists in the certainty of the principles on which they are founded, and the clearness and regularity with which all the subsequent conclusions obtained in them are deduced from those fundamental principles. There have been, however, other eminent Mathematicians who have supplied this great omission, and given us just and accurate demonstrations of this Theorem in some of the more obvi-

ous and important cases of it, though not, perhaps, in all its cases. And of these I consider Mr. *James Bernoulli* (who was Professor of Mathematicks at Basil, or Basle, in Switzerland, in the latter part of the last century,) as one of the most successful. For, in the 3d chapter of the second part of his excellent Treatise on the Doctrine of Chances, intitled, *Ars Conjectandi*, (which was published at Basil in the year 1713, in a small quarto volume, some years after his death,) there is a demonstration of this celebrated Theorem in the first, or simplest, case of it, (or when m , or the index of the power of the binomial quantity $a + b$, is an affirmative whole number,) that is deduced from the very nature of Multiplication and the properties of the Figurate numbers, in the clearest and most accurate manner possible. So that no demonstration of it ought to be expected, or need be desired, that shall exceed, or even equal, this in point of accuracy and perspicuity, though some others may, perhaps, be somewhat shorter. This demonstration I was therefore desirous of making more generally known to the Students of the Mathematicks; and with that view I resolved to republish it, together with so much of the concomitant text of Mr. Bernoulli's said valuable Treatise, as was necessary to the thorough understanding of it, in a volume of a moderate size and price. This was the inducement that gave rise to the present publication.

To answer this purpose in the most effectual manner, I thought it would be best to re-publish the whole of the three first chapters of the second Part of the said Treatise of Mr. James Bernoulli, together with the Preface to the said second Part; but without the first Part of the same work, because the said first part, (though in itself important and curious, and essential, I doubt not, to the full understanding of *the Doctrine of Chances*;) is not at all necessary to the understanding of the second Part, which treats of the Doctrine of Permutations and Combinations, and begins, in the most distinct and elementary manner, with the first foundations of that doctrine. And further, as there are many persons in England that are fond of the Mathematical Sciences without

without having much acquaintance with the Latin language, I have, in order to render the contents of these three valuable chapters accessible to such persons, translated these chapters into English, and subjoined the translation to the original text in Latin; so that the reader may chuse in which of the two languages he will peruse them. And in this translation I have expressed myself in a fuller manner than Mr. Bernoulli had adopted in the original, because I had observed that the great degree of brevity with which Mr. Bernoulli had expressed himself had rendered some parts of the original rather obscure. And I have likewise added a few notes both to the original and the translation, where the text seemed to me to require them.

And further, in the latter part of the translation of these chapters, I have also done something more than merely translate them. For, as I observed that Mr. Bernoulli's conclusions concerning the properties of the Figurate numbers, (which he had applied to the demonstration of the binomial theorem in the first, or simplest, case of it, or when m , or the index of the power of the binomial quantity $a + b$, was *an affirmative whole number*,) might easily be applied to the demonstration of the binomial theorem in another case of it, to wit, in that case of it in which m , or the index of the binomial quantity $a + b$, is *a negative whole number*, I drew up some additional articles, that are not contained in Mr. Bernoulli's text, for this purpose. These additional articles, (which contain a demonstration of the binomial theorem in *the case of integral and negative powers*, or in the case of the quantity $\overline{a + b}^{-m}$,) extend from page 123 to page 166; after which the translation of Mr. Bernoulli's text is resumed, and continues to page 213.

These three chapters contain a most accurate and distinct explanation of the fundamental parts of the Doctrine of Permutations and Combinations, and of the most remarkable properties of the Figurate numbers, which, it is well known, are of the most extensive use in various branches

of the Mathematicks. And they likewise contain an application of the properties of these important numbers to the summation of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to any proposed number n , and to the summation of the cubes, and of the fourth powers, and of the fifth powers, and of all the following powers, of the same numbers, (which is a matter of much nicety and difficulty, and was formerly a great object of inquiry to Mathematicians,) as well as to the demonstration of the binomial theorem in the case of integral and affirmative powers, and (with the articles I have added to it in pages 123, &c, to 166,) in the case of integral and negative powers. All which, together, makes a considerable body of very useful mathematical learning.

Immediately after these three chapters of Mr. James Bernoulli's *Ars Conjectandi*, I have re-published the tenth Mathematical Essay of the late very learned and ingenious Mathematician, Mr. Thomas Simpson, of Woolwich Academy, which is a solution of the following Problem, to wit, "*To find the sum of any series of powers whose roots are in*" "*arithmetical progression, as* $\overline{m+d}$ " ^{n} , $\overline{m+2d}$ " ^{n} , $\overline{m+3d}$ " ^{n} , " $\overline{m+4d}$ " ^{n} , $\overline{m+5d}$ " ^{n} , - - - - - x^n , *the letters* m , d , *and* " n , *denoting any numbers whatsoever.*" This Essay of Mr. Simpson had been alluded to in a note to the translation of the foregoing extract from Mr. Bernoulli's book, at the bottom of page 213; and it is so nearly connected with the subject of the latter part of that extract relating to the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, that I thought it would be agreeable to the reader to have it laid before him immediately after the said extract; and therefore I caused it to be re-printed in that place. It extends from page 214 to page 224.

The next Tract is one of my own composition, and contains *An Investigation and Demonstration of Sir Isaac Newton's Binomial*

P R E F A C E.

v

Binomial Theorem in the case of integral and affirmative powers; in which the law of the generation of the numeral co-efficients

of the terms of the series which is equal to the quantity $\overline{a + b^m}$, is discovered by a conjecture grounded on the observation of the law of the said co-efficients in some particular examples; but, when so discovered, is shewn to be true universally in all other integral and affirmative powers whatsoever of the said binomial quantity, by a strict and accurate demonstration. This Tract begins in page 227, and ends in page 268, and contains, as I believe, the best and most satisfactory demonstration of the Binomial Theorem in the case of integral and affirmative powers that has yet been given of it, next to that contained in the foregoing Extract from Mr. James Bernoulli's Treatise, intitled *Ars Conjectandi*. The conjectural investigation of the law of the numeral co-efficients of the terms

of the series that is equal to $\overline{a + b}^m$, given in this Tract, is suggested by Professor Saunderson, in the second volume of his Algebra, in the chapter on the Binomial Theorem; where (as I before observed,) the reader will find a good explanation and illustration of the said celebrated Theorem by a variety of examples, both in the case of integral powers and in the case of roots and other fractional powers, and even in the case of negative powers and of powers that are both fractional and negative; but no demonstration of it in any case, not even in that of integral and affirmative powers. And the following strict demonstration of this Theorem in the case of integral and affirmative powers, (which begins in page 252, and ends with page 264,) is nearly the same with that which is given by Mr. John Stewart, of Aberdeen, in the 6th section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis per Aequationes numero terminorum infinitas*, or *Analysis by Equations of an infinite number of Terms*. See his edition of Newton's Treatise on the Quadrature of Curves, and of the said Tract, intitled, *Analysis*, &c, with his learned comments on both, in one volume, quarto, published at London in the year 1745, page 471, art. 155.

This

This Tract, concerning the said conjectural investigation and subsequent general demonstration of the Binomial Theorem in the case of integral and affirmative powers, contains the substance of two Tracts published in the year 1792, in the second volume of the Collection of Mathematical Tracts, in quarto, called, *Scriptores Logarithmici*, to wit, the 15th Tract, which extends from page 153 to page 169, of the said second volume, and the 23d, or last, Tract in the said volume, which extends from page 587 to page 591.

Next to this Tract on the Investigation and Demonstration of Sir Isaac Newton's Binomial Theorem, I have re-published a Tract of the learned Dr. John Wallis, of Oxford, on the same Doctrine of Permutations and Combinations, which is the subject of the foregoing Extract from Mr. James Bernoulli's work above-mentioned. This Tract was published with Dr. Wallis's Algebra in the year 1685, under the title of *A Discourse of Combinations, Alternations, and Aliquot Parts*, and is mentioned by Mr. James Bernoulli in the foregoing Extract of his *Ars Conjectandi*, in the Scholium in pages 29 and 166, as a well-known and valuable Treatise on the properties of the Figurate numbers. And it does indeed contain a great deal of excellent and curious matter concerning those numbers, and the other subjects of which it treats, but without that accuracy and regularity in the manner of deducing the conclusions of it one from another, which distinguish the foregoing chapters of Mr. Bernoulli's work. However, on account of its intrinsic merit, and its relating to the same subjects, in a great measure, as the said Extract from Mr. Bernoulli's book, I thought it would be agreeable to my readers to see a re-publication of it in the same volume with the said Extract, and therefore I have given it a place in this Collection. It begins in page 271, and extends to page 351.

Much of this Discourse of Dr. Wallis relates to *Prime*, or *Incomposit*, numbers, and to curious arithmetical questions depending on them. And in one part of it, to wit, in page 318, the Doctor speaks of the great convenience of having at

at hand a Table of Prime Numbers set down in regular order, to be referred to when we want to know into what prime numbers a given odd number may be resolved. And he mentions a very useful Table of this kind that had been drawn up by a Mr. *Thomas Brancker*, M. A. and published by him in the year 1668, in an Appendix to an English translation, made by him, of *Rhoniſius's Algebra*, which had been published in the German language at Zurich in Switzerland, in the year 1659, under the title of *Algebra Rhonii, Germanicæ*. This English translation of Rhoniſius's *Algebra* was published by Mr. Brancker under the inspection, and with the assistance, of Dr. John Pell, an eminent Mathematician in the reign of King Charles the Second, and some considerable additions were made to the translation by Dr. Pell himself; which has given occasion to the book's being sometimes spoken of by subsequent writers of Mathematicks, and amongst others by Dr. Wallis himself in this Discourse, page 319, by the name of *Dr. Pell's Algebra*.

This Table of Prime Numbers Dr. Wallis set a high value on, inſomuch that he took the pains to examine it carefully throughout, and to correct the few errors that he found in it; ſo that now, with his corrections, it may be conſidered as very accurate. This Table therefore, together with the Appendix in which it is contained, I have here cauſed to be re-printed immediately after the foregoing Discourse of Dr. Wallis. It contains not only all the Prime numbers that are leſs than 100,000, but all the odd numbers whatſoever that are leſs than that number, (except ſuch odd numbers as end with the figure of 5, and are therefore evidently diſiſible by the number 5,) and it diſtinguiſhes the Prime numbers from the other odd numbers, by annexing to them the letter p; and it annexes likewise to every other odd number (that is not a Prime, or Incompoſit, number, but is the product of the multiplication of two, or more, leſſer numbers,) the leaſt of the prime numbers into which it may be reſolved. This Appendix, with the ſaid Table of odd numbers contained in it, extends from page 353 to page 416.

The

The next Tract in this Collection relates to the Rational Numbers that will express the Sides of Right-angled Triangles, and contains two methods of finding as many sets of numbers as we please that shall have this property. The first of these methods begins in page 417, and ends in page 431, and the second reaches from page 431 to page 448; after which I have inserted a Table of the Squares of the several natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 100, together with two additional columns adjoining to the column of the said squares, in the former of which I have set down the differences of the said squares, and in the latter the differences of those differences, or the second differences of the squares themselves; which second differences are all equal to each other, and to the number 2. This Table begins in page 449, and is accompanied with some remarks which extend to page 457. This Tract has a considerable resemblance to some parts of the foregoing Discourse of Dr. Wallis, and may afford some amusement to such readers as are fond of contemplating the properties of numbers.

The next Tract relates to the Cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the differences of the said cubes, and the differences of the said differences, or the second differences of the said cubes themselves, and to the differences of the said second differences, or the third differences of the cubes; which third differences are all equal to each other, and to the number 6. And in pages 460, 461, and 462, I have exhibited a Table of the Cubes of all the said natural numbers as far as 100, together with the 1st, 2d, and 3d differences of the said cubes in adjoining columns; after which follows an extract from a learned letter of the celebrated Mr. Leibnitz to Mr. Oldenburgh, the Secretary of the Royal Society of London, dated from London on the 3d day of February, 1672-3, relating to the subject of the differences of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the several successive orders of such differences, and to the ultimate equality

lity of the several successive differences in the second, or third, or fourth, or fifth, or other subsequent, order of the said differences, according to the height of the power to which the said numbers are raised, and relating to other curious properties of numbers. This letter is in Latin, and extends from page 463 to page 469, and is re-printed from the celebrated *Commercium Epistolicum* of Mr. John Collins, and other eminent Mathematicians of the latter part of the last century, that was first printed by the order of the Royal Society in the year 1712, and was afterwards re-printed in the year 1722. The remaining part of this Tract, from page 469 to page 504, relates to Monsieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and begins with shewing the usefulness of the foregoing Table of Cubes, in finding the first near value of the cube-root sought, to one, or two, places of figures, which is to be the basis of a further approximation to it by Mr. de Lagny's method. The four expressions given by Mr. de Lagny for the second near value of the cube-root that is sought, are stated in pages 470 and 471; and in the following pages to page 483, examples are given of the extraction of the cube-roots of the three numbers 2, 231, and 37,945, and of the long number 696,536,483,318,640,035,073,641,037, to a great degree of exactness, by means of some of the said expressions, after the first near values of the said cube-roots to one or two places of figures, have been obtained by the help of the foregoing table of the cubes of the first hundred numbers. These four examples (in which the several processes are stated very much at length,) will, I apprehend, be sufficient to make the reader familiarly acquainted with the method of using the said expressions of Mr. de Lagny in the extraction of the cube-roots of numbers, and at the same time to convince him of the great usefulness of these expressions for effecting that purpose; and they will likewise shew the usefulness of the foregoing Table of Cubes, in obtaining the first near values of the cube-roots sought, to one, or two, places of figures, from which the more accurate values of them are afterwards derived by means of Mr. de Lagny's expressions. After these four examples of Mr. de

b

Lagny's

Lagny's method of Approximation, follows a Scholium (in pages 484, 485, and 486,) concerning the invention of these expressions, and of Mr. Raphson's and Sir Isaac Newton's methods of extracting the Cube-roots, and other higher roots, of given numbers, and even the roots of affected equations of any order, by similar approximations; which methods were invented by those eminent Mathematicians before the publication of these expressions of Mr. de Lagny. And, then, (in pages 486, 487, 488, &c, to page 500,) I have given very full and accurate investigations of the foregoing expressions of Mr. de Lagny, which had been only stated in pages 470 and 471, and illustrated by examples in the following pages, from page 471 to page 483. And, lastly, in pages 501, 502, 503, 504, I have given a further illustration of the said expressions of Mr. de Lagny, by applying some of them to the extraction of some of the cube-roots which had been obtained in the foregoing examples by means of others of them; with a view to make a comparison between the different expressions given for the same purpose by Mr. de Lagny, and to discover which of them are the most exact, or the most easy to practice, and in which cases it will be most adviseable to resort to some of them in preference to the others. This Tract (which begins in page 459, and ends in page 504,) I consider as a very useful one to young students of Arithmetick and Algebra.

Having in the foregoing Tract very fully explained, and illustrated by examples, Mr. de Lagny's method of Extracting the Cube-roots of given numbers by Approximation, I proceed in the next Tract to state his general method of Extracting any higher Roots whatsoever of given numbers by similar Approximations. All these approximations are grounded on the same principle, and consist in putting some letter of the alphabet, as a , for the known part of the root sought, (which known part is found by conjecture, or otherwise, as the case may admit,) and putting some other letter, as z , for the unknown difference by which x , or the true root of the given number (which may be called N ,) exceeds, or falls short of, the first value a , (which is supposed to be known,) and then substituting $a + z$, or $a - z$, instead

instead of x , in the original equation $x^m = N$, or $x^r = N$, or, in general, $x^n = N$, (whereby the said equation will be transformed into another equation in which z will be the only unknown quantity,) and, lastly, in resolving this transformed equation (of which z is the root,) as if it was only a quadratick equation, or omitting, or expunging from it, all the terms that involve any higher power of z than the square. By such a resolution of this transformed equation Mr. de Cagny obtains a value of z that approaches nearly to its true value: and consequently, by substituting this near value of z , instead of z , in the binomial quantity $a + z$, or $a - z$, (which is equal to x , or the root sought, or the m th root of the given number N ,) he obtains a near value of $a + z$, or $a - z$, or a second near value of x , or $\sqrt[m]{N}$, which is much nearer to its true value than a , or its first near value, was. To explain in a full and distinct manner this method of extracting the m th root of any given number N , and to illustrate it by a few examples of the extraction of some high roots of given numbers, by means of the general expressions of the values of $a + z$ and $a - z$ derived from it, is the object of the present Tract. And, as the investigations necessary for this purpose are very general, and, from that circumstance, are rather more subtle and difficult than the investigations in the preceeding Tract, (which related only to the extraction of the cube-roots of given numbers,) I have taken great pains to set down all the steps in them in regular order, as clearly and plainly as I could; which may make them appear longer than might, perhaps, have been expected, but will, in fact, enable the reader to make himself perfect master of them in less time than if they had been compressed within a narrower compass. The general expressions that are thus investigated, are no less than four; to wit, two near values of $a + z$, obtained by considering the aforesaid transformed equation (arising from the substitution of $a + z$, instead of x , in the original equation $x = N$,) as a quadratick equation, and resolving it, as such, in two different manners, to wit, first, imperfectly,

imperfectly, and secondly, in an accurate manner; and two near values of $a - z$ obtained in like manner, by considering the other transformed equation, (arising from the substitution of $a - z$, instead of x , in the original equation $x^m = N$), as a quadratic equation, and resolving it, as such, in two different manners, to wit, first, imperfectly, and secondly, in an accurate manner. In order to perform these investigations the more easily and distinctly, I have divided the subject into two cases, with Problems corresponding to them, according as x , or $\sqrt[m]{N}$, is greater, or less, than its first value a , or is equal to $a + z$, or to $a - z$. The first case, or that in which x , or $\sqrt[m]{N}$, is equal to $a + z$, is considered in the first Problem; and the second case, or that in which x , or $\sqrt[m]{N}$, is equal to $a - z$, is considered in the second Problem: and from the Solution of the first Problem we obtain the two following

expressions, to wit, $a + \frac{2a \times N - a^m}{\overbrace{m-1} \times N + \overbrace{m+1} \times a^m}$ and

$$a + \sqrt{\frac{aa}{\overbrace{m-1}^2} + \frac{2 \times N - a^m}{m \times \overbrace{m-1} \times a^{m-2}}} - \frac{a}{m-1}, \text{ for}$$

near values of the binomial quantity $a + z$, or for second near values of x , or $\sqrt[m]{N}$; and from the Solution of the second Problem we obtain the two following expres-

sions, to wit, $a - \frac{2a \times a^m - N}{\overbrace{m-1} \times N + \overbrace{m+1} \times a^m}$ and

$$a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overbrace{m-1}^2} - \frac{2 \times a^m - N}{m \times \overbrace{m-1} \times a^{m-2}}}, \text{ for}$$

near values of the residual quantity $a - z$, or for second near values of x , or $\sqrt[m]{N}$. And the Solutions of both these Problems are illustrated by a few suitable examples, placed at the end of each solution respectively, of the extraction

tradition of different roots of given numbers by means of the said general expressions obtained in the preceeding solutions. The Solution of the first of these Problems begins in page 508, and ends in page 516; and is followed by three examples, which begin in page 516 and end in page 525: after which I have inserted a Scholium containing a comparison between Mr. de Lagny's aforesaid method of extracting the roots of given numbers, and Mr. Raphson's method of performing the same thing; which is somewhat simpler and easier than Mr. de Lagny's method, though not quite so exact. For the difference between the two methods consists only in this, that, whereas Mr. de Lagny resolves the transformed equation arising from the substitution of $a + z$ instead of x in the original equation $x^m = N$, as if it was a quadratick equation, omitting all the terms of it that involve any higher power of z than its square, Mr. Raphson resolves the same equation as if it was a mere simple equation, or omits all the terms of it that involve any higher power of z than its simple power, or z itself; which makes his expression of the near value of $a + z$, or of the second near value of x , or $\sqrt[m]{N}$, derived from the said transformed equation, a good deal simpler and easier to manage than those of Mr. de Lagny. This Scholium extends from page 525 to page 529, and is followed by a fourth example of the extraction of the root of a very long number by Mr. de Lagny's method, which extends to page 534. The Solution of the second of the said Problems begins in page 536, and extends to page 546, and is followed by two examples of the extraction of the roots of given numbers by means of the general expressions obtained in it, that extend from page 547 to page 554. And then the Tract concludes with some Observations, in pages 555 and 556, on the several different methods that may be taken for the extraction of the roots of numbers.

This Tract, as well as the last before it, concerning the Extraction of the Cube-roots of given Numbers, will, I hope, be found to be of great use to the Students of Arithmetick and Algebra.

The

The last Tract in this Collection is intitled, *Observations on Mr. Raphson's Method of resolving Affected Equations of all degrees by Approximation*. It begins in page 559, and ends in page 590; and its contents may be described as follows. The first part of it, as far as page 571, is intended, partly, to remove some difficulties that occur in reading Mr. Raphson's excellent Treatise on the Resolution of all Equations, (whether pure or affected,) by Approximation, intitled, *Analysis Equationum Universalis*; which difficulties are not inherent in the subject itself, or necessarily belonging to his method of resolving equations, but have arisen merely from his having unfortunately adopted the doctrine and language of *negative roots* of equations, by which the Science of Algebra, or Universal Arithmetick, has been disgraced and rendered obscure and difficult, and disgusting to men of a just taste for accurate reasoning, ever since its introduction by Harriot and Des Cartes. The first part of this Tract is, I say, intended, partly, to remove some difficulties of this kind, in the said Treatise of Mr. Raphson, and, partly, to illustrate his method of resolving high equations in other cases, or where no negative roots are mentioned, by performing the resolution of one of the equations given by him in his examples, to wit, of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, in a very full and distinct manner, with every step of the resolution, and the reasonings upon which it is grounded, set forth at length, agreeably to the principles laid down by him in the beginning of the said Treatise, instead of resorting (as he has done in his resolution of the same example, and in those of all his other examples,) to the repeated application of a general theorem, or canon, that he has deduced from the said principles: because that way of performing the said resolution, by means of a theorem, or canon, affords much less satisfaction to the mind of the reader, or operator, in the use of it, than he would receive by performing the resolution of the equation by the immediate application of the principles themselves, as I have done, in the resolution here given of the said equation. And the following part of this Tract contains a comparison between Mr. Raphson's method of Resolving Equations

Equations by Approximation, and Sir Isaac Newton's method of Resolving them also by Approximation, (which, after the first process of the approximation, or the discovery of the second near value of the root of the equation, differs a little from Mr. Raphson's method,) in order to discover which of the two methods deserves to be reckoned the most convenient. This comparison between these two methods of resolving equations by approximation, (the result of which is, that Mr. Raphson's method appears to me, upon the whole, more convenient than Sir Isaac Newton's,) reaches from page 571 to page 586: and the few remaining pages of this Tract, from page 586 to page 590, relate partly to the method of trying the exactness of the near value of x , or the root of the proposed equation, which has been obtained by either of the said two methods of Approximation, and, partly, to the method of finding a , or the first near value of x , or of the root of the proposed equation, to a moderate degree of exactness, in certain difficult cases, to wit, in those cases in which the proposed equation either has, or (from the changes of the signs of its terms from + to —, and from — to +,) seems to have, more than one real and affirmative root.

In the next place I have re-published a useful Table of Numbers, from a book intitled *The Calculator*, published in octavo in the year 1747, by the late learned Mr. James Dodson, being a Table of the Square-roots and Cube-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, to 180, carried to seven places of figures; which may often be the means of saving a Student of these Sciences some time and pains in performing the calculations that may occur in them. This Table is contained in pages 591 and 592.

And in the last place I have re-published a Table of the Square-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 1000, and likewise of the Reciprocals of all the said numbers, or of the values
of

of the fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9},$
 $\frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13},$ &c, as far as $\frac{1}{1000},$ expressed in decimal
 fractions, from the fourth volume of Dr. Charles Hutton's
Miscellanea Mathematica, published in the year 1775, in four
 little volumes, duodecimo. This Table begins in page 595,
 and ends in page 604, and, with Dr. Hutton's explanatory
 account of it in pages 605 and 606, concludes the present
 volume.

ARTIS CONJECTANDI

PARS SECUNDA,

CONTINENS

DOCTRINAM DE PERMUTATIONIBUS ET COMBINATIONIBUS.

PROOEMIUM.

INFINITAM varietatem, quæ cum in naturæ operibus, tum in actionibus mortalium elucet, quæque præcipuam hujus universi pulchritudinem constituit, non aliunde quam ex diversimodâ compositione, mixturâ & transpositione partium ejus inter se originem ducere palàm est. Sed, quia multitudo rerum ad effectum aliquem producendum concurrentium, sæpenumerò tanta est tamque varia, ut difficillimum sit recensere vias omnes, quibus earundem compositio, vel mixtura, fieri, vel non fieri, potest, hinc fit ut nullum sit vitium, in quod homines etiàm maximè prudentes & circumspecti frequentius incidant illò, quod Logici communiter appellant *insufficientem enumerationem partium*, adeò quidem ut non verear dicere, hanc unicam ferè scaturiginem esse infinitorum, eorùmque gravissimorum, errorum, quibz in conspectu nostris circa res tum cognoscendas tum agenda quotidie committimus. Quare merito suo utilissima censenda est ars, *combinatoria* dicta, quæ huic mentis nostræ defectui medetur, docetque sic enumerare, modos omnes possibiles, secundum

B

secundum quos res plures permisceri, transponi, vel conjungi, invicem possunt, ut certi simus, nos nullum eorum prætermisisse, qui instituto nostro conducere valent. Quamquam enim hoc negotii eatenus sit considerationis Mathematicæ, quatenus in subducendo calculo terminatur; si tamen usum & necessitatem spectes, universale prorsus est & ita comparatum, ut sine illo nec sapientia Philosophi, nec Historici exactitudo, nec Medici dexteritas, aut Politici prudentia, consistere queat. Argumento sit hoc unicum, quod omnis horum labor in *conjectando*, & omnis conjectura in trutinandis causarum complexionibus aut combinationibus versatur. Unde quoque nonnulli eximii viri, ac nominatim Schootenius, Leibnitijs, Wallisius, Prestetus, materiam hanc sibi tractandam sumpsere, ne quis existimet nova esse hæc omnia quæ prolaturi sumus; tametsi quædam non contemnenda de nostro adjecimus, imprimis demonstrationem generalem & facilem proprietatis numerorum figuratorum, cui cætera pleraque innituntur, & quam nemo, quod sciam, ante nos dedit eruisse. Cum itaque nondum plenum Artis systema habeamus, tum verò, ne illa quæ habemus aliunde petere sit opus, visum est totam Doctrinam ab ovo ordiri, ac ne quid indemonstratum relinquatur, ex primis fundamentis eruere; quod tamèn breviter fiet & succinctè, nec nisi in quantum instituti nostri ratio exigere videtur. Totam Tractationem ad duo summa capita referimus, quorum unum Permutationum, alterum Combinationum doctrinam persequitur; cui accedit tertium, quod utraqque mixtè contemplatur.

CAPUT I.

DE PERMUTATIONIBUS.

PERMUTATIONES rerum voco variationes, juxta quas, servatâ eadem rerum multitudine, ordo sitûsque inter ipsas diversimodè permutatur.

Itaque si queratur, quoties nonnullæ res transponi vel permisceri invicem possint, sic ut semper accipiantur. omne
sol

solo ordine sitive mutato, dicentur quæri omnes permutationes rerum illarum.

Res autem permutandæ vel omnes possunt esse diversæ, vel aliquot earum eadem; quæ quidem per totidem Alphabeti literas, sive diversas sive easdem, commodè designantur.

1. *Si res omnes permutandæ sunt diversa :*

CUM numerus permutationum in rebus pluribus iniri nequeat, nisi idem prius in omnibus aliis numero paucioribus compertus habeatur, liquet in hac inquisitione utendum viâ syntheticâ, hoc est, ordiendum nobis esse ab hypothesebus omnium primis & simplicissimis :

Unius rei, vel literæ, *a*, una tantum sumptio vel positio est.

Duarum ærum, aut literarum, *a* & *b*, vel *a* præcedit & *b* sequitur, vel præcedente *b* sequitur *a* ; unde duo ipsarum sunt ordines *a b* & *b a*.

Tres, porro, literæ *a*, *b*, *c*, ita collocari possunt, ut primus locus vel ipsi *a* vel *b* vel *c* concedatur : si *a* primum tenet locum, reliquæ duæ duobus, ut diximus, modis disponi queunt : si *b* in primum locum transferatur, reliquarum duarum duplex iisdem poterit esse positio ; quod & intelligendum, ubi tertia *c* primam sedem occupaverit. Unde trium literarum in universum ter duæ, seu 6, existunt permutationes *abc*, *acb* : *bac*, *bca* : *cab*, *cba*.

Similiter, si 4 extent literæ *a*, *b*, *c*, *d*, earum unaquæque primum obtinere locum potest, interea dum tres reliquæ, ut nunc ostensum, ter bis, seu sexies, ordinem variabunt : quare cum earum, quæ primo loco poni possunt, sint quatuor, sequitur omnes quatuor quater ter bis, seu quater sexies, hoc est, vicies quater situm inter se permutare posse.

Ob eandem rationem accedente 5^{ta} literâ *e* institui possunt quinquies tot variationes, quot in casu præcedenti, hoc est, quinquies 24, seu 120. Et generaliter, datis quocunque literis, numerus permutationum, quas subire possunt omnes, toties excedit numerum permutationum, quas recipiunt literæ unæ pauciores, quot sunt unitates in dato literarum numero. Unde sponte manat sequens

B 2

Regula

*Regula pro inveniendis omnibus permutationibus rerum
quocunque datarum.*

OMNES numeri ab unitate se consequentes naturali ordine, ad datum usque rerum numerum inclusive, ducantur in se invicem; productum manifestabit quæsitum.

Putà, si datus rerum numerus sit n , numerus permutationum erit 1. 2. 3. 4. 5. &c. usque ad n ; vel etiam (quia unitas non multiplicat) 2. 3. 4. 5. n . Nota, punctula numeris interjecta hîc et ubique in simili materiâ continuum numerorum in se ductum significant. Exempli gratiâ, septem rerum permutationes sunt 2. 3. 4. 5. 6. 7. = 5040. Ratio patet ex dictis, operatio ex adjunctâ Tabellâ :

Numerus Rerum. Permutationum.

1	-	-	-	1
				2
2	-	-	-	2
				3
3	-	-	-	6
				4
4	-	-	-	24
				5
5	-	-	-	120
				6
6	-	-	-	720
				7
7	-	-	-	5040
				8
8	-	-	-	40320
				9
9	-	-	-	362,880
				10
10	-	-	-	3,628,800
				3628800
11	-	-	-	39,916,800
				79833600
12	-	-	-	479,001,600

2. Si rerum permutandarum nonnullae sunt eadem:

Quod si literae una pluresve recurrant saepius, hoc est, si in dato rerum numero aliqua res similes sint sive eadem; ut, si datae sint *aaa b c d*, ubi litera *a* ter repetitur; numerus permutationum multo minor evadit: ad quem inveniendum cogitandum est, quod, si omnes essent diversae, puta, si loco *aaa* scriberetur *aaa*, possent hae tres literae etiam nullam ceterarum loco motam inter se sexies transponi, per praecedentem Regulam; unde totidem diversae nascerentur permutationes; at nunc cum sunt eadem, sex istae permutationes literarum *aaa* nullam universarum dispositioni variationem indicunt, ac proinde pro una eademque habendae sunt: quod cum de quacunque dispositione literarum pariter sit intelligendum, indicium praebet, numerum permutationum rerum datarum sexies, hoc est, toties minorem esse numero permutationum, quas subire possent si omnes essent diversae, quoties inter se permutari queunt res similes: sed si omnes 6 literae diversae existent, permutari possent, juxta praecedentem tabellam, 720. vicibus. Ergo nunc ubi tres ipsarum conveniunt, permutari duntaxat poterunt vicibus 120.

Iterum si datae sint 6 literae *aaa b b c*, ubi praeter literam *a* quae ter recurrit, etiam litera *b* bis repetitur; manifestum est, numerum permutationum adhuc bis minorem evadere, quam in praecedenti casu fuerat, adeoque

que solum ad 60 se extendere : quandoquidem binæ quælibet permutationes, quæ ex solâ transpositione duplici literarum *bb*, si diversæ essent, nascerentur, nunc cōcidunt Eodem pacto colligendum, si plures literæ repeterentur sæpius, pro singulis earum numerum permutationum minui toties, quoties seorsim inter se permutari possunt eadem literæ. Unde ratio habetur sequentis Regulæ.

Regula pro inveniendis rerum permutationibus, cū earum nonnullæ sunt eadem.

NUMERUS permutationum, quas admitterent datæ res si omnes differentes essent, dividatur per numerum permutationum, quas subire potest res similis secundum multitudinem suam, si una sit quæ sæpius repetatur : aut per productum ex numeris permutationum, quas seorsim recipere possunt singulæ res similes secundum multitudinem suam, si plures sint quæ sæpius recurrant ; & quotiens exhibebit quæsitum.

Ufus doctrinæ Permutationum insignis est in definiendo numero Anagrammatum alicujus vocis. Exempli gratiâ : Transpositiones omnes possibiles literarum in voce *Roma* sunt 1. 2.3.4. = 24, ob 4 differentes literas, per 1 Regulam : et in voce *Leopoldus* $\frac{361880}{2.3=4} = 90720$: et in voce *Studiosus* $\frac{361880}{2.6=12} = 30240$: ob 9 utrobique literas, intérque illas ibi geminum *l* et geminum *o*, hîc geminum *s* & triplex *s* ; per 2 Regulam.

Hûc pertinent versus nonnulli ob variationum multitudinem *Protei* dicti, quos inter celebrantur Lanfii, Scaligeri, Bauhusii. Thomæ Lanfio hoc distichon debemus :

Lex, Rex, Grex, Res, Spes, Jus, Tbus, Sal, Sol, (bona)

Lux, Laus :

Mars, Mens, Sors, Lis, Vis, Styx, Pus, Nex, Fax, (mala)

Crux, Fraus.

cujus singuli versus per Regulam primam, ob 11 monosyllaba (disyllabis vocibus *bona* & *mala* sive semper regioni affixis)

his) salva metri lege variari possunt 39,916,800 vicibus. Et quanquam aliàs contingat, ut pleræque variationes in metri leges arietent, nec non ut plerique Anagrammatismi sint non-significantes & barbari; levi tamen plerumque industria opus est ad discernendum utiles ab inutilibus, illorumque numerum seorsim ineundum, si aliquem in iis inquirendis ordinem observes. Quemadmodum cernere est in hexametro à Bernhardo Bauhusio, Jesuità Lovaniensì, in laudem Virginis Deiparæ constructo :

Tot tibi sunt dotes, Virgo, quot sidera celo;

quem dignum peculiari operà duxerunt plures viri celebres. Erycius Puteanus in libello, quem Thaumata Pietatis inscripsit, variationes ejus utiles integris 48 paginis enumerat, easque numero stellarum, quarum vulgò 1022 recensentur, accommodat, omisiss scrupulosius illis, quæ dicere videntur, tot sidera cælo esse, quot Mariæ dotes; nam Mariæ dotes esse multo plures. Eundem numerum 1022 ex Puteano repetit Gerhardus Vossius cap. 7. de Scientiis Mathematicis; Prestetus, Gallus, in primâ editione Elementorum Mathematicorum, paginâ 348, Proteo huic 2196 variationes attribuit; sed, factâ revisione in alterâ editione, tomio primo, paginâ 133, numerum earum dimidiò ferè auctum ad 3276 extendit. Industrii Auctorum Lipsiensium Collectores mense Junii 1686, in recensione Tractatûs Wallisiani de Algebrâ, numerum in quæstione (quem Auctor ipse definire non fuit ausus) ad 2580 determinant. Et ipse postmodum Wallisius in editione Latinâ operis sui, Oxoniæ anno 1693 impressâ, paginâ 494, eundem ad 3096 profert. Sed omnes adhuc à vero sunt deficientes, adeò ut delusam tot virorum, post adhibitas quoque secundas curas, in re levi perspicaciam merito mireris. Facto enim examine deprehendo, foetum hunc Bauhusianum, exclusis etiâ spondaicis, admissis verò iis qui cæsura destituti sunt, salvâ metri lege omninò tēr milliēs, tercentiēs, ac duodeciēs variabilem esse. At prolixius de his agere tanti non interest, nec institutum nostrum patitur.

TYPUS

TYPUS VARIATIONUM VERSUS BAUHIUSIANI :

Tot tibi sunt dotes, Virgo, quot sidera caelo.

Quintam Regionem Hexametri occupat
vel

<i>Sidera</i> , quam vocem excipit aut vox	
<i>Diffyllaba una</i> , nempe vel	
<i>Caelo</i> , ac tum vox <i>Tibi</i> inter sex reliquas occu- pat locum vel	
<i>Secundum</i> , præcedente voce nunc	
<i>Monosyllabâ</i> , eâque vel	
<i>Tot</i> , cui casui respondent - Variationes	24
<i>Sunt</i> , - - - - -	24
<i>Quot</i> , - - - - -	24
<i>Diffyllabâ Virgo</i> , - - - - -	24
<i>Tertium</i> , præeuntibus	
<i>Unâ monosyllabâ & undâ diffyllabâ</i> , primas tenente vel	
<i>Monosyllabâ, Tot</i> , quam excipit alterutra	
<i>Dotes: 6</i> }	
<i>Virgo: 6</i> }	12
<i>Sunt</i> , - - - - -	12
<i>Quot</i> , - - - - -	12
<i>Diffyllabâ, Dotes</i> , quam sequitur	
<i>Tot: 6</i> }	
<i>Sunt: 6</i> }	
<i>Quot: 6</i> }	18
<i>Virgo</i> , - - - - -	18
<i>Duabus diffyllabis</i> , nempe, <i>Dotes Virgo</i> ,	6
<i>Quartum</i> , præcedentibus	
<i>Tribus monosyllabis</i> , - - - - -	12
	<i>Duabus</i>

		<i>Duabus monosyllabis cum dissyllabâ Virgo,</i>	12
		<i>Una monosyllabâ & duabus dissyllabis,</i>	36
		<i>Quintum, præmissis</i>	
		<i>Tribus monosyllabis cum unâ dissyllabâ,</i>	48
		<i>Duabus monosyllabis cum totidem dissyllabis,</i>	
		<i>quarum posterior Virgo,</i>	18
		<i>Sextum,</i>	120
a	b	c	d
			420..420
		<i>Dotes, unde totidem variationes, quot in Cælo,</i>	
		<i>nempe</i>	420
		<i>Virgo, unde rursus totidem, quot in Cælo, excep-</i>	
		<i>tis solum illis 60 variationibus, ubi postrema</i>	
		<i>syllaba in Virgo correpta est; quibus proin-</i>	
		<i>demptis ex 420, remanent</i>	360
		<i>Monosyllabæ duæ, eæque</i>	
		<i>Quot sunt, vel Sunt quot; voce Tibi occupante</i>	
		<i>locum vel</i>	
		<i>Secundum, primo relicto voci</i>	
		<i>Monosyllabæ, Tot:</i>	12
		<i>Dissyllabæ, Virgo:</i>	12
		<i>Tertium, præcedentibus</i>	
		<i>Monosyllabâ cum dissyllabâ,</i>	24
		<i>Duabus dissyllabis, quarum post Virgo,</i>	8
		<i>Quartum, præeuntibus</i>	
		<i>Monosyllabâ cum duabus dissyllabis,</i>	36
		<i>Tribus dissyllabis, quarum ultima Virgo,</i>	4
		<i>Quintum,</i>	48
			144..144
		<i>Tot sunt, vel, Sunt tot, totidem</i>	144
		<i>Tot quot, aut, Quot tot, totidem</i>	144
			1632
			<i>Tibi, quam vocem sequitur vox</i>
			<i>Dissyllaba ana, eæque vel</i>
			<i>Cælo, voce Sidera occupante locum aut</i>

C

Primum

C A P. II.

DE COMBINATIONIBUS, IISQUE PRIMÒ CONSIDERATIS
SIMPLICITER.

COMBINATIONES rerum sunt conjunctiones, juxtà quas ex datâ rerum multitudine nonnullæ eximuntur, interque se conjunguntur nullo ordinis sitûs-ve ipsarum respectu habito.

Idcirco cùm quæritur, quoties ex dato rerum numero vel binæ, vel ternæ, vel quaternæ, &c. accipi possint, sic ut nunquam omnes eadem res sumantur sæpiùs quàm semel, dicentur quæri omnes combinationes diversæ rerum datarum.

Numerus, secundùm quem res datæ conjunguntur, dicitur *Exponens* Combinationis : Ita, si res binæ sumuntur, *Exponens* erit 2 ; si ternæ, 3 ; si quaternæ, 4. Res verò secundùm hos exponentes junctæ dicuntur *Binarii*, *Ternarii*, *Quaternarii*, &c. vel *Biniones*, *Terniones*, *Quaterniones*, &c. & consonantèr etiàm *Uniones*, vel *Unitates*, quando res sumuntur singulæ, & *Nulliones* cùm nulla planè sumitur.

Conjunctiones ipsas nonnulli vocant *Combinationes*, *Conternationes*, *Conquaternationes*, &c. quas omnes vulgò unâ voce *Combinationum* complecti solent, tametsi hæc vox strictiori significatu propriè non nisi illas conjunctiones indigitare videatur, quibus res binæ invicem junguntur. Quamobrem alii generaliori voce *Complicationum* vel *Complexionum* uti malunt : alii magis appositè *Elektiones* vocant, ut & illæ subintelligi possint rerum acceptiones, quibus res singulæ seorsim sumuntur, aut quibus etiàm nulla planè sumitur.

Res autèm quæ inter se combinandæ sunt, vel omnes possunt esse diversæ, vel aliquot ipsarum eadem ; eaque vel ita combinari debent, ut in nullâ combinatione res eadem sæpiùs contineatur, quàm ipsa reperitur in toto rerum numero : vel sic, ut in eadem combinatione res eadem etiàm sæpiùs recurrere, hoc est, ut secum ipsâ quoque combinari possit.

possit. Iterumque quæri potest numerus combinationum vel secundum omnes exponentes conjunctim, vel secundum singulos seorsim. Atque insuper circa unumquemque horum combinandi modorum plures formari possunt quæstiones & problemata, è quibus illa tantum delibabimus, quæ in sequentibus alicui usui fore judicamus.

1. *Si res omnes combinandæ sunt diversæ, inque nullâ combinatione eadem res bis occurrere debet, invenire omnes Combinationes simpliciter sive secundum omnes exponentes conjunctim.*

SUNTO combinandæ modis omnibus literæ a, b, c, d, e , &c. Fiant tot series quot literæ, hoc modo: In primâ seriæ ponatur sola litera a .

In secundâ ponatur b , nunc seorsim, nunc junctim cum a , ut habeatur ab vel ba . Eadem enim conjunctio est, quæ b cum a , & a cum b jungit, cum nullus ordinis, sitûs-ve ipsarum inter se, respectus haberi supponatur.

In tertiâ collocetur c , eaque primò sola, dein juncta, partim cum a & b , ut fiant biniones ac, bc ; partim cum ipso binione ab , ut fiat ternio abc .

a .

$b. ab.$

$c. ac. bc. abc.$

$d. ad. bd. cd. abd. acd, bcd, abcd.$

$e. ae. be. ce. de. abe. ace. bce. ade. bde. cde. abce. abde. acde. bcde. abcde.$

In quartâ ponatur d , primò sola, deinde juncta cum singulis præcedentium literarum a, b, c , singulisque earum tum binariis ab, ac, bc , tum ternario abc ; ut fiant novi biniones ad, bd, cd , terniones abd, acd, bcd , & quaternio $abcd$.

Similiter quintæ seriei agmen ducat litera e , quam primò ingrediatur sola, dein juncta cum omnibus præcedentium serierum electionibus. Eadêmque methodo procedendum esset,

fi

si plures essent datæ literæ. Quâ ratione satis manifestum est, datas literas in istis seriebus omnifariam inter se junctas esse, nullamque earum fieri posse electionem, quæ non in unâ harum serierum reperiatur, sed & nullam esse quæ alicubi bis occurrat; adeoque omnes unâ series suppeditaturas omnes electiones possibiles, quæ circa datas literas institui queunt.

Harum igitur numerus initur faciliè, si consideretur quòd in quolibet semper serie una ampliùs inveniri debeat electio, quàm in antecedentibus omnibus seriebus simul: quoniam litera, quæ illius seriei caput est, ibidem semel ponitur sola, & præterea unâ assumit secum omnes electiones præcedentium serierum. Hinc enim sequitur, quia in primâ serie est electio unica, fore in secundâ electiones duas, in tertiâ 4, in quartâ 8, & sic deinceps in progressionem geometricâ duplâ: quandoquidem progressionis duplæ ab unitate hanc quoque naturam esse constat*, ut summa terminorum quolibet unitate aucta sequentem terminum exhibeat. Quocirca summa electionum in seriebus omnibus æqualis est summæ terminorum totidem progressionis duplæ ab unitate, hoc est, per modò memoratam proprietatem, ipsi termino subsequenti

* Hoc autem ita demonstrari potest.

PROPOSITIO.

Sit series terminorum in geometricâ ratione unitatis ad numerum binarium continuò crescentium, scilicet, 1, 2, 4, 16, 32, 64, 128, 256, &c. usque ad n terminos. Horum terminorum summa vocetur S . Manifestum est ultimum, sive maximum, hujus seriei terminum fore æqualem 2^{n-1} . Augeatur jam hæc series uno adjecto termino, scilicet, $2 \times 2^{n-1}$, seu 2^n . Dico, quòd novus terminus 2^n erit æqualis $S + 1$, sive summæ S omnium priorum terminorum unâ cum unitate.

DEMONSTRATIO.

Duplicando terminos seriei S , sive $1 + 2 + 4 + 8 + 16 + 32 + 64 + \& + 2^{n-1}$, orietur series $2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c. + 2^n$, cujus termini omnes, excepto ultimo 2^n , sunt respectivè æquales terminis omnibus prioris seriei, excepto primo 1; hoc est, $2S$ erit $= S - 1 + 2^n$. Ergo $2S + 1$ erit $= S + 2^n$, et 2^n erit $= 2S + 1 - S$, seu $S + 1$.

Q. E. D.

ejusdem

ejusdem progressionis unitate multato; qui quidem terminus subsequens idem est cum producto binarii toties, sive tot vicibus, positi & in se ducti, quot ipsum in progressionem termini præcedunt, hoc est, quot sunt series, quarum electiones quærentur. Unde talis exurgit.

Regula pro inveniendis omnibus electionibus rerum datarum secundum omnes exponentes :

A Producto binarii toties, sive tot vicibus, positi & multiplicati in se, quot sunt datæ res, auferatur unitas : reliquum indicabit quæsitum.

Hoc est, posito rerum datarum numero n , numerus omnium electionum simpliciter, puta, omnium unionum, binionum, ternionum, &c. erit $2^n - 1$. Hinc si nullionem seu electionem, quâ ex rebus datis nulla sumitur, quæque in quâvis rerum multitudine una semper est & unica, simul comprehendas, fiet numerus ille 2^n : si cum nullione ipsos quoque uniones resecas, quorum numerus ipsi rerum numero perpetuò æquatur, erit numerus binionum, ternionum, cæterarumque complexionum $2^n - n - 1$. Exempli gratiâ. Septem planetarum conjunctiones, vel complicationes, omnes diversæ sunt $2^7 - 1 = 2.2.2.2.2.2.2 - 1 = 128 - 1 = 127$; unde si demas electiones 7, quibus singuli planetæ seorsim accipiuntur, quæque propriè non conjunctiones, sed disjunctiones planetarum sunt, relinquetur numerus omnium conjunctionum strictè dictarum, quibus planetæ vel bini, vel terni, vel quaterni, vel quini, vel seni, vel denique septeni junguntur, $2^7 - 7 - 1 = 120$. Sic etiâ duodecim, uti vocant, Registra, seu fistularum ordines, in organo pneumatico, quibus sonus, mox sibilans, mox tremebundus, efficitur, aut aliter modificatur, variari possunt $2^{12} - 1 = 4095$ vicibus.

Nota : Si quis examinet series combinationum suprâ in typo expositas, observabit in qualibet serie (solâ primâ exceptâ,

ceptâ, quæ unicum unionem a complectitur) numerum electionum secundum exponentes pares æquari numero electionum secundum impares: saltèm, cùm id in aliquot ab initio seriebus verumprehenderit, idem quoque in serie proximè sequente locum habere concludet. Nam litera, quæ illius seriei caput est, juncta præcedentium serierum electionibus iis, quæ impares exponentes habent, parium; & iis vicissim quæ pares habent juncta, imparium; exponentium complexiones efficit: adsciscens verò primæ seriei unionem a , paris; & ipsa per se sola accepta, imparis; exponentis electionem constituit: unde & in hac serie numerum harum numero illarum æquari constat. In omnibus igitur seriebus simul sumtis numerus electionum secundum impares exponentes numerum electionum secundum pares unitate superabit; aut, si his insupèr nullionem accenseas, æquabit. Quocircà, cùm numerus omnium electionum simpliciter, incluso nullione, ostensus sit 2^n , erit ejus semissis, sive potestas binarii proximè minor, 2^{n-1} , numerus electionum secundum solos impares; & dempto rursùm nullione, $2^{n-1} - 1$ numerus electionum secundum solos pares exponentes. Idem quoque demonstrabitur infrà in coroll. 6. cap. 4.

CAP.

C A P. III.

DE COMBINATIONIBUS SECUNDUM SINGULOS EXPONENTES
 SEORSIM; UBI DE NUMERIS FIGURATIS, EORUMQUE
 PROPRIETATIBUS AGITUR.

EX typo combinationum præcedentis capitis manifestum fit, literam quæ cujuscunque seriei caput est, adjunctam unionibus serierum præcedentium efficere suæ seriei biniones, adjunctam binionibus efficere terniones, ternionibus quaterniones, & sic porro: adeoque numerum binionum in quavis serie æquari summæ unionum in omnibus seriebus antecedentibus, numerum ternionum summæ binionum, numerum quaternionum summæ ternionum, & generaliter numerum combinationum secundum datum quemcunque exponentem in serie quâcunque æquari summæ combinationum omnium præcedentium serierum secundum exponentem unitate minorem dato. Sequitur hinc, quòd

Uniones, quia in singulis seriebus reperiuntur singuli, omnes inter se constituunt seriem 1.1.1.1.1. &c. seu seriem unitatum.

Biniones in primâ serie nulli sunt, in secundâ 1, in tertiâ $1+1=2$, in 4tâ $1+1+1=3$, in 5tâ $1+1+1+1=4$, &c. proinde omnes biniones inter se constituunt seriem 0.1.2.3.4.5. &c, hoc est, seriem *numerorum* arithmeticè progressionalium, sive *Lateralium*.

Terniones in primâ & secundâ serie nulli sunt, in 3tâ 1, in 4tâ $1+2=3$, in 5tâ $1+2+3=6$, in 6tâ $1+2+3+4=10$. &c. Omnes itaque ordine accepti seriem conficiunt 0.0.1.3.6.10.15. &c. hoc est, seriem *numerorum*, ut vocant, *Triangularium*, seu *Triangularium*.

Quaterniones in tribus primis seriebus nulli sunt, in 4tâ 1, in 5tâ $1+3=4$, in 6tâ $1+3+6=10$, in 7mâ $1+3+6+10=20$. &c. qui omnes ordine assumti seriem efficiunt 0.0.0.1.4.10.20. &c. seriem, videlicet, *Pyramidalium*.

Pari

Pari ratione Quiniones omnes seriem constituunt *Trianguli-pyramidalium* 0. 0. 0. 0. 1. 5. 15. 35. &c. Seniones seriem *Pyramido-pyramidalium* 0. 0. 0. 0. 0. 1. 6. 21. &c. aliæque combinationes secundum altiores exponentes efficiunt alias atque alias series numerorum figuratorum altioris generis in infinitum.*

Et sic occasione doctrinæ Combinationum in speculationem insperatam *numerorum figuratorum* incidimus; quâ appellatione vulgò insigniuntur numeri, qui ex continuâ arithmetice proportionalium, indeque ortorum numerorum, additione, vel collectione, generantur.

Ut verò hæ figuratorum numerorum series sub unum aspectum caderent, eoque facilius comprehenderentur quæ de illis dicenda supersunt, sequentem apposui tabellam, quam quis nullo negotio quousque voluerit tum deorsum tum

* De horum numerorum nominibus est inter auctores arithmeticos quædam variatio. Nam numeri 0.0.0.0.1.5.15.35, &c. qui hîc vocantur *Trianguli-pyramidales*, vocantur à quibusdam scriptoribus, et, inter alios, à Nicolao Mercatore, in celeberrimâ suâ Logarithmotechniâ, *Trigono-trigonales*; et numeri 0.0.0.0.0.1.6.21.56, &c. qui hîc vocantur *Pyramido-pyramidales*, ab illo vocantur *Trigono-pyramidales*. Nomina quibus diversi ordines numerorum figuratorum designantur apud *Mercatorem* sunt quæ sequuntur; scilicet, 1. unitates, 2. radices, 3. numeri trigonales, 4. pyramidales, 5. trigono-trigonales, 6. trigono-pyramidales, 7. pyramidi-pyramidales, 8. trigono-trigono-pyramidales, 9. trigono-pyramidi-pyramidales, 10. pyramidi-pyramidi-pyramidales. Vide Scriptores Logarithmicos, tom. 1^{um}, pag. 178.— Ad evitandam hanc confusionem nominum satius esse videtur diversos horum numerorum ordines post quartum ordinem, sive numerorum pyramidalium, distinguere solum per numeros exponentes ordinum designandorum, appellando eos sive numeros figuratos ordinis quinti, sive ordinis sexti, sive ordinis septimi, sive octavi, sive noni, sive decimi, aut alius cujuscunque ordinis.

D

dextrorsum

dextrorsum continuabit. Numeri barbari, seu Arabici, in sinistro tabulæ margine adscripti numerant columnas transversas, & simul rerum combinandarum multitudinem: nu-

TABULA COMBINATIONUM, SEU NUMERORUM
FIGURATORUM.

EXPONENTES COMBINATIONUM.

	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
1.	1	0	0	0	0	0	0	0	0	0	0	0
2.	1	1	0	0	0	0	0	0	0	0	0	0
3.	1	2	1	0	0	0	0	0	0	0	0	0
4.	1	3	3	1	0	0	0	0	0	0	0	0
5.	1	4	6	4	1	0	0	0	0	0	0	0
6.	1	5	10	10	5	1	0	0	0	0	0	0
7.	1	6	15	20	15	6	1	0	0	0	0	0
8.	1	7	21	35	35	21	7	1	0	0	0	0
9.	1	8	28	56	70	56	28	8	1	0	0	0
10.	1	9	36	84	126	126	84	36	9	1	0	0
11.	1	10	45	120	210	252	210	120	45	10	1	0
12.	1	11	55	165	330	462	462	330	165	55	11	1

Numeri Rerum Combinandarum.

meri verò Romani in supremo margine conspicui numerant columnas verticales & unâ exponentes combinationum innuunt. Columnarum verticalium prima est series monadum seu unitatum; secunda series numerorum naturalium, seu lateralium, ab unâ cyphrâ incipiens; tertia series trigonalium incipiens à cyphris duabus, quarta pyramidalium incipiens à tribus cyphris, quinta trianguli-pyramidalium incipiens à quatuor cyphris, & sic deinceps.

Habet hæc tabula proprietates planè eximias & admirandas; præterquàm enim quòd Combinationum mysterium in illâ latere jam ostendimus, notum est interioris geometriæ peritis, præcipua etiàm totius reliquæ matheseos arcana inibi delitescere. Nos proprietatum aliquas hîc delibabimus, & quidem delibabimus tantùm, nullius nisi primariæ illius, quæ proposito nostro inservit, demonstrationem accuratiorerum allaturi, cùm cæteræ vel ex hâc ostendi possint, vel ex ipsâ tabella

tabellæ constructione & numerorum figuratorum genefi fatiſ pateſcant.

Miſiſcæ Proprietates Tabulæ Combinationum :

1. Columnarum verticalium ſecunda incipit ab unâ cyphrâ, tertia à cyphris duabus, quarta à tribus : & generalitèr columna *c* à cyphris *c* — 1.

2. Columnarum verticalium termini primi ſignificativi à ſiniſtrâ dextrorſum obliquè deſcendendo ordine ſumpti reddunt ipſos terminos primæ columnæ verticalis, ſecundi ſecundæ, tertii tertiæ, & ita deinceps : putà, primi conſtituunt ſeriem monadum, ſecundi lateralium, tertii trigonalium, &c.

3. Secundus ab unitate terminus columnæ verticalis cujuſlibet æquatur ipſius columnæ numero.

4. Terminus quivis tabellæ æquatur ſummæ omnium ſuperiorum præcedentis columnæ verticalis.

5. Quilibet terminus æquatur duobus aliis immediatè ſuprà ſe poſitis, quorum unus eſt in eâdem verticali columnâ, alter in præcedente.

6. Columnæ cujuſvis tranſverſæ termini ab unitate aliquoſque creſcunt, deinde per eoſdem gradus rurſum decreſcunt. Idem intellige de ſummis columnarum verticalium æque-altarum, ceu terminis ſequentis columnæ tranſverſæ per quartam proprietatem.

7. Columnarum verticalium æque-altarum baſes, ſive termini columnæ tranſverſæ cujuſlibet, primus quidem & ultimus ſignificativus perpetuò inter ſe æquantur, ut et ſecundus & penultimus, tertius & antepenultimus, atque ita porrò, ſi columna pluribus terminis ſignificativis conſtet.

8. Quin & ſumptis ab initio columnis verticalibus quocumque cum totidem tranſverſis, collectiſque in unam ſummam qui in eâdem verticali ſibi reſpondent terminis, erit ſumma prima æqualis penultimæ, ſecunda antepenultimæ, tertia proantepenultimæ, & ſic deinceps. Exhibent enim hæ ſummæ ipſos columnæ tranſverſæ ſequentis terminos, primo excepto. Confer proprietates 4 & 7. Exempli gratiâ :

D 2

Quin-

Quinque primæ columnæ tum verticales tum transversæ sunt:

1.	0.	0.	0.	0.
1.	1.	0.	0.	0.
1.	2.	1.	0.	0.
1.	3.	3.	1.	0.
1.	4.	6.	4.	1.

5. 10. 10. 5. 1. Termini sextæ columnæ transversæ, primo excepto.

9. Columnæ transversæ ordine exhibent coëfficientes omnium potestatum à radice aliquâ binomiâ genitarum; nempe secunda coëfficientes radicis 1. 1. tertia quadrati 1. 2. 1. quarta cubi 1. 3. 3. 1. quinta biquadrati 1. 4. 6. 4. 1. & sic porro.

10. Summæ serierum transversarum progrediuntur in continuâ ratione duplâ: summarum verò summæ ab initio collectæ terminos constituunt progressionis duplæ unitate multatos; putà

$$\begin{array}{rcl}
 1 & = & 1 \\
 1 + 1 & = & 2 \\
 1 + 2 + 1 & = & 4 \\
 1 + 3 + 3 + 1 & = & 8 \\
 1 + 4 + 6 + 4 + 1 & = & 16
 \end{array}$$

$$\begin{array}{rcl}
 1 & = & 1 = 2 - 1 \\
 1 + 2 & = & 3 = 4 - 1 \\
 1 + 2 + 4 & = & 7 = 8 - 1 \\
 1 + 2 + 4 + 8 & = & 15 = 16 - 1 \\
 1 + 2 + 4 + 8 + 16 & = & 31 = 32 - 1
 \end{array}$$

fuit ex iis quæ in præcedente capite de Combinationibus simpliciter spectatis dicta sunt.

11. Termini seriei verticalis cujuscunque ordine divisi per terminos collaterales seriei præcedentis (initio vel ab unitate vel à suis respectivè cyphris facto) exhibent quotos arithmetice proportionales, quorum communis differentia est fractio, cujus numerator est unitas, & denominator ipse numerus, sive
secundus

secundus ab unitate terminus seriei dividendis. Exempli gratiâ :

<i>Divis.</i>	<i>divid.</i>	<i>(quot.</i>		<i>Divis.</i>	<i>divid.</i>	<i>(quot.</i>
1)	1	(2 : 2		1)	0	(0 : 2
2)	3	(3 : 2		2)	1	(1 : 2
3)	6	(4 : 2		3)	2	(2 : 2
4)	10	(5 : 2		4)	6	(3 : 2
5)	15	(6 : 2		5)	10	(4 : 2

<i>Divis.</i>	<i>divid.</i>	<i>(quot.</i>		<i>Divis.</i>	<i>divid.</i>	<i>(quot.</i>
1)	1	(3 : 3		1)	0	(0 : 3
3)	2	(4 : 3		3)	1	(1 : 3
6)	10	(5 : 3		6)	4	(2 : 3
10)	20	(6 : 3		10)	10	(3 : 3
15)	35	(7 : 3		15)	20	(4 : 3

Non difficulter hæc proprietas, si opus foret, deduci posset ex sequente.

12. Summa terminorum quocunque seriei verticalis cujuscunque à suis respectivè cyphris incipientis ad summam terminorum totidem ultimo æqualium eam habet rationem, quam habet unitas ad illius seriei numerum; hoc est, aggregatum numerorum quocunque lateralium ab unâ cyphrâ seriem auspicantium est ad aggregatum numerorum totidem maximo eorum, seu ultimo, æqualium, ut 1 ad 2; trigonali à cyphris duabus, ut 1 ad 3; pyramidalium à tribus, ut 1 ad 4; &c. Idem quoque valet de ratione, quam habet summa terminorum seriei cujuscunque ab unitate incipientis ad summam totidem maximum sequenti termino æqualium.

Exempli gratiâ :

0	3		1	5		0	6		1	15
1	3		2	5		0	6		3	15
2	3		3	5		1	6		6	15
3	3		4	5		3	6		10	15
						6	6			
6.	12 :: 1. 2		10.	20 :: 1. 2		10.	30 :: 1. 3		20.	60 :: 1. 3

0 10			
0 10	1	56	
0 10	4	56	
1 10	10	56	
4 10	20	56	
10 10	35	56	
15. 60 :: 1. 4	70. 280 :: 1. 4		&c.

Cum inter affectiones numerorum figuratorum hæc præcipua sit, eadémque scopo nostro primario inserviat, visum hîc est exponere methodum, quâ talem proprietatis ἀποδείξω exhibeo, quæ simul & scientifica sit, & propositum universali-
tèr concludat. Quem in finem sequentia præstruo lemmata :

LEMMA PRIMUM.

Summa terminorum quotlibet primæ seriei ad summam totidem terminorum ultimo æqualium rationem habet æqualitatis, five ut 1 ad 1.

DEMONSTRATIO.

Cum enim series meris constet unitatibus, erit summa terminorum quotlibet, summa tot unitatum, hoc est, tot terminorum ultimo æqualium, quot sunt termini.

Q. E. D.

LEMMA SECUNDUM.

In quâlibet serie à suis respectivè cyphris incipiente, si quota est ipsa inter series, tot ab initio sumantur termini, erit summa terminorum omnium ad summam totidem ultimo æqualium, ut 1 ad seriei numerum.

DEMONSTRATIO.

Numerus enim cyphrarum quamcunque seriem auspicantium unitate minor est seriei numero, per proprietatem primam.

nam. His igitur si accedat sequens terminus, numerus terminorum seriei numero æquabitur. Sed terminus, qui proximè cyphras sequitur, est unitas, per proprietatem secundam. Unde terminorum aggregatum æquatur unitati, & aggregatum totidem ultimo æqualium æquatur ipsi seriei numero. Quare constat Propositio.

LEMMA TERTIUM.

In quâcunque numerorum serie, si summa terminorum ab initio sumptorum ad summam totidem ultimo æqualium perpetuò eandem habeat rationem, quotcunque accipiantur termini, putà ut 1 ad R, ita ut summa terminorum æquetur summæ totidem ultimo æqualium divisæ per R; erit numerus terminorum assumptorum ablato R ad eundem numerum unitate multatum, ut sumptorum penultimus ad ultimum.

DEMONSTRATIO.

Sumpti sint ab initio termini quotlibet A.B.C.D. quorum numerus sit N, penultimus C, & ultimus D. Est utique $A + B + C = A + B + C + D - D$, hoc est, (per hypothesein) $\frac{C \text{ in } N - 1}{R}$ est $= \frac{D \text{ in } N}{R} - D$, & proinde, æque-multiplicando, C in $N - 1$ erit $= D \text{ in } N - D \text{ in } R = D \text{ in } N - R$, adeoque $N - R : N - 1 :: C : D$. Q. E. D.

LEMMA QUARTUM.

In tabulâ numerorum figuratorum si duæ sint columnæ verticales contiguæ, in quarum priore quotlibet ab initio termini ad totidem ultimo eorum æquales habeant constantem rationem, ut 1 ad r; habeant verò in posteriore termini aliquot ab initio sumpti ad totidem sumptorum ultimo æquales rationem ut 1 ad $r + 1$: habebit quoque, addito sequenti termino,

termino, summa omnium terminorum unà cum adjecto ad tot terminos adjecto æquales, quot sunt cum adjecto termini, rationem ut 1 ad $r + 1$.

DEMONSTRATIO.

Sumpti sint in posteriore columnâ termini E.F.G.H, quos proximè sequatur I; atque sumantur in columnâ immediatè præcedente termini totidem A.B.C.D; sumptorum verò utrinque numerus sit n . Erit r H = (ex numerorum figuratorum generi per proprietatem quartam) r in $A + B + C$ = (per hypothesin) $n - 1$ in C = (per lemma tertium) $n - r$ in D; quare $n - r : H :: r : D ::$ (per hypothesin) $n : A + B + C + D ::$ (ex numerorum figuratorum generi per proprietatem quartam) n . I. Unde $n - r$ in I = n H = (per hypothesin) $r + 1$ in $E + F + G + H$; adeoque $n - r : r + 1 :: E + F + G + H : I$, &, componendo, $n + 1 : r + 1 :: E + F + G + H + I : I$, hoc est, $E + F + G + H + I : n + 1$ in I :: $1 : r + 1$ *. Q. E. D.

Cum

* Hæc demonstratio præ nimia brevitate mihi videtur esse obscura. Potest verò explicari et, ut opinor, satis perspicua reddi, modo sequente.

Sumpti sint in posteriore columnâ termini E.F.G.H; quos proximè sequatur I; atque sumantur in columnâ immediatè præcedente termini totidem A.B.C.D: sumptorum verò utrinque numerus sit n . Et sit summa quotlibet terminorum A.B.C.D. ad totidem ultimos eorum æquales in ratione 1 ad r ; et sit summa terminorum E.F.G.H. ad n terminos ipsi H, eorum ultimo, æquales, hoc est, ad quantitatem $n \times H$, in ratione 1 ad $r + 1$. Dico, quòd summa omnium terminorum E. F. G. H. I erit ad $n + 1$ terminos ipsi I æquales, hoc est, ad $n + 1 \times I$, ut 1 ad $r + 1$.

DEMONSTRATIO.

Ex numerorum figuratorum generi, per proprietatem quartam suprâ memoratam, erit $r \times H$ æqualis $r \times A + B + C$, ideòque (per hypothesin) æqualis $n - 1 \times C$, atque idcirco (per lemma tertium) æqualis $n - r \times D$. Erit igitur $n - r$ ad H ut r ad D. Sed (per hypothesin) $A + B + C + D$ est ad $n \times D$ ut 1 ad r ; et proinde (permutando) $A + B + C + D$ erit ad 1 ut $n \times D$ ad r , et (invertendo) 1 erit ad $A + B + C + D$ ut r ad $n \times D$.
Est

Cum omnium horum Fratri * copiam fecissem, animadvertit ille posse demonstrationem elegantèr abbreviari, postremis tribus lemmatibus in unum conflatis, hoc modo:

LEMMA.

In tabulâ numerorum figuratorum si summa terminorum ab initio seriei verticalis cujusvis ad summam totidem maximo æqualium ubique rationem habeat ut 1 ad r , habebit summa terminorum seriei proximè sequentis ad summam totidem maximo æqualium rationem ut 1 ad $r + 1$.

DEMONSTRATIO.

Sint series sequentes $a. b. c. d. \&c. \& o. g. h. i. \&c.$ numerus terminorum prioris sit n , posterioris $n + 1$. Est primò $q + p + l + i + b + g + o =$ (ex hypothesi & genesi numerorum figuratorum per proprietatem quartam) $\frac{nf}{r} + \frac{n-1.e}{r} + \frac{n-2.d}{r} + \frac{n-3.c}{r} + \frac{n-4.b}{r} + \frac{n-5.a}{r} = \frac{n.f + e + d + c + b + a}{r}$

Est autem $n \times 1$, seu n , ad 1, ut $n \times r$ est ad r . Ergò, ex æquo, $n \times 1$, seu n , erit ad $A + B + C + D$ ut $n \times r$ ad $n D$, hoc est, ut r ad D . Erit igitur $n - r$ ad H ut n ad $A + B + C + D$.

Sed (ex numerorum figuratorum genesi, per proprietatem quartam suprâ memoratam) terminus I est æqualis $A + B + C + D$.

Erit igitur $n - r$ ad H ut n ad I , et proinde $(n - r) \times I$ erit $= n \times H$.

Sed, per hypothesin, $E + F + G + H$ est ad $n \times H$ ut 1 ad $r + 1$; atque ideò $(E + F + G + H) \times (r + 1)$ est æqualis $n \times H \times 1$, seu $n \times H$.

Erit igitur $(n - r) \times I = (E + F + G + H) \times (r + 1)$; atque ideò erit $n - r$ ad $r + 1$ ut $E + F + G + H$ ad I , et proinde (componendo) erit $n - r + r + 1$, seu $n + 1$, ad $r + 1$ ut $E + F + G + H + I$ ad I , et (permutando) $n + 1$ ad $E + F + G + H + I$ ut $r + 1$ ad I , et (invertendo) $E + F + G + H + I$ ad $n + 1$ ut I ad $r + 1$, et (multiplicando consequentes per I) $E + F + G + H + I$ ad $n + 1$ ut I ad $r + 1$ ut I ad $r + 1$, hoc est, ut 1 ad $r + 1$.

Q. E. D.

* Johanni Bernoullio.

E

—

$\frac{-e-2d-3c-4b-5a}{r} = (\text{ex genesi numerorum figuratorum})$
 $\frac{nq-p-l-i-b-g}{r}$. Ergò $rq + r \cdot \overline{p+l+i+b+g} = nq - p - l - i - b - g$; factâque translatione convenienti, $\overline{r+1} \times \overline{p+l+i+b+g} = nq - rq$. Dividatur utrinque per $r+1$, erit $\overline{p+l+i+b+g} = \frac{nq-rq}{r+1}$; additôque q habebitur $q + \overline{p+l+i+b+g} = \frac{nq-rq}{r+1} + q = \frac{n+1 \times q}{r+1}$, hoc est, $g+b+i+l+p+q$ erit ad $n+1 \times q$ ut 1 ad $r+1$. Q; E. D.

Sequitur nunc *Propositio principalis*, quæ talis est.

PROPOSITIO PRINCIPALIS.

In tabulâ numerorum figuratorum summa terminorum quolibet à suis respectivè cyphris incipientium ad summam totidem ultimo æqualium: Item summa terminorum quotvis incipientium ab unitate ad summam totidem ultimum sequenti æqualium: in serie primâ, seu monadum, est ut 1 ad 1; in serie secundâ, seu lateralium, ut 1 ad 2; in tertiâ seu trigonalium, ut 1 ad 3; in quartâ, seu pyramidalium, ut 1 ad 4, & generalitèr in serie quâcunque ut 1 ad illius seriei numerum.

DEMONSTRATIO PRIMÆ PARTIS HUIUSCE PROPOSITIONIS,

De primâ serie constat ex primò lemmate: de secundâ tertiâ, quartâ, &c. è reliquis. Nam, quia summa terminorum quolibet ad summam totidem ultimo æqualium in primâ serie est ut 1 ad 1, erit, vi horum lemmatum, in secundâ ut 1 ad $1+1 = 2$; &, quia in secundâ est ut 1 ad 2, erit in tertiâ ut 1 ad $2+1 = 3$; & propterea etiâ in quartâ ut 1 ad $3+1 = 4$; in quintâ ut 1 ad $4+1 = 5$; & generalitèr in serie c ut 1 ad c . Q; E. D.

DEMON

DEMONSTRATIO SECUNDÆ PARTIS HJUSCÆ
PROPOSITIONIS.

Quia rationem 1 ad $r+1$ memoratam in ultimo lemmate hic interpretamur per rationem 1 ad c , erit $r = c - 1 =$ (per proprietatem primam 1) numero cyphrarum, à quibus columna c incipit. Quare, cum in dicto lemmate repertum fit $g+b+i+l+p = \frac{n-r \times q}{r+1} = \frac{n-r \times q}{c}$, sequitur quòd $g+b+i+l+p$ (summa terminorum quorum numerus est n) se habet ad q in $n-r$ (numerus terminorum minùs numero cyphrarum) sicut 1 ad c ; hoc est, summa terminorum quotlibet ab unitate incipientium ad totidem terminos sequenti ultimum æquales, ut 1 ad c^* . Q. E. D.

CONSECTARIUM.

Ex hac offensâ proprietate facile nunc est invenire tùm terminum optatum, tùm summam terminorum seriei cujuslibet. Sumpti intelligantur termini æque-multi ex pluribus continuè columnis, & sit numerus sumptorum ab initio cujusque columnæ n , adeoque numerus terminorum ab unitate (exclusis cyphris initialibus) in secundâ columnâ $n-1$, in tertiâ $n-2$, in quartâ $n-3$, atque ita deinceps, per primam proprietatem: quo posito, quæsitum ita colligo. Summa terminorum n primæ columnæ, nempe, n unitates, seu $\frac{n}{1}$, æquatur termino $n+1$.no, hoc est, termino sequenti ultimum, secundæ columnæ, per quartam proprietatem, ex tabulæ genesi. Quare termini hujus in $n-1$ (numerus terminorum ab unitate secundæ columnæ) ducti subduplum, seu $\frac{n \cdot n - 1}{1 \cdot 2}$, per duodecimam proprietatem æquale est aggregato terminorum secundæ columnæ, & simul (per quartam proprietatem) ipsi termino sequenti ultimum tertiæ columnæ.

* Vide super hanc materiâ opera ipsius Johannis Bernoullii, edita Lausannæ anno Domini 1742, Tomum tertium, paginam 521, in 47mâ Lectione de Calculo Integralium.

Unde similiter hujus termini in $n - 2$ (numerum terminorum ab unitate tertiæ columnæ) ducti subtripulum, nempe

$$\frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3},$$

æquatur (per duodecimam proprietatem) aggregato terminorum tertiæ columnæ, insimulque (per quartam proprietatem) ipsi termino sequenti ultimum quartæ.

Quocirca & hujus termini in $n - 3$ (numerum terminorum ab unitate quartæ columnæ) ducti subquadruplum, puta

$$\frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4},$$

exhibet summam terminorum quartæ columnæ, unâque terminum qui sequitur ultimum quintæ; & rursus istius termini in $n - 4$ ducti subquintuplum

$$\frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

producit summam terminorum columnæ quintæ, & simul terminum qui excipit ultimum sextæ; atque ita consequenter. E quibus igitur infertur, quod summa terminorum n primæ columnæ sit $\frac{n}{1}$, secundæ

$$\frac{n \cdot n - 1}{1 \cdot 2},$$

$$\text{tertiæ } \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3},$$

$$\text{quartæ } \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4},$$

$$\text{quintæ}$$

$$\frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

& generalitèr columnæ c ,

$$\frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4 \dots n - c + 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots c}.$$

Et, quia quælibet harum quantitatum etiâ exprimit terminum $n + 1$ sequentis columnæ, sequitur quod ipse illius terminus optatus, seu ultimus, n habeatur mutato solummodò ubique n in $n - 1$; adeoque quod terminus optatus, secundæ columnæ sit

$$\frac{n - 1}{1},$$

$$\text{tertiæ } \frac{n - 1 \cdot n - 2}{1 \cdot 2},$$

$$\text{quartæ } \frac{n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3},$$

$$\text{quintæ } \frac{n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4},$$

&, generalitèr, columnæ c ,

$$\frac{n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4 \dots n - c + 1}{1 \cdot 2 \cdot 3 \cdot 4 \dots c - 1}.$$

SCHOLIUM.

SCHOLIUM.

Multi, ut hoc in transitu notemus, numerorum figuratorum contemplationibus vacârunt (quos inter Faulhaber & Remmelini Ulmenſes, Walliſius, Mercator in Logarithmo-techniâ, Preſtetus, aliſque); ſed qui proprietatis hujus demonſtrationem univerſalem dederit & ſcientificam, novi neminem. Walliſius in Arithmeticâ Infinitorum fundamentum ſuæ methodi jacturus, rationes quas habent ſeries quadratorum, cuborum, aliarumque poteſtatum, numerorum naturalium ad ſeriem totidem maximo æqualium, inductione inveſtigat; indè in propoſitione 176, ad contemplationem numerorum trigonalium, pyramidalium, reliquorumque figuratorum tranſit. Sed ſatius fuiſſet fortèque naturæ rei convenientius, ſi vice verſâ tractationem numerorum figuratorum, eamque univerſali & accuratâ demonſtratione munitam, præmiſiſſet, ac tum demùm ad poteſtatum ſummas inveſtigandas perrexiſſet. Præterquam enim quòd modus demonſtrandi per inductionem parùm ſcientificus eſt, inſuperque pro quâlibet ſerie peculiarem operam depoſcit; illa utique omnium judicio præcedere debent, quæ cæteris naturæ ſunt priora & ſimpliciora, quales videntur eſſe numeri figurati præ poteſtatibus, tùm quòd illi additione, hæ multiplicatione generantur, tùm, & præcipuè, quòd ſeries figuratorum à ſuis reſpectivè cyphris incipientes ad ſeries æqualium rationem habent exactè ſubmultiplicem, qualem non habere poſſunt ſeries poteſtatum (ſaltèm in terminis numero finitis) abſque aliquo exceſſu vel defectu, quicumque cyphrarum numerus ipsis præfigatur. De cætero namque ex cognitis figuratorum ſummis nihilo difficiliùs inveſtigari poterunt poteſtatum ſummæ, atque ex his priores collegit auctor; quod quomodo fiat, paucis oſtendam.

Inveſtigatio

Investigatio Summarum quæ proveniunt ex additione quadratorum, cuborum, quadrato-quadratorum, et sequentium potestatum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. ex numerorum figurarum summis derivata.

Proponatur series numerorum naturalium ab unitate 1. 2. 3. 4. 5. &c. usque ad n , & quærantur omnium ipsorum, item omnium quadratorum, cuborum, et sequentium potestatum ex ipsis, summæ. Quoniam in tabulâ combinationum terminus secundæ columnæ indefinitè est $n-1$, & summa omnium terminorum, hoc est, summa omnium $n-1$, seu $\int (n-1)$, per confectarium præcedens inventa, est $\frac{n \cdot n - 1}{1 \cdot 2} = \frac{nn-n}{2}$, erit $\int (n-1)$, five $\int n - \int 1$, $= \frac{nn-n}{2}$, & proinde $\int n = \frac{nn-n}{2} + \int 1$; sed $\int 1$ (summa omnium unitatum) est n ; quare summa omnium n , seu $\int n$, erit $= \frac{nn-n}{2} + n = \frac{1}{2} nn + \frac{1}{2} n$.

Porrò cum terminus, tertiæ columnæ indefinitè acceptus per idem confectarium fit $\frac{n-1 \cdot n-2}{1 \cdot 2} = \frac{nn-3n+2}{2}$, & summa omnium terminorum (hoc est), omnium $\frac{nn-3n+2}{2}$ $\frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} = \frac{n^3 - 3nn + 2n}{6}$; erit $\int \frac{nn-3n+2}{2}$ five $\int \frac{1}{2} nn - \int \frac{3}{2} n + \int 1 = \frac{n^3 - 3nn + 2n}{6}$, & $\int \frac{1}{2} nn = \frac{n^3 - 3nn + 2n}{6} + \int \frac{3}{2} n - \int 1$; sed $\int \frac{3}{2} n = \frac{3}{2} \int n =$ (per modò ostensa) $\frac{3}{2} nn + \frac{3}{2} n$, & $\int 1 = n$: unde his substitutis fit $\int \frac{1}{2} nn = \frac{n^3 - 3nn + 2n}{6} + \frac{3nn + 3n}{4} - n = \frac{1}{6} n^3 + \frac{1}{4} nn + \frac{1}{12} n$, ejusque duplum $\int nn$ (summa quadratorum ex omnibus n) $= \frac{1}{3} n^3 + \frac{1}{2} nn + \frac{1}{6} n$.

Rursus,

Rursûs, quia terminus n quartæ columnæ est $\frac{n-1.n-2.n-3}{1.2.3}$
 $= \frac{n^3-6nn+11n-6}{6}$, & summa omnium terminorum

$$\frac{n.n-1.n-2.n-3}{1.2.3.4} = \frac{n^4-6n^3+11nn-6n}{24}, \text{ erit utique } f$$

$$\frac{n^3-6nn+11n-6}{6}, \text{ hoc est, } f\frac{1}{6}n^3 - fnn + f\frac{1}{6}n - f1 =$$

$$\frac{n^4-6n^3+11nn-6n}{24}, \text{ indéque } f\frac{1}{6}n^3 = \frac{n^4-6n^3+11nn-6n}{24} + fnn -$$

$f\frac{1}{6}n + f1$. Et quoniam per modò inventa $fnn = \frac{1}{4}n^3 + \frac{1}{2}nn + \frac{1}{6}n$; nec non $f\frac{1}{6}n$ five $\frac{1}{6}fn = \frac{1}{12}nn + \frac{1}{12}n$, & $f1 = n$; hinc, factâ horum substitutione, emerget $f\frac{1}{6}n^3 =$

$$\frac{n^4-6n^3+11nn-6n}{24} + \frac{1}{4}n^3 + \frac{1}{2}nn + \frac{1}{6}n - \frac{1}{12}nn - \frac{1}{12}n +$$

$n = \frac{1}{24}n^4 + \frac{1}{12}n^3 + \frac{1}{4}nn$, ejúsque proin sextuplum fn^3 (summa cuborum) $= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{2}nn$. Atque sic porro ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet:

Summe

Summa Potestatum.

$$\begin{array}{rcl}
 \sum n & = & \frac{1}{2}n(n+1) \\
 \sum n^2 & = & \frac{1}{3}n(n+1)(2n+1) \\
 \sum n^3 & = & \frac{1}{4}n^2(n+1)^2 \\
 \sum n^4 & = & \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + \frac{1}{4}n^2 \\
 \sum n^5 & = & \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + \frac{1}{4}n^3 \\
 \sum n^6 & = & \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{3}{2}n^5 + \frac{2}{3}n^4 + \frac{1}{4}n^3 \\
 \sum n^7 & = & \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{4}n^6 + \frac{7}{4}n^5 + \frac{7}{8}n^4 \\
 \sum n^8 & = & \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 + \frac{7}{15}n^6 + \frac{2}{5}n^5 + \frac{1}{3}n^4 \\
 \sum n^9 & = & \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{2}n^8 + \frac{3}{2}n^7 + \frac{8}{15}n^6 + \frac{3}{5}n^5 + \frac{1}{2}n^4
 \end{array}$$

Quin imò qui legem progressionis terminorum in hoc latereculo descriptorum attentius inspexerit, eundem etià continuare poterit absque his ratiociniorum ambagibus. Sumptà enim c pro potestatis cujuslibet exponente, fit summa

$$\begin{aligned} \text{ma omnium } n^c, \text{ seu } f n^c, &= \frac{1}{c+1} n^{c+1} + \frac{1}{2} n^c + \\ &\frac{c}{2} A n^{c-1} + \frac{c \cdot c-1 \cdot c-2}{2 \cdot 3 \cdot 4} B n^{c-3} + \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} C n^{c-5} \\ &+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4 \cdot c-5 \cdot c-6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} D n^{c-7} \dots \dots \text{ \& ita dein-} \end{aligned}$$

ceps, exponentem potestatis ipsius n continuè minuendo binario, quousque perveniat ad n vel nn . Literæ capitales A, B, C, D, &c. ordine denotant coëfficientes ultimorum terminorum pro $f n^c$, $f n^4$, $f n^6$, $f n^8$, &c. nempe $A = \frac{1}{2}$, $B = -\frac{1}{3}$, $C = \frac{1}{4}$, $D = -\frac{1}{5}$. Sunt autem hi coëfficientes ita comparati, ut singuli cum cæteris sui ordinis coëfficientibus complere debeant unitatem; sic D valere diximus $-\frac{1}{5}$, quia $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} (+ D) - \frac{1}{7} = 1$. Hujus laterculi beneficio intrà semi-quadrantem horæ reperi, quòd potestates decimæ, sive quadrato-sursolidæ, mille primorum numerorum ab unitate in summam collecta efficiunt

9, 1409924241424243424241924242500.

E quibus apparet, quàm inutilis censenda sit opera Ismaelis Bullialdi, quam conscribendo tam spisso volumini Arithmeticæ suæ Infinitorum impendit, ubi nihil præstitit aliud, quàm ut primarum tantum sex potestatum summas (partem ejus quod unicâ nos consecuti sumus paginâ) immenso labore demonstratas exhiberet.

De seriebus serierum figuratarum analogis.

Antequam caput hoc finiamus, paucis adhuc indicare lubet quomodo, suppositis iis quæ de seriebus figuratis ostensa sunt, possint quævis etiàm aliæ series figuratarum analogæ (quæ, scilicet, differentias suas primas, secundas, tertias, &c. æquales habent, adeoque ex continuâ additione terminorum alicujus seriei æqualium generantur) ad homologas figuratas reduci, ac proinde summari, vel postremi ipsarum termini inveniri. Sit series quævis æqualium D, ex cujus additione nascatur series C, & ex hujus additione series B, & ex hujus tandem collectione series A, sumptis ad arbitrium primis se-

F

rierum

rierum terminis d, c, b, a . Vocabitur series A figuratarum analoga, cujus differentiae primæ constituunt seriem B, secundæ seriem C, tertiæ seriem D, &c. Et quoniam appa-

D	C	B	A.
d	c	b	a
d	$c + d$	$b + c$	$a + b$
d	$c + 2d$	$b + 2c + d$	$a + 2b + c$
d	$c + 3d$	$b + 3c + 3d$	$a + 3b + 3c + d$
d	$c + 4d$	$b + 4c + 6d$	$a + 4b + 6c + 4d$
d	$c + 5d$	$b + 5c + 10d$	$a + 5b + 10c + 10d$

ret, seriem A componi ex seriebus unitatum 1, 1, 1, 1, &c. lateralium 1, 2, 3, 4, &c. trigonalium 1, 3, 6, 10, &c. pyramidalium 1, 4, 10, 20, &c. in primos differentiarum terminos a, b, c, d , seorsim ductis, quarumque omnium postremi termini & summæ per ante dicta habentur, ipsius quoque hinc seriei A postremum terminum & summam terminorum obtineri posse constat; nimirum, si numerus terminorum vocetur n , erit ultimus terminus seriei A $= a + \overline{n-1}$.

$b + \frac{\overline{n-1} \cdot \overline{n-2}}{2} c + \frac{\overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}}{2 \cdot 3} d$; & summa omnium terminorum $= na + \frac{\overline{n \cdot n-1}}{2} b + \frac{\overline{n \cdot n-1} \cdot \overline{n-2}}{2 \cdot 3} c + \frac{\overline{n \cdot n-1} \cdot \overline{n-2} \cdot \overline{n-3}}{2 \cdot 3 \cdot 4} d$.

A TRANS.

A
T R A N S L A T I O N

OF THE
²
THREE FIRST CHAPTERS

OF THE
SECOND PART, OR BOOK,

OF
MR. JAMES BERNOULLI'S EXCELLENT TREATISE,

INTITLED

A R S C O N J E C T A N D I;

OR

“THE ART OF FORMING PROBABLE CONJECTURES
CONCERNING EVENTS THAT DEPEND ON CHANCE.”

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THE PROCEMIUM, OR PREFACE, TO THE SECOND PART OF THE SAID
TREATISE

DE ARTE CONJECTANDI.

IT is easy to perceive that the prodigious variety which
appears both in the works of nature and in the actions of
men, and which constitutes the greatest part of the beauty of
the universe, is owing to the multitude of different ways in
which its several parts are mixed with, or placed near, each
other. But, because the number of causes that concur in pro-
ducing a given event, or effect, is oftentimes so immensely
great, and the causes themselves are so different one from an-
other, that it is extremely difficult to reckon up all the different
ways in which they may be arranged, or combined together,

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it often happens that men, even of the best understandings and greatest circumspection, are guilty of that fault in reasoning which the writers on logick call *the insufficient, or imperfect enumeration of parts, or cases*: inasmuch that I will venture to assert, that this is the chief, and almost the only, source of the vast number of erroneous opinions, and those too very often in matters of great importance, which we are apt to form on all the subjects we reflect upon, whether they relate to the knowledge of nature, or the merits and motives of human actions. It must therefore be acknowledged, that that art which affords a cure to this weakness, or defect, of our understandings, and teaches us so to enumerate all the possible ways in which a given number of things may be mixed and combined together, that we may be certain that we have not omitted any one arrangement of them that can lead to the object of our inquiry, deserves to be considered as most eminently useful and worthy of our highest esteem and attention. And this is the business of *the art, or doctrine of combinations*.

Nor is this art or doctrine to be considered merely as a branch of the mathematical sciences. For it has a relation to almost every species of useful knowledge that the mind of man can be employed upon. It proceeds indeed upon mathematical principles in calculating the number of the combinations of the things proposed: but by the conclusions that are obtained by it, the sagacity of the natural philosopher, the exactness of the historian, the skill and judgment of the physician, and the prudence and foresight of the politician, may be assisted; because the business of all these important professions is but *to form reasonable conjectures* concerning the several objects which engage their attention, and all wise conjectures are the results of a just and careful examination of the several different effects that may possibly arise from the causes that are capable of producing them. And, I presume, it was from a sense of the great and general utility of this doctrine that several very eminent mathematicians have undertaken to treat of it in their public writings; and particularly Mr. Van Schooten (the learned commentator on Des Cartes's geometry), Mr. Leibnitz, Dr. Wallis, and Monsieur

Monfieur Prefter: fo that the reader is not to confider every thing he will meet with in this treatife as entirely new and of my invention. I have, however, made fome improvements on the fubject, and thofe too of confiderable importance, which I may juftly call my own: and particularly I have difcovered a general and eafy demonftration of the principal and moft remarkable property of the figurate numbers, to wit, "that of the proportion between the fum of any number of terms of a feries of figurate numbers of any order whatfoever to the fum of the fame number of terms all equal to the laft term of the feries;"* upon which property many of the following propofitions in this book are founded: for of this property I believe no other writer has ever before given a demonftration.

Indeed, none of the tracts hitherto published on this fubject, can be faid to contain a full and fatisfactory account of it. And therefore I have thought it would be agreeable to my readers to fee it here treated in a regular manner, from the firft and moft fimple principles on which it is founded, to the higher and more extenfive propofitions which have been built upon them, without being under the neceffity of referring to other books upon the fubject. But, though, for thefe reafons, I have laid down the very firft elements of the doctrine, and have endeavoured to demonftrate every thing as I went on, to the end that the chain of reasoning might be uniform and compleat, I have done it in as concise a manner as I could, and only as far as was neceffary to prepare the way to the fubfequent and more important parts of the book. The greater part of the treatife confifts of two principal heads, of which the firft contains the doctrine of *permutations*, and the fecond contains the doctrine of *combinations*; which is followed by a third branch, which fprings out of the two former, and treats of *permutations* and *combinations* joined together.

Figurate. adg. of a certain and determinate form; resembling any thing of a determinate form.

CHAP.

CHAPTER I.

CONCERNING PERMUTATIONS.

ARTICLE I. **B**Y the *permutations* of a number of things, I mean the several variations that may be made in their relative situations, or positions, or in the order in which they may be made to follow each other, while their number continues the same. So that, when it is proposed to find in how many different ways a given number of things may be ranged, or disposed, without omitting any of them, this is said to be requiring the *number of their permutations*.

2. The things of which we are required to discover the number of permutations, may be either all distinguished from each other by some plain mark, such as a difference of shape or colour, as cubes from spheres, or black balls from white balls; or they may be exactly like each other, so as to be liable to be mistaken one for another, as two spherical black balls of exactly the same size and weight. In the former case it will be proper to denote the several things by as many different letters of the alphabet; and in the latter case it will be convenient to denote so many of the things as are exactly like each other, by the same letter of the alphabet, repeated as often as any of the said things which are like each other shall occur, as will be seen in the course of the following pages. We will first consider the former of these cases, or that in which all the things are distinguished from each other.

The first Case of Permutations, in which all the things whose permutations are required to be assigned, are distinguished from each other.

3. As it is obvious that the number of changes of position that may happen in a great number of things cannot be determined without first knowing the number of the like changes of position that may happen in all lesser numbers of them, it is manifestly necessary, in treating of this subject, to proceed in the synthetick method, and begin our reasonings from the first and most simple cases: which may be done as follows.

4. If there is only one thing to be arranged, which is denoted by the letter a , it can be taken, or ranged, only in one manner.

5. If there are two things clearly distinguished from each other, which are denoted by the letters a and b , it is evident that we may either place a before b , or b before a ; so that there will be two different ways of arranging them, to wit, ab and ba ; or, in other words, there will be two *permutations* of them. Q. E. I.

6. If there be three things distinguished from each other, and denoted by the three different letters a , b and c , it is evident that either of the three letters may be placed before the other two. Now, if a is placed first, the other two letters b and c may undergo two permutations, by what has been seen in the last article, and the three letters may be placed in these two positions, abc , and acb ; and in like manner, if b is placed first, the other two letters a and c may undergo two permutations, and the three letters may be placed in the two following positions, to wit, bac and bca ; and,
lastly,

lastly, if c is placed first, the other two letters a and b may undergo two permutations, and the three letters may be placed in the two following positions, to wit, cab and cba . Therefore the whole number of permutations which the order, or position, of the three letters, a , b , and c may undergo, is three times 2, or 6, to wit, abc , acb , bac , bca , cab , and cba . Q. E. I.

7. In like manner, if there are four different things clearly distinguished from each other, and denoted by the four different letters a , b , c , and d , it is evident that either of the four may be placed before the other three, and that, while each of them is placed first, the other three may undergo 6 permutations, by what has been just now shewn in art. 6. Therefore the whole number of permutations which these four things, or letters, may undergo, will be four times 6, or 24. Q. E. I.

8. And, for the same reason, if there were five things denoted by the five different letters a , b , c , d and e , the number of their permutations would be five times as great as in the last case; or would be 5 times 24, or 120. And in general, whatever be the number of things or letters, the number of permutations, or changes of position, which they may be made to undergo, will be equal to the product that arises by multiplying the number of permutations of the next smaller number of things by the given number of them. So that, if the whole number of things, or letters, be n , and the number of permutations in $n - 1$ things, or letters, be N , the number of permutations in all the n letters, will be equal $n \times N$. And hence arises the following

Rule

Rule for discovering the whole number of permutations, or relative changes of position, which any given number n , of things, may be made to undergo.

9. Let all the numbers 1, 2, 3, 4, 5, 6, 7, &c, in their natural order, beginning from unity, up to the given number n , of things, or letters, whose permutations are to be investigated, be multiplied one into the other; and the product $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times \&c \dots \times n$ will be the number of permutations that is required. Q. E. D.

10. It will be convenient sometimes to use a full point [.] instead of the common mark of multiplication $\times$; and then $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times \&c. \times n$ will be $= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \cdot n$, or (because 1 has no effect in multiplication) $= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \cdot n$; which will therefore be equal to the whole number of permutations, or changes of position, which n things may be made to undergo.

11. According to this rule, the number of permutations, or changes of position, which 7 different things may be made to undergo, is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, or 5040. Thus, for example, the different changes that may be rung upon seven different bells is 5040. The multiplications of these numbers into each other will appear in the following table:

The Number of Things. *The Number of Permutations,
or Changes of Position.*

1	—	—	1
			2
			—
2	—	—	2
			3
			—
3	—	—	6
			4
			—
4	—	—	24
			5
			—
5	—	—	120
			6
			—
6	—	—	720
			7
			—
7	—	—	5040
			8
			—
8	—	—	40,320
			9
			—
9	—	—	362,880
			10
			—
10	—	—	3,628,800
			11
			—
			3 628 800
			36 288 00
			—
11	—	—	39,916,800
			12
			—
			79 833 600
			399 168 00
			—
12	—	—	479,001,600

12. We may see by this table how very fast the number of permutations increaseth, as the number of things to be arranged becomes greater and greater. The four letters that compose the word *Roma* may be arranged in 24 different ways; but the six letters that compose the word *Romani* may be arranged in 720 different ways; and the seven letters that compose the word *Romanis* may be arranged in no less than 5040 different ways. We are now to consider the second case of permutations, in which some of the things to be arranged are exactly like others of them, so as not to be distinguished from them.

The Second Case of Permutations, in which some of the things, the permutations of which are required to be assigned, are like others of them, so as not to be distinguished from them.

13. If some of the things of which we are required to find the permutations, are exactly like others of them, so as not to be distinguished from them, the number of permutations, or changes of position, which they may be made to undergo, will be much smaller than in the former case. Thus, for example, if there are six different things, whereof we are required to assign the permutations, but three of them are exactly like each other, so that it is impossible to distinguish either of them from the other two; as is the case with the six letters *a a a b c d*, in which the letter *a* occurs three times; the number of permutations which these six things, or letters, can undergo, will be much less than the number of permutations they could undergo, if they were all distinguishable from each other, as they were supposed to be in the former case. And the way of finding out how much less the number of permutations will be in this case than in the former case, will be to consider how many permutations;

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mutations, or changes of position, the three things which are exactly alike, and are denoted by the same letter *a*, might undergo, if they were unlike each other, and distinguishable one from the other, and then to substitute an unit, or one single position, in lieu of all those several permutations. Thus, for example, if, instead of the three things exactly alike which are denoted by the same letter *a*, we were to take three things that were unlike each other, and denote them by the three letters *a*, *α*, and *α*, that is, by an Italick *a*, a Greek *α*, and a Roman *a*, it is evident from what has been shewn in art. 6, that, without making any change in the position of the other letters, *b*, *c*, *d*, these three letters *a*, *α*, and *α*, might be placed in six different positions, instead of the one position *aaa* in which alone the three things perfectly alike, that were denoted by the same letter *a*, could be arranged. The number of permutations therefore in the six things denoted by the letters *a*, *α*, *α*, *b*, *c*, *d*, will be six times as great as that of the six things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, in which three of the things are alike, and denoted by the same letter *a*. And therefore, to find the number of permutations of the six things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, we must first find the whole number of permutations which they might undergo if they were all unlike each other, and denoted by the letters *a*, *α*, *α*, *b*, *c*, *d*, and then we must divide the said number by 6, or the number of permutations which the three things denoted by the same letter *a* might undergo if they were unlike each other, and denoted by the three different letters *a*, *α*, and *α*. Now the whole number of permutations of six different things unlike each other, that are denoted by the letters *a*, *α*, *α*, *b*, *c*, *d*, has been shewn to be 720. Therefore the number of permutations of six different things, whereof three are perfectly like each other,

and

and denoted by the same letter *a*, or of six different things denoted by the letters *a, a, a, b, c, d*, will be $\frac{720}{6}$, or 120*.

14. Again, if the six letters whereof we were required to find the permutations, were *a a a b b c*, in which, besides the letter *a*, which is repeated three times, the letter *b* is also

* The truth of this article may be made visible to the eye in the following manner :

Let us (to avoid a great number of permutations, which would take up a great deal of room, and tend to confound the subject) suppose the three different letters *a, a*, and *a*, to be connected only with one more letter, to wit, *b*. Then, by art. 7, the whole number of permutations of these letters will be 24, to wit,

<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>
<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>
<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>
<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>
<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>
<i>a, a, a, b,</i>	<i>a, a, b, a,</i>	<i>a, b, a, a,</i>	<i>b, a, a, a,</i>

Now, let the Italic letter *a*, and the Greek letter α , be converted into the Roman letter *a*. And the foregoing permutations will thereby be converted into the following ones, to wit,

<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>
<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>
<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>
<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>
<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>
<i>a a a b,</i>	<i>a a b a,</i>	<i>a b a a,</i>	<i>b a a a,</i>

of which the first fix are all exactly alike, to wit, *a a a b*, and therefore must be reckoned as only one position, or permutation; and, in like manner, the next fix are also all alike, to wit, *a a b a*, and therefore must be reckoned as only one position, or permutation; and the third fix are also all alike, to wit, *a b a a*, and therefore must be reckoned as only one position, or permutation; and lastly, the fourth fix are also all alike, to wit, *b a a a*, and therefore must be reckoned as only one position, or permutation. So that, by the coincidence of six permutations into one in each of the four sets of six permutations, the said twenty-four different permutations will be reduced to only four, or $\frac{1}{6}$, different permutations, to wit, *a a a b*, *a a b a*, *a b a a*, *b a a a*.

And it is easy to see that the like reduction must take in the whole number of permutations that may happen amongst any other given number of things that are all different and distinguishable from each other, when any other and lesser number of the said things are rendered like to, and undistinguishable from, each other.

repeated

repeated twice, it is evident that the number of the permutations which the said letters could undergo, would be but half the number of the permutations of the six letters $a a a b c d$; because every two permutations of these letters which would be distinguishable from each other when the two different letters b and d are made use of, will coincide, or become undistinguishable from each other, when b is inserted instead of d . And therefore the number of the permutations of the six letters $a a a b b c$ will be only $\frac{120}{2}$, or 60.

15. And in the same manner it may be shewn that, when several of the letters, of which we are required to assign the number of permutations, are repeated, or taken more than once, we must, for every such repetition of the same letter, divide the number of permutations of the whole number of letters by the number of the permutations of so many different letters as there are repetitions of the same letter. And hence arises the following

Rule for discovering the number of permutations, or relative changes of position distinguishable from each other, which any given number n , of things, whereof some are exactly like others, and cannot be distinguished from them, may be made to undergo.

16. Let the whole number of permutations, or changes of position, which the said things might be made to undergo, if they were all unlike each other, and could be clearly distinguished one from the other, be divided by the number of permutations, or changes of position, which the two, or more, things which are like each other, and are denoted by the same letter, might be made to undergo, if they were

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unlike

unlike to each other, and clearly distinguished from each other. And the quotient will be the number of permutations that is required. This is upon a supposition that, amongst the things that are given, and of which we are required to find the number of permutations, there is only one set of things that are exactly like each other, and therefore denoted by the same letter.

But, if, amongst the things of which we are required to find the number of permutations, there should be two, or more, sets of things that are exactly like each other, and therefore denoted by the repetition of the same letters, we must multiply the number of all the permutations which the first set of like things, denoted by the first letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, into the number of all the permutations which the second set of like things, denoted by the second letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, and further into the number of all the permutations which the third set of like things, denoted by the third letter that occurs more than once in the notation, might be made to undergo, if they were all unlike each other, and into the numbers of all the permutations which the fourth set, and the fifth set, and all the following sets, of like things, denoted by the repetition of the same letters, might be made to undergo, if the things in each set were unlike each other: and the whole number of permutations, which all the n things that are given (and whereof we are required to find the number of permutations distinguishable from each other) might be made to undergo, if they were all unlike each other, must be divided by the product of the said multiplication. The quotient will be the number of permutations distinguishable from each other, of the given number n of things, which was required to be found.

17. This doctrine of permutations is of great use in determining the number of anagrams that may be made of any proposed word, or the number of different ways in which the letters that compose it may be arranged. Thus, for example,

ample; the letters that form the word *Roma* may be arranged in 2 . 3 . 4, or 24, different ways; and those of the word *Romani* (which are six in number) may be arranged in 2 . 3 . 4 . 5 . 6, or 720, different ways; and those of the word *Romanis* (which are seven in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7, or 5040, different ways; as we have seen in art. 12. In like manner the letters of the word *Trojanum* (which are eight in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8, or 40,320, different ways; and those of the word *Doctrinam* (which are nine in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9, or 362,880, different ways. But the letters of the word *Leopoldus*, though they are also nine in number, cannot be arranged in so many different ways, because of the repetition of the letters *l* and *o*, each of which occurs twice. The number of different ways in which the letters of this word can be arranged is $= \frac{362,880}{2 \cdot 2}$, or $\frac{362,880}{4}$, or 90,720; because the two *ls*, if

they were different letters, would admit of two permutations, and the two *os*, if they were different letters, would likewise admit of two permutations, and consequently these numbers of permutations, to wit, 2 and 2, must (according to the foregoing rule) be multiplied into each other, so as to make the product 4, and then the number 362,880 (which is the whole number of permutations which nine different letters may be made to undergo) must be divided by it, which gives the quotient 90,720. And the letters of the word *Studiosus*, though likewise nine in number, will admit of only 30,240 permutations, because of the repetition of the letter *s* twice, and the letter *i* three times. For the permutations which the two *ss* might be made to undergo, if they were different letters, are 2, and the permutations which the three *ies* might be made to undergo, if they were different letters, is 6; and the product that arises by multiplying 2 into 6 is 12. We must therefore divide 362,880 (which is the whole number of permutations of nine different letters) by 12; and the quotient 30,240 will be the number of all the permutations of the nine letters of the word *Studiosus* that will be different, or distinguishable from each other.

18. It is only by the assistance of this doctrine of permutations that all those questions can be determined, which some learned and ingenious men have proposed concerning the number of the variations, or transpositions of the words contained in certain verses, which, on account of the great number of such transpositions which may be made in them, have been called *Proteus verses*, in allusion to the Egyptian sea-god of that name mentioned in Homer's *Odyssæy*, who was so famous for assuming many different shapes. The most celebrated of these verses are those which have been given us by Thomas Lanſius, and the learned Joseph Scaliger, and Bernard Bauhuſius, a Jesuit of the college at Louvain[†], in the Austrian Netherlands[‡]. The following two verses we have from Thomas Lanſius:

Lex, Rex, Grex, Res, Spes, Jus, Tbus, Sal, Sol, (bona)
Lux, Laus:
Mars, Mors, Sors, Lis, Vis, Styx, Pus, Nox, Fax, (mala)
Crux, Fraus.

In each of these verses there are eleven words of one syllable, and one word of two syllables, to wit, *bona* in the first, and *mala* in the second. These two words of two syllables must always remain in the same place, or within two words of the end of the lines, in order to preserve the measure of the verses, which requires that the fifth foot in each verse should be a dactyl. But the other eleven words in each verse may be placed in any order, with respect to each other, that we please, without altering the measure of the verses. Now the number of permutations, or changes of position, that eleven different things can undergo is 39,916,800, as appears from the table in art. 11. It follows therefore that the words of each of the two foregoing verses may be transposed in 39,916,800 different ways, without spoiling the measure of them.

19. In some other instances of these *Proteus* verses that have been given by ingenious writers on this subject, it happens that many of the transpositions of the words contained

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in

in them are incompatible with the measure of the verses, and some of them, from the irregular and ungrammatical order in which the words follow each other, seem to convey no sense or meaning whatsoever, or, perhaps, in some cases, a different sense from that which the author intended. But in all these cases a little attention and care will enable us to distinguish the useful transpositions from the absurd ones, and to determine the numbers of transpositions of each sort separately, if we proceed by regular steps according to some order, or plan of admission or exclusion, in making the enquiry. An instance of this kind occurs in the following Hexameter Latin verse, which was made by the above-mentioned Bernard Bauhufius, the Jesuit of Louvain, in honour of the blessed Virgin Mary, the mother of our Saviour Jesus Christ; to wit,

Tot tibi sunt dotes, Virgo, quot sidera cælo.

On this celebrated verse several men of great learning and reputation have bestowed a great deal of attention. For, in the first place, *Ericius Puteanus*, in a little book which he published under the title of *Tbaumata Pietatis*, has employed no less than 48 pages in reckoning up the several useful, or rational, transpositions that may be made of the words contained in it, and makes them amount to as many at least as there are stars in the heavens, the number of which is usually said to be 1022; leaving out (through a religious reverence for the character of the Virgin Mary) all those transpositions which seem to affirm that there are as many stars in the heavens as there are virtues in the Virgin's character, because he thinks the number of the latter to be much greater than that of the former. And, 2dly, Gerard Vossius, in the 7th chapter of his treatise intitled, *De Scientiis Mathematicis*, has affirmed the number of the transpositions which may be made in the words of this verse without spoiling the sense or the measure, to be 1022, as *Puteanus* had made it before him. And, 3dly, Monsieur *Preftet*, a French mathematician, in the first edition of a book called "*The Elements of the Mathematicks*," page 348, has examined this *Proteus* verse, and declared it to admit of 2196 transpositions

positions of its words without spoiling the sense or the measure : and afterwards, in the second edition of his said work, vol. i. page 133, having re-considered the subject, has increased the number of these transpositions to almost half as many more, or 3276. And, 4thly, the industrious compilers of the *Leipscic Acta Eruditorum*, in the month of June, 1686, in giving an account of Dr. Wallis's *Treatise of Algebra*, have fixed the number of these transpositions (which Dr. Wallis himself had not in that treatise ventured to assign) at 2580. And, lastly, Dr. Wallis himself, in a Latin edition of his works which he published afterwards in the year 1693, page 494, has carried the number of these transpositions to 3096. But all these writers have been mistaken in their calculations, and have assigned wrong numbers for the solution of this question ; which cannot but seem rather surprising, as some of them had examined the subject twice over, and corrected their first conclusions. The true number of transpositions of its words which this famous hexameter verse will admit of without spoiling either the sense or the measure of it, that is, without admitting a spondee in the fifth place, but admitting such transpositions as only destroy the *caesura* of the verse, I have found, upon a careful examination, to be 3312.

20. I here conclude the chapter on the doctrine of permutations, of which I hope the fundamental principles have been sufficiently explained ; and I proceed to consider the doctrine of combinations, which is of no less use and importance than the former.

CHAPTER II.

CONCERNING COMBINATIONS.

DEFINITION I.

21. **B**Y the *combinations* of things, I mean the several different ways in which any given number of things may be joined, or connected with each other, without any regard to their relative positions, or the order in which they follow one another. So that, when a certain number n of things is given, and we are required to find in how many different ways these n things may be taken, by taking, first, two of them at a time, then three of them at a time, then four of them at a time, and so on in all other possible conjunctions, so that no one heap, or parcel, of them shall be taken more than once, we are said to be required to find all the possible *combinations* of the said given number of things.

DEFINITION 2.

22. The number of the things given which is directed to be joined together in one heap, or parcel, is called *the exponent* of the combination : so that if we are directed to combine them by pairs, or in parcels containing two a-piece, the exponent of the combination will be 2 ; if we are directed to combine them by triplets, or in parcels containing three a-piece, the exponent of the combination will be 3 ; and if we are directed to combine them by quadruplets, or in parcels containing four a-piece, the exponent of the combination will be 4 ; and, in general, if we are directed to combine them in parcels containing m a-piece, the exponent of the combination will be the number m .

DEFINITION

DEFINITION 3.

23. And the several things combined in these different manners are called *pairs*, or *couplets*, and *triplets*, and *quadruplets*, &c, or *binaries*, *ternaries*, and *quaternaries*, &c, or *binions*, *ternions*, and *quaternions*, &c; that is, all the different conjunctions, or combinations, of any given number of things in parcels consisting of two things each, are called all the *pairs*, or *couplets*, or *binaries*, or *binions*, in the said number of things; and all the different conjunctions or combinations of them in parcels consisting of three things each, are called all the *triplets*, or *ternaries*, or *ternions*, in them; and all the different conjunctions or combinations of them in parcels consisting of four things each, are called all the *quadruplets*, or *quaternaries*, or *quaternions*, in them. And similar names may be found for these combinations, when the number of things contained in a single parcel is greater than 4.

24. And when the things are taken singly, or separately, or one by one, it will be convenient to denominate them *unaries*, or *unities*, or to give them some name that bears a resemblance to the names by which we distinguish the several combinations of them with each other in parcels of two, or of three, or of four, or more together: because, though, when they are taken singly, they cannot, in a strict sense, be said to be *combined*, or the taking them singly cannot, in strictness, be called a *combination* of them, yet in this doctrine of combinations it is often necessary to take into consideration the number of them when taken singly, in order to determine all the variations that can be made upon them; and therefore, in a looser and more extensive sense of the word *combinations*, the things, when taken separately, are considered as undergoing one species of combination, or forming one class of the several classes of combinations which they may be made to undergo. This is a small inaccuracy of language, similar to that by which a unit is often called a *number*, though in strictness a *number* means *two* or *more* *units*,

units, or single quantities, joined together. But when due notice is given of what is meant by such inaccurate expressions (which are often convenient for the purpose of avoiding a multiplicity of words) no mistakes can arise from the use of them.

25. And for a like reason it will be convenient to give a name to the act of omitting to take them at all, either singly, or combined with each other, or to consider such omission as one species of their combinations. Such an omission of them may be called a *nullienation* (from the word *nullies*, which means *no-times*); and the *notbings*, or cyphers, set down, instead of the things themselves, on these occasions may be called *nullenaries* (like *binaries*, *ternaries*, and *quaternaries*) or *nullions*. The use of this sort of odd language will appear in the course of the following pages.

26. Some writers have confined the word *combination* to the strict original sense of "taking things by *binaries*, or *pairs*, or *couplets*, only, or parcels consisting of two things;" and have called the taking them by *ternaries*, or parcels consisting of three things, *conternation*; and the taking them by *quaternaries*, or parcels consisting of four things, *conquaternation*; and have denominated the parcels consisting of two things each, that may be formed out of a given number of things, the *combinations* of the said given things; and the parcels consisting of three things each, which may be formed out of the same given number of things, the *conternations* of the said given things; and the parcels consisting of four things each, which may be formed out of the same given number of things, the *conquaternations* of the said given things. But this degree of accuracy in our expressions would evidently lead to the composition of an immense number of new words, in order to express the variety of conjunctions that may be made of the things given in parcels of different sorts, such as parcels consisting of two things, parcels consisting of three things, parcels consisting of four things, parcels consisting of five things, parcels consisting of six things, and the like; the use of which multitude of new words might

might be found inconvenient. And therefore other persons, who were apprehensive of this inconvenience, and yet were desirous of avoiding the inaccuracy of employing the word *combinations* for parcels consisting of more than two things, have proposed to make use of the more general words *complications* or *complexions* (derived from the Latin verb *complere*, which signifies *to fold up together*) for parcels consisting of three things, or four things, or five, or more, things, each, made out of a given number of things successively so united together : and some authors, with great sagacity and judgment, have recommended the word *elections* to be used on this occasion, in order to comprehend those methods of reckoning and classing the things under consideration by which the things when taken separately, or one by one, are admitted as one species of combinations of them ; and even, when nothings, or cyphers, are taken in their stead, those nothings, or cyphers, are admitted as another species of their combinations, or elections. But the generality of writers who have treated of this subject, make use of the word *combinations* to denote all the different parcels of things, whether consisting of two things, or of three things, or of four things, or of any greater number of things, which can be formed out of a given parcel of things ; and even to denote the given things, when taken singly or separately ; and also the nothings, or cyphers, which are set down instead of them, when they are not taken at all : nor does there seem to be any necessity for inventing new words on the occasion.

These definitions of the words that will occur most frequently in this doctrine of combinations being premised, I now proceed to consider the doctrine itself.

27. Now when we are enquiring into the number of combinations of a given number of things, the said things may either be all unlike and clearly distinguishable from each other, or some of them may be exactly like others of them, so as not to be distinguishable from them. And the said things may either be so combined together that no one thing shall be contained oftener in any of the proposed combinations than it occurs in the original number of things which are proposed

proposed to be combined together; or they may be combined together without this restriction, or so that in some of the proposed combinations the same thing may occur oftener than it does in the original number of things which are proposed to be combined together, to wit, by being combined with itself *. And different suppositions may be made, and different

* I am not quite certain that this last sentence is a faithful translation of the original, which I shall therefore here subjoin for the reader's attentive consideration. *Eaque vel ita combinari debent, ut in nulla combinatione res eadem sepius continetur quam ipsa reperitur in toto rerum numero; vel sic, ut in eadem combinatione res eadem etiam sepius recurrere, hoc est, ut secum ipsa quoque combinari, possit.* The meaning of this obscure sentence (as far as I can understand it) may be illustrated by the two following examples.

In the first place, let us suppose that the things that are to be combined together are six in number, all clearly distinguishable from each other, and denoted by the six letters *a, b, c, d, e, f*. And let us suppose that these six letters are to be combined together in quaternions, or quadruplets, or parcels consisting of four letters each. Then, says the author, these quadruplets may be either restrained to those only which consist of four different letters, or in which the same letter does not occur oftener than once, or than it occurs in the original enumeration of the six things, *a, b, c, d, e, f*, out of which these quadruplets are to be formed, such as the quadruplets *abcd, acde, adef*, &c; or these quadruplets may be formed without this restriction, so as to admit the same letter to be contained in them, more than once, or to be, as it were, combined with itself, as happens in the quadruplets *abbc, aacd, aade, aaef, aabd, aabe, aabf, aace, qaab, aaac, aaad*, &c.

In the second place, let us suppose that the things that are to be combined together are, as before, six in number, but that two of them are exactly like each other, and are therefore denoted by the same letter *a*, and that three of them are also exactly like each other, and therefore denoted by the same letter *b*, and the sixth is different both from those of the first set, and from those of the second set, and is therefore denoted by the letter *c*; so that the six things that are to be combined together, are denoted by the letters *a, a, b, b, b, c*. And let it be required to combine these six things, or letters, together in quaternions, or quadruplets, or parcels consisting of four letters each. Then, says the author, these quadruplets may be either so restrained that they shall not contain either of the three letters *a, b*, and *c* oftener than it is contained in the original enumeration of them, to wit, *a, a, b, b, b, c*, that is, that no quadruplet shall contain the letter *a* oftener than twice, or the letter *b* oftener than three times, or the letter *c* oftener than once; as is the case with the quadruplets *abbc, aabc, aabb, abbb, bbbc*; or the said quadruplets may be formed without this restriction, so as to admit the letter *a* to be repeated more than twice, and the letter *b* to be repeated more than three times, and the letter *c* to be repeated more than

different questions consequently may be proposed, concerning the manner in which the quantities are combined together. For it may either be required to find the number of all the possible combinations of a given number of things, by taking them first singly, then in couplets or parcels of two, and then in triplets or parcels of three, and then in quadruplets or parcels of four, and so on according to all the exponents they will admit of; or it may be required only to find the number of all the combinations that may be made of the same given number of things according to one, or more, of their exponents, separately; as, for instance, by disposing them in parcels of two things, or in parcels of three things, or in parcels of four things, each. In each of these ways of combining the things under consideration, a great variety of questions and problems may be proposed concerning them, the full discussion of which would lead us into a very ample field of speculation. But of these we shall only select a few of the most curious and important, which we conceive to be necessary to the solution of the questions concerning the doctrine of chances; or the art of forming reasonable conjectures concerning future events depending on chance, which will be considered in the subsequent part of this treatise.

28. Let it then be required, in the first place, to find the number of all the possible combinations that can be formed of a given number of things according to all the exponents they will admit of, upon a supposition that all the things that are to be combined together, are unlike to, and clearly distinguishable from, each other, and consequently are denoted by different letters.

Let the things that are to be thus combined be denoted by the several small letters *a, b, c, d, e, &c.* Let these

than once; as is the case in the quadruplets *aaab, aaac, aaaa, bbbb, aacc, accc, &c.*

This is the only meaning that I can find for the foregoing passage; but I cannot help entertaining some doubt whether it is the true one. *Idea* *gaure,*

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letters

letters be set down in separate lines, or rows, one under another, in the manner following.

In the first line we must place the first letter *a*, by itself.

In the second line we must place the letter *b*; first, by itself; and then in conjunction with *a*, so as to form the combination, or couplet, *ab*, or *ba*. For *ab* and *ba* are, in this doctrine of combinations, to be considered as only one combination, because in this doctrine no regard is to be had to the order in which the letters are placed, as there was in the doctrine of permutations.

In the third line we must place the third letter *c*; first, by itself; and then in conjunction with the preceeding letters *a* and *b*, so as to form the binions, or couplets, *ac*, *bc*; and, lastly, with the preceeding couplet *ab*, so as to form the triplet *abc*.

a.

b, ab.

c, ac, bc, abc.

d, ad, bd, cd, abd, acd, bcd, abcd.

e, ae, be, ce, de, abe, ace, bce, ade, bde, cde, abce, abde, acde, bcde, abcde.

In the fourth line we must place the fourth letter *d*; first, by itself; and, 2dly, in conjunction with each of the three preceeding letters *a*, *b*, and *c*, so as to form the three pairs or couplets of letters, *ad*, *bd*, and *cd*; and, 3dly, in conjunction with each of the three foregoing couplets, *ab*, *ac*, *bc*, so as to form the three triplets *abd*, *acd*, and *bcd*; and, 4thly, in conjunction with the foregoing triplet, *abc*, so as to form the quadruplet, *abcd*.

And in like manner the fifth letter *e* must be placed in the beginning of the fifth line; first, by itself; and, 2dly, in conjunction with each of the four preceeding letters *a*, *b*, *c*, *d*, so as to form the four pairs, or couplets, *ae*, *be*, *ce*, and *de*; and, 3dly, in conjunction with each of the six foregoing couplets,

couplets, ab, ac, bc, ad, bd, cd , so as to form the six triplets $abe, ace, bce, ade, bde, cde$; and, 4thly, in conjunction with each of the foregoing triplets, abc, abd, acd, bcd , so as to form the four quadruplets $abce, abde, acde, bcde$; and, 5thly, in conjunction with the foregoing quadruplet $abcd$, so as to form the quintuplet, $abcde$.

And in the same manner must every following letter, f, g, h , &c, be combined with each of the preceeding letters, and with every preceeding combination of them, if the number of things, or letters, to be combined together, was more than five.

29. And from this manner of combining any given number of things, or letters, together, it is plain that we shall thereby obtain all the possible combinations of them, so that no combination of them whatsoever can be formed, or conceived, that will not be contained in one or other of the successive lines, or rows, of quantities so generated from each other: and likewise it is plain that each of the combinations so obtained will be different from every other, or that no combination will occur in the said lines more than once. And consequently the sum-total of all the combinations set down in the lines, or rows, of quantities so formed out of any given number of quantities, will be the number of all the possible combinations which the said given number of quantities will admit of. We must therefore endeavour to find the number of all the combinations of a given number of quantities that will be contained in an equal number of lines, or rows, of quantities formed, or generated, from each other in the manner above described.

30. Now, in order to discover the number of the combinations contained in a given number of the foregoing lines, or rows, of quantities, it will be proper to observe, "that every new line, or row, must contain as many combinations as all the preceeding lines, or rows, added together, and one combination over;" and for this reason, to wit, because the letter which is at the beginning of every new line, is placed

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there,

there, first, by itself, and afterwards in conjunction with all the letters and their several combinations in all the preceding lines. Thus, for example, the letter *e* is placed in the beginning of the fifth line, first by itself, and afterwards in conjunction with each of the foregoing letters *a*, *b*, *c*, *d*, and with every combination of them in couplets, triplets, and quadruplets, contained in all the four foregoing lines; and consequently the number of combinations contained in this fifth line (reckoning the letter *e* by itself for one of them) will be equal to the number of all the combinations contained in all the four preceding lines, beginning with the letters *a*, *b*, *c*, and *d*, and one combination over. This observation is of great use. For from it we may deduce the whole number of combinations contained in any given number of these lines, or which may be made with any given number of letters, by reasoning in the manner following.

A LEMMA.

31. If in the increasing geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c$, of which the first term is 1, and the common ratio is that of 1 to 2, we take any number of terms whatever, as, for example, n terms, and call the sum of the said n terms S , and afterwards add another term to the said series, the said new term will be equal to $S + 1$, or to the sum of all the former n terms together with an unit.

DEMONSTRATION.

Since the terms in the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c$, increase continually in the proportion of 1 to 2, it is evident that all the terms of it after the first term 1 will be the several powers of 2 in their natural order,

order, to wit, 2^1 , 2^2 , 2^3 , 2^4 , 2^5 , 2^6 , 2^7 , &c, and consequently that the n th, or last, term of it will be 2^{n-1} . If therefore to these n terms we add another term, the said new term will be $= 2 \times 2^{n-1}$, or 2^n . We are therefore to shew that 2^n is $= S + 1$.

Now, since S is $= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + 2^{n-1}$, $2S$ will be $= 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + 2^n$; which series consists of n terms as well as the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + 2^{n-1}$, or S . Therefore (adding 1 to both sides) $2S + 1$ will be equal to the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + 2^n$, that is, to the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + 2^{n-1}$ together with the new term 2^n , or to $S + 2^n$. Therefore (subtracting the series S from both sides) the new term 2^n will be equal to $S + 1$, or to the sum of all the n terms of the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + 2^{n-1}$, together with 1. Q. E. D.

COROLL. 1. It follows therefore that the second and third and other following terms of the increasing geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c$, may be generated, or derived, from the first term 1, not only by doubling it continually, but by the application of the property that has just now been shewn to belong to the terms of such a series, to wit, by adding together all the preceeding terms, and increasing their sum by an unit. Thus, $1 + 1$ will be $= 2$, which is the second term; and $1 + 2 + 1$, will be $(= 3 + 1) = 4$, which is the third term; and $1 + 2 + 4 + 1$, will be $(= 3 + 4 + 1 = 7 + 1) = 8$, which is the fourth term; and $1 + 2 + 4 + 8 + 1$ will be $(= 7 + 8 + 1 = 15 + 1) = 16$, which is the fifth term; and $1 + 2 + 4 + 8 + 16 + 1$ will be $(= 15 + 16 + 1 = 31 + 1) = 32$, which is the sixth term; and $1 +$
 $2 + 4$

$2 + 4 + 8 + 16 + 32 + 1$ will be ($= 31 + 32 + 1 = 63 + 1$) $= 64$, which is the seventh term; and $1 + 2 + 4 + 8 + 16 + 32 + 64 + 1$ will be ($= 63 + 64 + 1 = 127 + 1$) $= 128$, which is the eighth term. And in the same manner may all the following terms of this progression, however numerous, be generated from those that precede them by means of the aforesaid property.

COROLL. 2. And hence it follows, *à converso*, that, if there be a series of terms beginning from 1, the terms of which are generated one from the other by means of the foregoing property, or by adding together all the preceeding terms, and increasing their sum by an unit, the said series will be the geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c.$

32. Now it has been shewn above in art. 28, that the numbers of quantities, or combinations, contained in the above-mentioned lines, or rows, beginning with the letters $a, b, c, d, e, \&c.$ are generated from the first quantity a , and from each other in the manner just now described, or that the number of quantities in every new line is equal to the sum of the numbers of all the quantities in all the preceeding lines, together with an unit. It follows therefore, from coroll. 2, of the foregoing lemma, that the numbers of quantities contained in the said several lines must constitute the geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c.$ or $1 + 2^1, + 2^2, + 2^3, + 2^4, + 2^5, + 2^6, + 2^7, + \&c + 2^{n-1}$, supposing the number of lines to be n . Therefore the sum of the numbers of quantities contained in all the n lines beginning with the letters $a, b, c, d, e, \&c.$ will be equal to the sum of the first n terms of the increasing geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c.$ or to the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + 2^{n-1}$, or $1 + 2^1, + 2^2, + 2^3, + 2^4, + 2^5, + 2^6, + 2^7, + \&c + 2^{n-1}$. But, by the foregoing lemma, this series together with an unit is equal

equal to 2^n . Therefore this series alone is equal to $2^n - 1$. And consequently the number of all the quantities contained in all the said n lines, or the number of all the possible combinations of the n letters a, b, c, d, e , &c, (reckoning the said letters, when taken singly, among the said combinations) will be $= 2^n - 1$. And hence arises the following

Rule for finding the number of all the possible different combinations of a given number of things according to all their different exponents.

33. Raise the number 2 to the power of which n or the given number of things, that are to be combined together, is the index; and subtract 1 from the said power. The remainder $2^n - 1$ will be the number of combinations that was required.

34. COROLL. 1. From this rule it follows, in the first place, that, if we consider the total omission of all the n letters as one way of combining them, the number of all the possible combinations will be greater than it was before by an unit, and therefore will be $= 2^n$; and it follows, in the second place, that, if we exclude this case of the omission of all the letters (which may be called the *nullion*, or the combination of them by *nones*), and likewise exclude the several letters when taken singly, or separately, (which are not in strictness combinations of them), the number of the remaining combinations of n different things, or letters, in binions and ternions and quaternions, or in couplets, and triplets, and quadruplets, and in parcels of more than four letters in each, will be $2^n - 1 - n$, or $2^n - n - 1$.

Thus,

Thus, for example, the number of all the different conjunctions, or combinations, that may be made of the seven planets, to wit, Saturn, Jupiter, Mars, Venus, Mercury, the Earth, and the Moon, (taking the word *combinations* in the extent given to it in the foregoing rule) will be $= 2^7 - 1 (= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 128 - 1) = 127$; from which if we subtract 7, which is the number of the planets taken singly, or separately, (in which cases there are not properly any conjunctions of them), the remaining number $2^7 - 1 - 7$, or $127 - 7$, or 120, will be the number of all the possible conjunctions, or combinations, of them by conjoining two together, or three together, or four together, or five together, or six together, or all the seven together; or, it will be the number of all the possible conjunctions of them properly so called. And the twelve registers, as they are called, or rows of pipes in a musical organ, by means of which the sound of it is made to change so remarkably from a soft and gentle sound to a very loud and solemn one, may be made to undergo $2^{12} - 1$, or $4096 - 1$, or 4095 combinations, or variations.

35. CORROLL. 2. If we examine the number of combinations of the letters *a*, *b*, *c*, *d*, and *e*, in the several lines, or rows of quantities, set down above in art. 28, we shall find that the number of combinations that have even numbers for their exponents contained in each of the said lines after the first line (which contains only the single quantity *a*) is equal to the number of combinations that have odd numbers for their exponents contained in the same line. Thus, in the second line, which begins with the letter *b*, there is one quantity, namely *b*, with an odd number, to wit, 1, for its exponent, and one quantity, namely, *ab*, with an even number, to wit, 2, for its exponent. And in the third line beginning with the letter *c*, there are two quantities, namely, *c* and *abc*, with odd numbers, to wit, 1 and 3, for their exponents, and two quantities, namely, *ac* and *bc*, with an even number, to wit, 2, for their exponent. And in the fourth line beginning with the letter *d*, there are four quantities, namely, *d*, *abd*, *acd*, and *bcd*, which have the odd numbers,

numbers 1 and 3 for their exponents ; and there are four other quantities, namely, ad , bd , cd , and $abcd$, which have the even numbers 2 and 4 for their exponents. And in the fifth line, beginning with the letter e , there are eight quantities, namely, e , abe , ace , bce , ade , bde , cde , and $abcde$, which have the odd numbers 1, 3, and 5 for their exponents ; and eight other quantities, namely, ae , be , ce , de , and $abce$, $abde$, $acde$, $bcde$, which have the even numbers 2 and 4 for their exponents. And the same thing must happen in the sixth line of quantities beginning with the letter f , and in every following line, because every new line is formed by setting down the new letter first by itself, and then combining it with the first letter a , and afterwards with all the quantities contained in the second, third, and other following lines that precede the new line. Now the combination of the new letter with each of the quantities in the second, third, and other following lines, will turn all the quantities that have odd numbers for their exponents into quantities that have even numbers for their exponents, and all the quantities that have even numbers for their exponents into quantities that have odd numbers for their exponents. And therefore, as the number of quantities with odd numbers for their exponents in each of the said second, third, and other following lines, was equal to the number of quantities with even numbers for their exponents, it follows that of the new quantities in the new line arising from the combination of the new letter with all the quantities contained in the second, third, and other following lines, there will be as many that have odd numbers for their exponents as there will be that have even numbers for their exponents. And, if we add to these quantities the new letter itself, which is to be placed in the beginning of the new line, and of which the exponent is 1, and the combination of the new letter with the first letter a , of which combination the exponent is the even number 2, whereby we shall obtain all the quantities set down in the new line, it is evident that the addition of these two quantities (of which the first has the odd number 1, and the second has the even number 2, for its exponent) will not alter the equality of the numbers of combinations,

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binations, or quantities, of each kind, but that the number of quantities in the new line that have odd numbers for their exponents will still be equal to the number of quantities in the same line that have even numbers for their exponents.

36. COROLL. 3. If therefore we add all the quantities contained in all the lines except the first line (which contains only the single quantity a) together, it is evident that the number of quantities in such sum that will have odd numbers for their exponents will be equal to the number of quantities that will have even numbers for their exponents.

37. COROLL. 4. And, if we add together all the quantities contained in all the lines, including the first line, which contains the single quantity a (the exponent of which is the odd number 1), the number of quantities in such sum that will have odd numbers for their exponents will exceed by an unit the number of quantities that will have even numbers for their exponents.

38. COROLL. 5. And, if to all the quantities contained in all the lines together we add, as another combination, the case denoted by a cypher 0, or the case of the omission of all the letters, which we have above in art. 34, called the combination by *nonas*, or the *nullion*, and consider 0, or the exponent of this combination as an even number, the number of quantities in the said sum that will have odd numbers for their exponents will be exactly equal to the number of quantities that will have even numbers for their exponents.

39. COROLL. 6. It has been shewn in Coroll. 1, that the number of all the quantities in all the n lines taken together, and the case of the nullion is equal to 2^n . It follows therefore from Coroll. 5, that the number of quantities in this sum that will have odd numbers for their exponents will be equal to half of 2^n , or to $\frac{2^n}{2}$, or 2^{n-1} , and the number of
quantities

quantities in the same sum that will have even numbers for their exponents, including the nullion, will likewise be 2^{n-1} ; and consequently the number of quantities in the said sum that will have even numbers for their exponents, without including the nullion, will be $2^{n-1} - 1$. The same thing will be demonstrated in another manner here below in the 6th Corollary of Chapter 4.

CHAPTER III.

Of the numbers of combinations that may be made of a given number of things in parcels consisting of two things, or of three things, or of four things, or of any other particular number of things, each; and of the numbers known by the name of the figurate numbers, and their properties, with which the investigation of the said combinations is connected.

40. **F**ROM an attentive consideration of the five lines, or rows, of quantities in art. 28, of the foregoing chapter, beginning with the letters *a*, *b*, *c*, *d*, and *e*, and which exhibit all the different combinations that can be formed out of those five letters, it will be evident, that the *binions*, or *couplets*, or parcels consisting of two letters, in every new line of quantities, are formed by combining the single letter which is placed in the beginning of such new line, with each of the single letters contained in all the foregoing lines of quantities; and that the *ternions* or *triplets*, or parcels consisting of three letters, in such new line, are formed by combining the single letter which is placed in the beginning of such new line, with each of the *binions* or *couplets*, contained in all the foregoing lines; and the *quaternions*, or *quadruplets*, or parcels consisting of four letters,

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in

in such new line, are formed by combining the single letter which is placed in the beginning of such new line, with each of the *ternions*, or *triplets*, contained in all the foregoing lines; and, in like manner, in all higher combinations than *quaternions*, the combinations denoted by any exponent m in such new line, are formed by combining the said single letter which is placed in the beginning of such new line, with all the combinations denoted by the next lower exponent $m - 1$ contained in all the foregoing lines. It follows therefore that the number of *binions*, or *couplets*, of letters in every new line will be equal to the number of all the single letters in all the foregoing lines taken together; and that the number of ternions, or triplets, of letters in such new line, will be equal to the number of all the binions, or couplets, of letters in all the foregoing lines taken together; and the number of quaternions, or quadruplets, of letters in such new line; will be equal to the number of all the ternions, or triplets, of letters in all the foregoing lines taken together; and in like manner, that the number of combinations of any higher order than quaternions, denoted by the exponent m , in such new line, will be equal to the number of combinations of the next lower order, which is denoted by the exponent $m - 1$, in all the foregoing lines taken together. From these observations we may derive the following conclusions:

41. *First Conclusion.* As there is only one single letter in each of the said lines, or rows, of quantities, to wit, the letter in the beginning of the line, the single letters in all the lines successively will exhibit a set of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, &c, which are the figurate numbers of the first order,

42. *Second Conclusion.* As there is no binion, or couplet, of letters in the first line (which contains only the letter a), and there is only one binion in the second line, to wit, ab ; and two binions in the third line, to wit, ac , bc ; and three binions in the fourth line, to wit, ad , bd , cd ; and, in general, as the number of binions in every new line is

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equal

equal to the number of the single letters in all the preceding lines taken together; it follows that the numbers of binions, or couplets, of letters in the first, second, third, and fourth, and other following lines, will be 0, 1, $1 + 1$, $1 + 1 + 1$, $1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1$, &c, or 0, 1, 2, 3, 4, 5, &c, or the series of numbers 1, 2, 3, 4, 5, &c, in their natural order, with a cypher, 0, prefixed to them.

These numbers form an arithmetical progression, in which the common difference of the terms is 1; and they are often called *the natural numbers*, or a series of *lateral numbers*, or the figurate numbers of the second order.

43. *Third Conclusion*. As there are no ternions, or triplets, of letters in the two first lines; and there is one ternion, or triplet, to wit, abc , in the third line; and $1 + 2$, or 3 ternions, to wit, abd , acd , and bcd , in the fourth line; and $1 + 2 + 3$, or 6 ternions, to wit, abe , ace , bce , ade , bde , cde , in the fifth line; and, in general, there are as many ternions in every new line as there are binions in all the foregoing lines together; it follows that the numbers of ternions, or triplets, of letters in the first, second, third, fourth, fifth, and other following lines, of rows, of quantities, will be 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, which are formed by the continual addition of the numbers of the binions contained in the said lines, or of the terms of the series 0, 1, 2, 3, 4, 5, 6, 7, &c, with a new cypher, 0, prefixed to them.

These numbers 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, are often called the *trigonal*, or *triangular*, numbers, or the figurate numbers of the third order.

44. *Fourth Conclusion*. As there are no quaternions, or quadruplets, in the three first lines; and there is one quaternion, or quadruplet, to wit, $abcd$, in the fourth line; and $1 + 3$, or 4, quaternions, to wit, $abce$, $abde$, $acde$, and $bcde$, in the fifth line; and as, in general, there are as many quaternions in every new line as there are ternions in all the foregoing lines together; it follows that the number of quaternions, or quadruplets, of letters in the first, second, third,

third, fourth, fifth, and other following lines, or rows, of quantities, will be 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, respectively; which numbers are formed by the continual addition of the ternions, or triplets, contained in the said lines, or of the terms of the last preceeding series 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, with a new cypher, 0, prefixed to them.

These numbers 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, are often called the *pyramidal numbers*, or the figurate numbers of the fourth order.

45. *Fifth Conclusion.* In like manner the numbers of the quinions, or quintuplets, of letters contained in the several successive lines, or rows, of quantities beginning with the letters *a, b, c, d, e, &c*, will be 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, respectively; which are formed by the continual addition of the quaternions, or quadruplets, contained in the said lines, or of the terms of the last preceeding series 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, with a new cypher, 0, prefixed to them.

These numbers 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, are often called the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the fifth order. And they are also sometimes called the *triangulo-triangular*, or *trigono-trigonal*, numbers.

46. *Sixth Conclusion.* And, in like manner, the numbers of the *senions*, or *sextuplets*, of letters contained in the said several successive lines, or rows, of quantities beginning with the letters *a, b, c, d, e, f, g, h, &c*, will be 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, respectively; which are formed by the continual addition of the quinions, or quintuplets, contained in the said lines, or of the terms of the last preceeding series 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, with a new cypher, 0, prefixed to them.

These numbers 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, are often called the *pyramido-pyramidal* numbers, and sometimes the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the sixth order.

47. And

47. And in the same manner the numbers of the *septenaries*, or *septuplets*, and *octonaries*, or *octuplets*, and other higher combinations of the letters *a, b, c, d, e, f, g, h, &c.*, contained in the said several successive lines, or rows, of quantities beginning with the said letters respectively, will form the seventh and eighth and other following higher orders of the figurate numbers respectively.

48. And thus we have unexpectedly been led by the consideration of the nature of combinations to the contemplation of the *figurate* numbers, or of the numbers that are formed from a series of equal numbers, or units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, by the continual addition of all the terms, and by the like addition of all the terms of every following series so obtained. For these are the numbers to which arithmeticians have given the name of the *figurate numbers*.

49. In order to represent the several orders, or serieses, of these figurate numbers in one view, and thereby to render what I have further to observe concerning them more easy to be apprehended, I have subjoined the following table of them, containing the first twelve terms of the first twelve orders, or serieses, of the said numbers; which the reader, if he chooses it, may easily continue to a greater extent, both downwards, or towards the bottom of the page, and sideways towards the right hand. In this table the Indian, or Arabian, figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (that are placed on the left side of the table in a direction parallel to the side of the page, and separated from the table by a double black line) express the places, or numbers, of the several horizontal rows of numbers to which they are adjacent respectively, and also the numbers of letters, or things that are to be combined together. And the capital Roman figures I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, (that are placed directly over the table, parallel to the top of the page) express the places, or numbers, of the several vertical columns, and are likewise the exponents of the combinations of the letters *a, b, c, d, e, f, g, h, &c.*, which

which are represented by the said vertical columns respectively. And the said vertical columns themselves are the several orders, or serieses, of figurate numbers, or of the several combinations of the first, second, third, and other following orders, as far as the twelfth order, of which the Roman numerals I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, at the top of the table, are the exponents, with the proper number of cyphers, 0, prefixed to them. Thus, the first vertical column on the left-hand side of the table, under the Roman numeral I, is a series of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the first order of figurate numbers, which Dr. Wallis calls *Monadicks*, and represents the numbers of the letters *a, b, c, d, e, f, g, h, i, k, l*, and *m*, that occur singly, or without being joined with any other letter, in the several lines, or rows, of quantities that are set down in chap. 2, art. 28, and which are supposed to be continued to the twelfth line; and the second vertical column on the left-hand side of the table, under the Roman numeral II, is the series 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the second order of figurate numbers called *the natural*, or *lateral*, numbers*, with a cypher, 0, prefixed to them, and represents the numbers of binions, or couplets, of the letters *a, b, c, d, e, f, g, h, i, k, l*, and *m*, or combinations of them by two in a parcel, that occur in the said twelve successive lines, or rows, of quantities; and the third vertical column, under the Roman numeral III, is the series 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers, called the *trigonal*, or *triangular*, numbers, with two cyphers prefixed to them, and represents the numbers of ternions, or triplets, of the letters *a, b, c, d, e, f, g, h, i, k, l*, and *m*, or combinations of them by three in a parcel, that occur in the said twelve successive lines: and the fourth vertical column, under the Roman numeral IV, is the series 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, 165, or the fourth order of figurate numbers (called the *pyramidal numbers*) with three cyphers prefixed to them, and repre-

* See Dr. John Wallis's *Discourse of Combinations, Alterations, and Ali-*
quot Parts, bound up with his *Algebra*, page 109.

sents the numbers of quaternions, or quadruplets, of the letters $a, b, c, d, e, f, g, h, i, k, l$, and m , or combinations of them by four in a parcel, that occur in the said twelve successive lines : and the fifth vertical column, under the Roman numeral V, is the series 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, ~~210~~, ~~330~~, or the fifth order of figurate numbers (called the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers) with four cyphers prefixed to them, and represents the numbers of quintuplets, or quinions, of the letters $a, b, c, d, e, f, g, h, i, k, l$, and m , or combinations of them by five in a parcel, that occur in the said twelve successive lines : and the sixth vertical column, under the Roman numeral VI, is the series 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, or the sixth order of figurate numbers (called the *pyramido-pyramidal* numbers) with five cyphers prefixed to them, and represents the numbers of sextuplets, or senions, of the letters $a, b, c, d, e, f, g, h, i, k, l$, and m , or combinations of them by six in a parcel, that occur in the said twelve successive lines *. And, in like manner, the six following vertical columns, under the Roman numerals VII, VIII, IX, X, XI, and XII, contain the seventh, eighth, ninth, tenth, eleventh, and twelfth orders of figurate numbers, with six, seven, eight, nine, ten, and eleven, cyphers prefixed to them respectively, and represent the numbers of septuplets,

* Dr. John Wallis, in his *Discourse of Combinations, Alternations, and Aliquot Parts*, bound up with his *Algebra*, page 109, and Mr. Nicholas Mercator, in his *Logarithmotechnia*, published in the first volume of this collection of tracts, called *Scriptores Logarithmici*, page 178, call the 5th order of figurate numbers, to wit, 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, 330, &c, the *triangulo-triangular*, or *trigono-trigonal*, numbers, and the 6th order of figurate numbers, to wit, 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, instead of calling the former the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, and the latter the *pyramido-pyramidal* numbers, as our author calls them. So that there appears to be a variation amongst different writers on this subject with respect to the names to be given to the figurate numbers of the 5th and 6th, and other higher orders. And therefore, to avoid ambiguity, it seems to be most convenient to denote the figurate numbers of the 5th and 6th, and all higher orders, only by the numbers or exponents, of their orders, calling them the figurate numbers of the 5th, and the 6th, and the 7th, and the 8th, and other following higher orders.

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octuplets,

octuplets, noncuplets, decuplets, undecuplets, and duodecuplets, or of septenions, octonions, novenions, denions, undenions, and duodenions, of the letters *a, b, c, d, e, f, g, h, i, k, l, and m*, respectively, or combinations of them by seven in a parcel, and by eight in a parcel, and by nine in a parcel, and by ten in a parcel, and by eleven in a parcel, and by twelve in a parcel, that occur in the said twelve successive lines.

A Table of the first twelve orders, or serieses, of figurate numbers, or of all the different combinations that may be made of twelve different letters a, b, c, d, e, f, g, h, i, k, l, and m, by taking them, first, singly, and then combining them with each other in parcels consisting of two letters, of three letters, of four letters, of five letters, of six letters, of seven letters, of eight letters, of nine letters, of ten letters, of eleven letters, and of twelve letters.

The Roman numerals at the top of the table are the exponents of the combinations of the letters a, b, c, d, e, f, g, h, i, k, l, m, exhibited by the several vertical columns in it.

lett.	num	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.
a.	1.	1	0	0	0	0	0	0	0	0	0	0	0	0
b.	2.	1	1	0	0	0	0	0	0	0	0	0	0	0
c.	3.	1	2	1	0	0	0	0	0	0	0	0	0	0
d.	4.	1	3	3	1	0	0	0	0	0	0	0	0	0
e.	5.	1	4	6	4	1	0	0	0	0	0	0	0	0
f.	6.	1	5	10	10	5	1	0	0	0	0	0	0	0
g.	7.	1	6	15	20	15	6	1	0	0	0	0	0	0
h.	8.	1	7	21	35	35	21	7	1	0	0	0	0	0
i.	9.	1	8	28	56	70	56	28	8	1	0	0	0	0
k.	10.	1	9	36	84	126	126	84	36	9	1	0	0	0
l.	11.	1	10	45	120	210	252	210	120	45	10	1	0	0
m.	12.	1	11	55	165	330	462	462	330	165	55	11	1	0
a.	13.	1	12	66	220	495	792	924	792	495	220	66	12	1
l.	14.	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13

The first vertical column on the left hand contains the letters that are to be combined together.

The second column expresses the numbers of letters that are to be combined together.

50. The properties of the numbers exhibited in the foregoing table are truly curious and surprising. For it not only contains in it (as we have seen in the foregoing pages) the clue to the mysterious doctrine of combinations, but it is also the ground, or foundation, of most of the important and abstruse discoveries that have been made in the other branches of the mathematicks, as is well known to those persons who are skilled in the higher parts of geometry. We shall here give a slight sketch, and but a slight one, of some of the said properties, without a formal demonstration of any of them except the twelfth and last, which is that which is most immediately connected with the subject of combinations which we are here inquiring into; the other eleven properties being either easy consequences of the said 12th property, or being sufficiently evident from the manner in which the foregoing table was constructed, or the several orders of figurate numbers were generated from each other.

Some wonderful properties of the foregoing table of combinations.

51. The first property. The second of the vertical columns of numbers in the said table, or that which is placed under the exponent II, begins with one cypher; the third column begins with two cyphers; the fourth column with three cyphers; and, in general, every column with as many cyphers, wanting one, as there are units in the exponent of the combinations represented by it, so that, if the exponent

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of

of the column is c , the number of cyphers in the beginning of the column will be $c-1$.

This property is too evident to need, or, perhaps, to admit of, any proof.

52. The second property. The first significant terms of the several vertical columns, taken in their order in a slanting line downwards from the top of the table on the left hand to the bottom of it on the right hand, are the same with the significant terms of the first vertical column; and the second significant terms of the several vertical columns, taken in the same manner, are the same with the significant terms of the second vertical column; and the third significant terms of the several vertical columns, taken in the same manner, are the same with the significant terms of the third vertical column; and, in like manner, the fourth, fifth, sixth, and other following, significant terms of the several vertical columns are the same with the significant terms of the fourth, fifth, sixth, and other following, vertical columns, respectively: so that the first of those oblique lines of terms constitutes a series of units, or 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the first order of figurate numbers; and the second of those oblique lines of terms constitutes a series of the natural, or lateral, numbers, or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the second order of figurate numbers; and the third of those oblique lines of terms constitutes a series of the trigonal or triangular numbers, to wit, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers; and, in like manner, the fourth, fifth, sixth, and other following oblique lines of terms, constitute the fourth, fifth, sixth, and other following orders of figurate numbers respectively.

53. The third property. The second term in every vertical column, beginning from 1, is the same number as the exponent of the combinations exhibited by the said column, or as the number denoted by a Roman numeral at the top of the column, which denotes its place or order in the table. Thus, the second term of the fourth vertical column 1, 4, 10, 20, 35, 56, 84, 120, 165, reckoning from 1, is 4;
and

and the second term of the fifth vertical column is 5, and the second term of the sixth vertical column is 6; and the same thing may be observed in all the following columns.

This property is too evident to need a proof.

54. The fourth property. Every term in the foregoing table is equal to the sum of all the terms that stand above it in the next preceeding vertical column. Thus, for example, 56 (which is the sixth significant term in the fourth vertical column) is equal to the sum of 1, 3, 6, 10, 15, and 21, which are the first six significant terms of the next preceeding, or third, vertical column, and which all stand above the said term 56, or above the term 28 in the third vertical column, which stands even with the said term 56 in the fourth column.

This property is manifest from art. 40, 41, 42, &c. — 48.

55. The fifth property. Every term in the table, after the first term 1 in each horizontal row of terms, is equal to the sum of the two terms that stand immediately above it in the same vertical column and in the next preceeding vertical column. Thus, for example, 56 (which is the 9th term in the fourth vertical column, including the cyphers, or the sixth term exclusive of the cyphers) is equal to the sum of 35, which is the term next above 56 in the same vertical column, and 21, which is the term next above 56 in the next preceeding or third vertical column. And, from the manner in which the table is formed, the same thing is evident of every other term in the table.

56. The sixth property. The terms of every transverse, or horizontal, column increase gradually from 1 to a certain magnitude, and then decrease again by the same degrees to 1, so as to make the terms that are equidistant from the beginning and the end of the column be equal to each other. Thus, for example, the terms of the 7th transverse, or horizontal column are 1, 6, 15, 20, 15, 6, 1, in which the first and the last term are, both of them, 1, the second term

term and the last term but one, are, both of them, 6, the third term and the last term but two are, both of them, 15, and the middle term is 20, which is greater than any of the others. And the same thing may be observed in all the other transverse or horizontal columns.

57. This property may be shewn to be general or to extend to all the transverse, or horizontal columns; how numerous soever, to which we may suppose the table to be continued, as well as to the twelve columns set down in the foregoing table, by proving that, if it is true in any one horizontal column, (as we have seen that it is in all the horizontal columns set down in the foregoing table), it will also be true in the next following horizontal column. Now this may be proved in the manner following.

It appears from the fifth property already mentioned, that the second and other following terms of every new transverse, or horizontal, column are equal to the sums of the two terms that stand immediately above them in the same vertical column, and in the next preceeding vertical column, respectively, or to the successive sums of the terms of the next preceeding horizontal column, taken two by two. Thus, for example, the second, third, and other following terms of the 8th horizontal column are 7, 21, 35, 35, 21, 7, and 1; of which 7 is $= 1 + 6$, or the sum of the first and second term of the next preceeding, or 7th horizontal column 1, 6, 15, 20, 15, 6, 1; and 21 is $= 6 + 15$, or the sum of the second and third terms of the said seventh horizontal column; and 35 is $= 15 + 20$, or the sum of the third and fourth terms of the said seventh horizontal column; and the second 35 is $= 20 + 15$, or the sum of the fourth and fifth terms of the said seventh horizontal column; and the second 21 is $= 15 + 6$, or the sum of the fifth and sixth terms of the said seventh horizontal column; and the second 7 is $= 6 + 1$, or the sum of the sixth and seventh terms of the said seventh horizontal column; and the second 1 is $= 1 + 0$, or the sum of the seventh and eighth terms of the said 7th horizontal column. So that the terms of the said 8th horizontal column, 1, 7, 21, 35, 35,

35,

35, 21, 7, 1, may be derived from the terms of the next preceeding, or 7th, horizontal column, 1, 6, 15, 20, 15, 6, 1, by setting down the latter terms twice following in two parallel horizontal rows, with the terms in the lower row advanced one step to the right hand beyond those in the upper row, so that the first term of the second row shall be immediately under the second term of the first row, and the second term of the second row under the third term of the first row, and the third term of the second row under the fourth term of the first row, and every following term of the second row under the next higher term of the first row, and then adding the correspondent terms, or the terms which stand in the same vertical lines, together, in the manner following.

$$\begin{array}{r}
 1, 6, 15, 20, 15, 6, 1 \\
 1, 6, 15, 20, 15, 6, 1 \\
 \hline
 1, 7, 21, 35, 35, 21, 7, 1
 \end{array}$$

Now from this manner of deriving the terms of the 8th horizontal row of numbers from the 7th horizontal row of numbers, it is manifest that, since the terms of the seventh row that are equidistant from the two extreme terms 1 and 1, are equal to each other, the terms of the 8th row which are equidistant from the two extreme terms 1 and 1, must likewise be equal to each other, being the sums of equal numbers that are added together in an opposite order, to wit, $6 + 1$ and $1 + 6$, $15 + 6$ and $6 + 15$, and $20 + 15$ and $15 + 20$. And this method of reasoning will prove in like manner that, since the terms of the 8th horizontal row of numbers that are equidistant from the two extreme terms 1 and 1 are equal to each other, the terms of the 9th horizontal row of numbers that are equidistant from the two extreme terms 1 and 1 will also be equal to each other; and, consequently that, to whatever extent the table be supposed to be continued, the terms of every following horizontal row of numbers that are equidistant from the two extreme terms 1 and 1 will be equal to each other.

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58. The

58. The seventh property. If we take a certain number of vertical columns of numbers in the foregoing table, and continue the terms in each column till they are as many as there are columns, and then add up the several numbers in each column, and place the sums thereby obtained in a new horizontal line, or row, at the bottom of the said columns, the first of these sums will be equal to the last but one, the second of them to the last but two, the third of them to the last but three, and in general, the m th term to the last but m . Thus, for example, if we take the first eight vertical columns, and continue them to eight terms each (including the cyphers in the beginning of all but the first column), and then add the numbers in each separate column into one sum, the sums thereby obtained will be 8, 28, 56, 70, 56, 28, 8, and 1; of which the first sum 8 is equal to the last but one, the second sum 28 is equal to the last but two, and the third sum 56 is equal to the last but three, and the fourth term 70 is itself the last term but four.

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
1	3	3	1	0	0	0	0
1	4	6	4	1	0	0	0
1	5	10	10	5	1	0	0
1	6	15	20	15	6	1	0
1	7	21	35	35	21	7	1
8	28	56	70	56	28	8	1

59. This property follows from the fourth property, which is mentioned above in art. 54. For, by that property, each of the said sums is equal to the next following term of the next horizontal row of numbers, that is, in the example here given, of the ninth horizontal row of numbers in the foregoing table, which is 1, 8, 28, 56, 70, 56, 28, 8, and 1; to wit, the sum of the units in the first vertical column is equal to the second term 8 of the said ninth horizontal row of numbers, and the sum of the numbers 1, 2, 3, 4, 5, 6, 7, in the second vertical column is equal to the third
8 term

term 28 of the said ninth horizontal row of numbers; and the sum of the numbers 1, 3, 6, 10, 15, 21, in the third vertical column is equal to the fourth term 56 of the said ninth horizontal row of numbers; and in like manner, the sums of the numbers 1, 4, 10, 20, 35, in the fourth vertical column, and of the numbers 1, 5, 15, 35, in the fifth vertical column, and of the numbers in the sixth, seventh, and eighth vertical columns, are respectively equal to the numbers 70, 56, 28, 8, and 1, or the fifth, sixth, seventh, eighth, and ninth, terms of the said ninth horizontal row of numbers. But, by the sixth property (which has been mentioned above in art. 56, and demonstrated in art. 57) the terms of the said ninth horizontal row of numbers, 1, 8, 28, 56, 70, 56, 28, 8, and 1, that are equidistant from the two extreme terms 1 and 1, are equal to each other, and must be so from the manner in which they are generated. Therefore the said sums of the numbers contained in the said eight vertical columns, being equal to the second, third, fourth, and other following terms of the said ninth horizontal row of numbers, must be such that, if an unit be prefixed to them (whereby their number will be increased to nine terms), the terms that are equidistant from the two extreme terms 1 and 1 will be equal to each other. And consequently, if an unit be not prefixed to them, the first of those sums will be equal to the last of them but one, and the second of them will be equal to the last but two, and the third of them will be equal to the last but three, and the m th of them will be equal to the last but m . And this, it is evident, will be true, if instead of eight vertical columns continued to eight terms each, we were to take any other number of vertical columns, how great soever, and continue them till the number of the terms in each column (including the cyphers) was equal to the number of the columns. And therefore it is true universally. Q. E. D.

60. The eighth property. The horizontal rows of numbers in the foregoing table of combinations, beginning with the second row, exhibit the co-efficients of the several successive powers of a binomial quantity, as $a + b$. Thus, the

M numbers

numbers in the second horizontal row, to wit, 1 and 1, are the co-efficients of the two members a and b of the said binomial quantity $a + b$ itself, or (as it is sometimes called) of the first, or simple, power of the said binomial quantity. The numbers in the third horizontal row, to wit, 1, 2, and 1, are the co-efficients of the several terms aa , $2ab$, and bb , of the compound quantity $aa + 2ab + bb$, which is the square, or second power, of the said binomial quantity. The numbers in the 4th horizontal row, to wit, 1, 3, 3, and 1, are the co-efficients of the several terms of the compound quantity $a^3 + 3a^2b + 3ab^2 + b^3$, which is the cube, or third power, of the said binomial quantity. The numbers in the 5th horizontal row, to wit, 1, 4, 6, 4, and 1, are the co-efficients of the several terms of the compound quantity $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which is the 4th power of the said binomial quantity. And, in like manner, the numbers in the 6th, 7th, 8th, 9th, and every following horizontal row to the n th row (n being any whole number whatsoever) will be the co-efficients of the terms of the 5th, 6th, 7th, 8th, and every following power of the said binomial quantity $a + b$, to the $n - 1$ th power, respectively; and the numbers in the $n + 1$ th horizontal row of terms in the said table will be the co-efficients of the terms of the n th power of the said binomial quantity.

61. This property will appear from the manner in which the powers of the binomial quantity $a + b$ are generated from each other by multiplication, which is as follows :

$$a + b$$

$$\begin{array}{r}
 a+b \\
 a+b \\
 \hline
 aa+ab \\
 +ab +bb \\
 \hline
 aa+2ab+bb = \overline{a+b}^2 \\
 a+b \\
 \hline
 a^3+2a^2b+abb \\
 +a^2b+2abb+b^3 \\
 \hline
 a^3+3a^2b+3ab^2+b^3 = \overline{a+b}^3 \\
 a+b \\
 \hline
 a^4+3a^3b+3a^2b^2+ab^3 \\
 +a^3b+3a^2b^2+3ab^3+b^4 \\
 \hline
 a^4+4a^3b+6a^2b^2+4ab^3+b^4 = \overline{a+b}^4.
 \end{array}$$

Or, if, for brevity's sake, we substitute 1 + 1 instead of $a+b$, the multiplication will be as follows :

$$\begin{array}{r}
 1+1 \\
 1+1 \\
 \hline
 1+1 \\
 +1+1 \\
 \hline
 1+2+1 = \overline{1+1}^2 \\
 1+1 \\
 \hline
 1+2+1 \\
 +1+2+1 \\
 \hline
 1+3+3+1 = \overline{1+1}^3 \\
 1+1 \\
 \hline
 1+3+3+1 \\
 +1+3+3+1 \\
 \hline
 1+4+6+4+1 = \overline{1+1}^4.
 \end{array}$$

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Here

Here we see that every new power of the binomial quantity $1 + 1$ is formed by adding together the terms of the next preceeding power of it by two at a time, or in setting down the terms of the preceeding power twice following in two parallel horizontal rows, with the terms in the lower row advanced one step to the right hand beyond those in the upper row; which is the manner in which the several horizontal rows of numbers in the foregoing table of combinations are, or may be, derived from each other, as appears from the 5th and 6th properties above-mentioned in art. 55, 56, and 57. Therefore, since these co-efficients of the terms of the powers of the binomial quantity $1 + 1$ are generated from $1 + 1$, in the same manner as the 3d, 4th, 5th, and other following rows of numbers in the foregoing table of combinations are generated from the same numbers 1 and 1 in the second horizontal row, it follows that the numbers contained in the 3d, 4th, 5th, and other subsequent horizontal rows must coincide with, or be the same with, the co-efficients of the terms of the square, cube, fourth power, and other following correspondent powers of the said binomial quantity. Q. E. D.

62. The ninth property. The sums of the numbers contained in the several successive horizontal rows in the foregoing table of combinations increase continually in the proportion of 1 to 2; or the sum of the numbers in every new horizontal row is double of the sum of the numbers in the next preceeding horizontal row. These sums are as follows:

In the 1st horizontal row $1 + 0 + 0$ &c are = 1.
 In the 2d horizontal row $1 + 1 + 0 + 0$ &c are = 2.
 In the 3d row $1 + 2 + 1 + 0 + 0$ &c are = 4.
 In the 4th row $1 + 3 + 3 + 1 + 0 + 0$ &c are = 8.
 In the 5th row $1 + 4 + 6 + 4 + 1 + 0 + 0$ &c are = 16.
 In the 6th row $1 + 5 + 10 + 10 + 5 + 1$ &c are = 32.
 In the 7th row $1 + 6 + 15 + 20 + 15 + 6 + 1$ are = 64.
 In the 8th row $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$ are = 128.

Each

Each of these sums is double of the sum immediately preceeding it. And the same thing is true of the sums of the terms of the four following horizontal rows in the foregoing table, and of the sums of the terms of all the following horizontal rows that would belong to it, if it were continued to any greater number of vertical columns and horizontal rows whatsoever.

63. This property follows from what has been shewn above in art. 57, to wit, that every new horizontal row of numbers in the said table may be derived from the next preceeding horizontal row of numbers by setting down the numbers of the said preceeding row twice following in two parallel horizontal rows, with the terms in the lower row advanced one step further to the right hand than the terms in the upper row, and then adding the terms of the two rows that stand in the same vertical lines together. For the sum of the numbers contained in the new horizontal line arising from the addition of the said two lines together, must evidently be double of the sum of the numbers in only one of the lines so added. Thus, if we set down the numbers of the 7th horizontal row of numbers, to wit, 1, 6, 15, 20, 15, 6, 1, twice following in two parallel rows one under the other, as follows,

1, 6, 15, 20, 15, 6, 1

1, 6, 15, 20, 15, 6, 1,

and then add the two rows together, so as to make a new line of numbers, to wit,

1, 7, 21, 35, 35, 21, 7, 1,

it is evident that the sum of the numbers contained in this new line must be double of the sum of the numbers contained in either of the two former lines. Q. E. D.

64. This property may also be derived from the last or 8th property, set forth and proved in art. 60 and 61. For, since every new power of the binomial quantity $1 + 1$ must be greater than the next preceeding power of it in the proportion of $1 + 1$, or 2, to 1; and it has been shewn in art. 61, that the several horizontal rows of numbers in the foregoing

going table exhibit the members of the several successive powers of the binomial quantity $1 + 1$; it follows that the said horizontal rows of numbers must be greater, one than the other, in the same proportion of $1 + 1$, or 2, to 1.

Q. E. D.

65. The tenth property. If the sums of the numbers contained in the several horizontal rows of the foregoing table of combinations be continually added to each other, the new sums thence arising, or the sums of the former sums, will form a series of numbers, which will be equal to the several powers of 2, with an unit subtracted from them, or to $2 - 1$, $2^2 - 1$, $2^3 - 1$, $2^4 - 1$, $2^5 - 1$, $2^6 - 1$, $2^7 - 1$, &c. or to $2 - 1$, $4 - 1$, $8 - 1$, $16 - 1$, $32 - 1$, $64 - 1$, $128 - 1$, &c. or 1, 3, 7, 15, 31, 63, 127, &c.

Thus, for example, the sums of the eight first horizontal rows of numbers are 1, 2, 4, 8, 16, 32, 64, and 128, respectively, as we have seen in art. 62. Now, if these sums are added together, we shall have

$$1 = 1 = 2 - 1$$

$$1 + 2 = 3 = 4 - 1 = 2^2 - 1,$$

$$1 + 2 + 4 = 7 = 8 - 1 = 2^3 - 1,$$

$$1 + 2 + 4 + 8 = 15 = 16 - 1 = 2^4 - 1,$$

$$1 + 2 + 4 + 8 + 16 = 31 = 32 - 1 = 2^5 - 1,$$

$$1 + 2 + 4 + 8 + 16 + 32 = 63 = 64 - 1 = 2^6 - 1,$$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 = 127 = 128 - 1 = 2^7 - 1, \text{ and}$$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255 = 256 - 1 = 2^8 - 1.$$

And universally, if the number of the horizontal rows be n , the sum of the sums of all the numbers contained in them will be $2^n - 1$.

66. This is evident from art. 33. For the sum of all these sums is the number of all the possible combinations of n letters; which is shewn in art. 33 to be $= 2^n - 1$. Therefore the successive sums of the sums of all the numbers contained

tained in the said horizontal rows will form the series $2 - 1, 2^2 - 1, 2^3 - 1, 2^4 - 1, 2^5 - 1, 2^6 - 1, 2^7 - 1, 2^8 - 1, \&c \dots 2^{n-1} - 1$. Q. E. D.

67. The eleventh property. If we divide the terms of the second, or any other of the vertical columns of the foregoing table of combinations by the corresponding terms (or terms situated in the same horizontal line) of the next preceding vertical column, the several quotients thence arising will be equal to the terms of an arithmetical progression consisting of fractions, of which the common difference is a fraction of which an unit is the numerator, and the number which is the exponent of the first of the said vertical columns (by the terms of which the terms of the other vertical column are divided) is the denominator.

Thus, for example, if we divide each of the terms of the second vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 (omitting the first term, which is a cypher, or 0), by the corresponding term of the first vertical column 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, the quotients will be $\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{7}{1}, \frac{8}{1}, \frac{9}{1}, \frac{10}{1}, \frac{11}{1}$, which differ from each other by the common difference $\frac{1}{1}$.

And, if we divide each of the terms of the third vertical column 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 (omitting the two cyphers) by the corresponding term of the second vertical column, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, the quotients will be $\frac{1}{2}, \frac{3}{3}, \frac{6}{4}, \frac{10}{5}, \frac{15}{6}, \frac{21}{7}, \frac{28}{8}, \frac{36}{9}, \frac{45}{10}$, and $\frac{55}{11}$, which are respectively equal to $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \frac{8}{2}, \frac{9}{2}$, and $\frac{10}{2}$, which differ from each by the fraction $\frac{1}{2}$, of which 1 is the numerator, and 2, or the exponent of the second vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

10, 11, (by the terms of which the terms of the third vertical column are divided), is the denominator.

If we divide the terms of the fourth vertical column (omitting the cyphers at the beginning) by the corresponding terms of the third column, the quotients will be as follows; to wit,

$$\begin{array}{r} \frac{1}{3} = \frac{1}{3} \\ \frac{4}{6} = \frac{2}{3} \\ \frac{10}{10} = \frac{3}{3} \\ \frac{20}{15} = \frac{4}{3} \\ \frac{35}{21} = \frac{5}{3} \\ \frac{56}{28} = \frac{6}{3} \\ \frac{84}{36} = \frac{7}{3} \\ \frac{120}{45} = \frac{8}{3} \end{array}$$

and $\frac{165}{55} = \frac{9}{3}$, which differ from each other

by the fraction $\frac{1}{3}$, of which 1 is the numerator, and 3, or the exponent of the third vertical column (by the terms of which the terms of the fourth vertical column are divided), is the denominator.

In like manner the quotients that arise by dividing the terms of the fifth vertical column by the corresponding terms of the fourth vertical column are the following, to wit,

$\frac{1}{4}$, $\frac{5}{10}$, $\frac{15}{20}$, $\frac{35}{35}$, $\frac{70}{56}$, $\frac{126}{84}$, $\frac{210}{120}$, and $\frac{330}{165}$, which are respectively equal to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$, and $\frac{8}{4}$.

And the quotients that arise by dividing the terms of the sixth vertical column by the corresponding terms of the fifth

fifth vertical column are $\frac{1}{5}$, $\frac{6}{15}$, $\frac{21}{35}$, $\frac{56}{70}$, $\frac{126}{126}$, $\frac{252}{210}$, and $\frac{462}{330}$, which are respectively equal to $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, $\frac{6}{5}$, and $\frac{7}{5}$.

The quotients that arise by dividing the terms of the seventh vertical column by the corresponding terms of the sixth column are $\frac{1}{6}$, $\frac{7}{21}$, $\frac{28}{56}$, $\frac{84}{126}$, $\frac{210}{252}$, and $\frac{462}{462}$, which are respectively equal to $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$.

The quotients that arise by dividing the terms of the eighth vertical column by the correspondent terms of the seventh column are $\frac{1}{7}$, $\frac{8}{28}$, $\frac{36}{84}$, $\frac{120}{210}$, and $\frac{330}{462}$, which are respectively equal to $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, and $\frac{5}{7}$.

The quotients that arise by dividing the terms of the ninth vertical column by the corresponding terms of the eighth column are $\frac{1}{8}$, $\frac{9}{36}$, $\frac{45}{120}$, and $\frac{165}{330}$, which are respectively equal to $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, and $\frac{4}{8}$.

The quotients that arise by dividing the terms of the tenth vertical column by the correspondent terms of the ninth column are $\frac{1}{9}$, $\frac{10}{45}$, $\frac{55}{165}$, which are respectively equal to $\frac{1}{9}$, $\frac{2}{9}$, and $\frac{3}{9}$.

And the quotients that arise by dividing the terms of the eleventh vertical column by the correspondent terms of the tenth column are $\frac{1}{10}$ and $\frac{11}{55}$, which are respectively equal to $\frac{1}{10}$ and $\frac{2}{10}$.

68. This property of these numbers might, if it were necessary to the main object of this Treatise, be derived from
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the following, or 12th, property of the said numbers, which we will now proceed to set forth and to demonstrate.

69. The twelfth property. The sum of all the numbers contained in any one of the vertical columns of the foregoing table of combinations, is to the sum of the like number of terms that should be all equal to the last term of the column, in the same proportion as 1 to the exponent of the column, or the number which denotes the place of the column, and which is marked by a Roman numeral figure at the top of it.

Thus, in column 1st, which consists entirely of units, the sum of all the twelve terms is 12, which is to the sum of twelve terms all equal to the last term as 1 to 1, or the exponent of the first column. This proposition is self-evident.

In the 2d column the twelve terms are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; the sum of which is 66. And the sum of twelve terms equal to the last, or greatest, term 11, is 132. Now 66 is to 132 as 1 is to 2, or the exponent of this second column.

In the 3d column the twelve terms are 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55; the sum of which is 220. And the sum of twelve terms equal to the last, or greatest, term 55, is 660. Now 220 is to 660 as 1 is to 3, or the exponent of the third column.

In the 4th column the twelve terms are 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, and 165; the sum of which is 495. And the sum of twelve terms equal to the last, or greatest, term 165, is 1980. Now 495 is to 1980 as 1 is to 4, or the exponent of the fourth column.

In the 5th column the twelve terms are 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, and 330; the sum of which is 792. And the sum of twelve terms equal to the last, or greatest, term 330, is 3960. Now 792 is to 3960 as 1 is to 5, or the exponent of the fifth column.

In the 6th column the twelve terms are 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, and 462; the sum of which is 924.

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And

And the sum of twelve terms equal to the last, or greatest, term 462, is 5544. Now 924 is to 5544 as 1 is to 6, or the exponent of the sixth column.

In the 7th column the twelve terms are 0, 0, 0, 0, 0, 0, 1, 7, 28, 84, 210, and 462; the sum of which is 792. And the sum of twelve terms equal to the last, or greatest, term 462, is 5544. Now 792 is to 5544 as 1 is to 7, or the exponent of the seventh column.

In the 8th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 1, 8, 36, 120, and 330; the sum of which is 495. And the sum of twelve terms equal to the last, or greatest, term 330, is 3960. Now 495 is to 3960 as 1 is to 8, or the exponent of the eighth column.

In the 9th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 1, 9, 45, and 165; the sum of which is 220. And the sum of twelve terms equal to the last, or greatest, term 165, is 1980. Now 220 is to 1980 as 1 is to 9, or the exponent of the said ninth column.

In the 10th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 10, and 55; the sum of which is 66. And the sum of twelve terms all equal to the last, or greatest, term 55, is 660. Now 66 is to 660 as 1 is to 10, or the exponent of the said tenth column.

In the 11th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, and 11; the sum of which is 12. And the sum of twelve terms all equal to the last, or greatest, term 11, is 132. Now 12 is to 132 as 1 is to 11, or the exponent of the said eleventh column.

In the 12th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, and 1; the sum of which is 1. And the sum of twelve terms all equal to the last, or greatest, term 1, is 12. Now 1 is to 12 as 1 is to the exponent of the said twelfth column, that exponent being 12.

And the same thing will be found to be true, if, instead of taking twelve terms in each of the said vertical columns, we were to take any lesser number, as five, or six, or seven, terms, or any greater number of terms whatsoever, as fif-

teen, or twenty, or a hundred terms, continuing the table both downwards and sideways for that purpose, namely, that, if the exponent of any one of the vertical columns be called c , the sum of all the terms in the said column, continued to any number of terms whatsoever, will be to the sum of as many terms all equal to the last, or greatest, term of the said column, as 1 is to c .*

70. This

* The attentive reader may perhaps have observed, in reading the foregoing translation of the twelve surprising properties of the numbers contained in the table of combinations, exhibited in page 74, (which properties are set forth in pages 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, and 91) that the 7th property (which is set forth in art. 58, page 80) is not the same with the 7th property in the author's original, (which is contained in page 19), but answers to the 8th property set forth in the said original; and that there is no property in the translation that exactly answers to the said 7th property in the original. The reason of this omission is, that the 7th property set forth in the author's original, seemed to me to be the same with the next preceeding, or 6th, property, and therefore to be an unnecessary repetition. But, whether it is so, or not, must be referred to the reader's judgment. And therefore I will here set down both the 6th and the 7th property, as they are expressed in the original.

The sixth property is expressed in these words. *Columnæ cujusvis transversæ termini ab unitate aliquotusque crescunt, deinde per eisdem gradus rursùm decrescunt. Idem intellige de summis columnarum verticalium æquè-altarum, seu terminis sequentis columnæ transversæ, per quartam proprietatem.*

And the seventh property is expressed in these words. *Columnarum verticalium æquè-altarum bases, sive termini columnæ transversæ cujuslibet, primus quidem et ultimus significativus perpetuò inter se æquantur, ut et secundus, et penultimus, tertius et antepenultimus, atque ita porro, si columna pluribus terminis significativis consistit.*

Now this 7th property seems to me to be a mere repetition of the foregoing 6th property, and particularly of the first sentence of it, to wit, *Columnæ cujusvis transversæ termini ab unitate aliquotusque crescunt, deinde per eisdem gradus rursùm decrescunt.* These words, "*ab unitate crescunt, deinde per eisdem gradus rursùm decrescunt*" seem only to be paraphrased, or more fully explained, by the words of the 7th property, to wit, *primus quidem et ultimus perpetuò inter se æquantur, ut et secundus et penultimus, tertius et antepenultimus, atque ita porro.* Therefore, as I could find no new meaning to the words of the 7th property, whereby it could be distinguished from the 6th property, I thought it better to omit it.

Yet we may observe that there are twelve properties of the figurate numbers, or numbers contained in the foregoing table of combinations, set down in the translation as well as in the author's original. This is owing to my having divided the 10th property of these numbers mentioned in the author's original

70. This is the most important property belonging to the figurate numbers, and that which will be of most use to us in treating of the doctrine of chances, or the art of forming probable conjectures concerning events that depend on chance, which is the subject of this treatise. I shall therefore now endeavour to demonstrate this property of the said numbers in a scientific and satisfactory manner, and so as to convince my readers that it must be true in all cases whatsoever (however great the number of vertical columns, and that of the terms in each vertical column, may be supposed to be taken) as well as in the small number of cases exhibited in the foregoing table of combinations. And in order to this I shall proceed to lay down the four following preliminary propositions, or lemmas, as the ground-work of the following demonstration.

LEMMA I.

71. The sum of any number of terms whatsoever in the first vertical column in the foregoing table of combinations, is equal to the sum of an equal number of terms that are all equal to the last term.

original (which consists of two branches) into two separate properties, calling the first branch of the said 10th property in the original (which is expressed in these words, "*Summæ serierum transversarum progrediuntur in continuâ ratione duplâ*") the 9th property in the translation; and the second branch of it (which is expressed in these words, "*Summarum verò summæ, ab initio collectæ, terminos constituunt progressionis duplæ unitate multatos*") the 10th property in the translation.

The differences therefore between the properties of these numbers, as expressed in the translation, and as expressed in the original, are as follows. The six first properties of these numbers in the translation answer to the six first properties of them in the original respectively; the 7th property in the translation answers to the 8th property in the original: the 8th property in the translation answers to the 9th property in the original: the 9th and 10th properties in the translation answer to the first and second branches of the 10th property in the original: and the 11th and 12th properties in the translation answer to the 11th and 12th properties in the original, respectively.

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DEMONSTRATION.

This is evident, because all the terms in the first column are units, or equal to the last term. Therefore the sum of all the said terms is the sum of the same number of terms equal to the last term. Q. E. D.

LEMMA II.

72. If in any one of the vertical columns of numbers in the foregoing table of combinations, after the first column, we take as many terms (including the cyphers in the beginning of the column) as there are units in the exponent of the column, the sum of all the said terms will be to the sum of the same number of terms that are all equal to the last of them in the same proportion as 1 is to the exponent of the said column.

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By the first property of these figurate numbers, set forth in art. 51, the number of cyphers at the beginning of each of the said vertical columns is less by an unit than the exponent of the said column. And, by the second property of these numbers, set forth in art. 52, the first term in every column after the cyphers is an unit. Therefore the sum of all the terms of the vertical column that are supposed in this lemma to be taken (which are only as many as there are units in the exponent of the column) will be the sum of some cyphers and an unit, and consequently will be equal only to an unit. And the sum of the same number of terms all equal to the last will be equal to the sum of the same number of terms all equal to an unit, or will be equal to the exponent of the column. Therefore the sum of all the terms in the said vertical column will be to the sum of as many terms

terms all equal to the last term (which is an unit) in the same proportion as 1 is to the exponent of the column.

Q. E. D.

Thus, for example, in the 4th vertical column, if we take only the four first terms 0, 0, 0, 1, the sum of these terms will be = 1, and the sum of four terms all equal to the last term, which is 1, will be ($= 1 + 1 + 1 + 1$) = 4. And therefore the former sum is to the latter as 1 is to 4, which is the exponent of the said column. And, in general, if the exponent of the column be c , and we take c terms in it, the $c-1$ terms will be all cyphers, and the c th term will be 1. Therefore the sum of the said c terms will be 1. And the sum of c terms all equal to the last term (which is 1) will be c . Therefore the former sum will be to the latter sum as 1 is to c .

This lemma is the same with the general proposition hereafter to be proved, or the 12th property of the figurate numbers, in the case of taking only the first significant term in each of the vertical columns, which first term is always an unit.

LEMMA III.

73. If the above described 12th property of the figurate numbers (which we are preparing to demonstrate the truth of) should be found to be true in any one of the vertical columns of numbers contained in the foregoing table of combinations, or the sum of any number of terms taken in the said column should be to the sum of the same number of terms all equal to the last, or greatest, term, always in the same constant proportion, whatever be the number of terms so taken; and this proportion be that of 1 to a certain number denoted by r , so that the sum of the terms so taken shall always be equal to the quotient that arises by dividing the latter sum (or the sum of the same number of terms all equal to the last term) by the number r ; the excess
of

of the number of the terms so taken above the number r will be to the excess of the number of terms so taken above 1 in the same proportion as the last term but one of the terms so taken to the last term of all.

DEMONSTRATION.

Let the terms supposed to be taken in the said vertical column be $A, B, C, D, \&c \dots K, L$, of which L is the last, and K the last but one. And let the number of the terms so taken be n . We are then to prove that $n-r$ will be to $n-1$ as K is to L .

Now, since the sum of all the n terms $A, B, C, D, \&c \dots K, L$ is supposed to be to the sum of n terms all equal to the last term L , or to $n \times L$, in the proportion of 1 to r ; and the sum of all the terms $A, B, C, D, \&c \dots K, L$, except the last term L , or the sum of all the $n-1$ terms $A, B, C, D, \&c \dots K$, is also supposed to be to the sum of $n-1$ terms all equal to the last term K , or to $(n-1) \times K$, in the same proportion of 1 to r ; it follows that the sum of all the n terms $A, B, C, D, \&c \dots K, L$ will be $= \frac{n \times L}{r}$, and the sum of all the $n-1$ terms $A, B, C, D, \&c, K$, will be $= \frac{(n-1) \times K}{r}$. But this latter sum, or $A + B + C + D + \&c + K$, is less than the former sum, or $A + B + C + D + \&c + K + L$, by L . Therefore $\frac{(n-1) \times K}{r}$ is $= \frac{n \times L}{r} - L = \frac{n \times L - r \times L}{r} = \frac{(n-r) \times L}{r}$, and consequently $(n-1) \times K$ is $= (n-r) \times L$. Therefore $n-r$ is to $n-1$ as K is to L .
Q. E. D.

LEMMA

LEMMA IV.

74. If in the foregoing table of combinations, or figurate numbers, we take two contiguous vertical columns; and the numbers in the first of the two columns are found to have the twelfth property above-described, or the sum of any number of terms of it is to the sum of as many terms all equal to the last, or greatest, term, in the same proportion as the sum of any other number of its terms is to the sum of as many terms all equal to the last, or greatest, of this latter number of terms; and the said proportion is that of 1 to the number r ; and in the second of the said two contiguous vertical columns it be found that for a certain number of terms the numbers in the said column are likewise possessed of the same 12th property, and that the sum of the said number of terms is to the sum of as many terms all equal to the last, or greatest, term in the proportion of 1 to $r + 1$, and that the sum of any lesser number of its terms is to the sum of as many terms all equal to the last, or greatest, of the said lesser number of terms, in the same proportion of 1 to $r + 1$; I say, then, that, if we take another term in the said second vertical column above the number before taken, and in which the said 12th property has been found to take place, the said 12th property will take place likewise with respect to the numbers in the said second column, when increased by the said new term, and the sum of all the terms in the said column, including the said new term, will be to the sum of as many terms all equal to the said new term, in the same proportion of 1 to $r + 1$.

DEMONSTRATION.

Let n be the number of terms that are taken in the second of the two vertical columns; and let the same number of terms be taken in the first of them. Let the terms in the said first column be $A, B, C, D, \&c \dots K, L$, and those in the second column be $a, b, c, d, \&c \dots, k, l$. Then, by the supposition, the sum of the n terms $A, B, C, D, \&c$
 $\qquad\qquad\qquad O \qquad\qquad\qquad \dots K,$

... K, L, will be to the sum of n terms, all equal to the last term L, or to $n \times L$, in the proportion of 1 to r ; and the sum of all the n terms $a, b, c, d, \&c \dots k, l$, will be to the sum of n terms all equal to l , or to $n \times l$, in the proportion of 1 to $r+1$. Now let another term m be added to the former terms $a, b, c, d, \&c \dots k, l$, of the second of the said two vertical columns. We are then to prove that the sum of all the terms $a, b, c, d, \&c \dots k, l$, and m (the number of which is $n+1$) will be to the sum of as many terms all equal to the last term m , or to $n+1 \times m$, in the same proportion of 1 to $r+1$.

Now, by the 4th property of the figurate numbers above set forth in art. 54, it is manifest that l , or the n th term of the second of the two vertical columns, will be equal to the sum of all the terms in the preceding vertical column except the last term L, or to the sum of the $n-1$ terms A, B, C, D, &c ... K. But, by the supposition, the sum of these terms is less than the sum of as many terms equal to the last term K in the proportion of 1 to r , or is equal to $\frac{n-1 \times K}{r}$. Therefore l is $= \frac{n-1 \times K}{r}$.

But, because the above-described 12th property is supposed to belong to the numbers of the first of the said two vertical columns, to wit, A, B, C, D, &c ... K, L, and the sum of any number of terms in the said column is supposed to be to the sum of as many terms all equal to its last term in the constant proportion of 1 to r , it follows from lemma 3d, art. 73, that $n-r$ will be to $n-1$ as K is to L. Therefore $n-1 \times K$ will be $= n-r \times L$; and consequently $\frac{n-1 \times K}{r}$ will be $= \frac{n-r \times L}{r}$.

Therefore l (which has been shewn to be $= \frac{n-1 \times K}{r}$) will be $= \frac{n-r \times L}{r}$; and consequently $n-r$ will be to l as r is to L.

But, by the supposition, the sum of the terms A, B, C, D, &c ... K, L, is to the sum of the same number of terms all

all equal to the last term L , or to $n \times L$, in the proportion of 1 to r . Therefore $r \times A + B + C + D + \&c + K + L$ is $= n \times L$; and consequently r is to L as n is to $A + B + C + D + \&c + K + L$.

Therefore $n - r$ will be to l as n is to $A + B + C + D + \&c + K + L$.

But, by the 4th property of the figurate numbers above set forth in art. 54, m (which is the $n+1$ th term of the second vertical column) is equal to the sum of $A, B, C, D, \&c, K, L$, or the n first terms of the preceding column. Therefore $n - r$ will be to l as n is to m ; and consequently $\overline{n - r} \times m$ will be $= n \times l$.

But, by the supposition concerning the numbers in the second vertical column, the sum of the first n terms of it, to wit, $a + b + c + d + \&c + k + l$, is to the sum of as many terms all equal to the last term l , or to $n \times l$, in the proportion of 1 to $r + 1$. Therefore $r + 1 \times a + b + c + d + \&c + k + l$ is $= n \times l$; and consequently $\overline{n - r} \times m$ (which is equal to $n \times l$) will be $= r + 1 \times a + b + c + d + \&c + k + l$. Therefore $n - r : r + 1 :: a + b + c + d + \&c + k + l : m$. Therefore, *componendo*, we shall have $(n - r + r + 1, \text{ or } n + 1 : r + 1 :: a + b + c + d + \&c + k + l + m : m$; and *permutando*, $n + 1 : a + b + c + d + \&c + k + l + m :: r + 1 : m$; and, *invertendo*, $a + b + c + d + \&c + k + l + m : n + 1 :: m : r + 1$. But $n + 1$ is to $n + 1 \times m :: r + 1 : \overline{r + 1} \times m$. Therefore, *ex æquo*, $a + b + c + d + \&c + k + l + m : n + 1 \times m :: m : \overline{r + 1} \times m :: 1 : r + 1$; that is, the sum of the first $n + 1$ terms of the second vertical column will be to $n + 1$ times the last, or $n + 1$ th, term, m , of the said column in the same proportion of 1 to $r + 1$ in which the sum of the first n terms of it was to n times the last, or n th, term. Q. E. D.

COROLL. It follows from this lemma, that, if the number of terms in the second of the two vertical columns be increased

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increased from n terms to any other number of terms whatsoever denoted by $n + p$, it will be true with respect to the column, when so increased, that the sum of all its terms will be to $n + p$ times the last, or greatest, term of it in the same proportion of 1 to $r + 1$. For the lemma may be successively extended from a column consisting of $n + 1$ terms, to a column consisting of $n + 2$ terms, and to a column consisting of $n + 3$ terms, and to a column consisting of $n + 4$ terms, and so on till we come to the column of $n + p$ terms; the reasonings being exactly the same in this extension of it to these several columns of $n + 2$ terms, $n + 3$ terms, $n + 4$ terms, &c, as in the lemma itself, in which, upon a supposition that the sum of n terms of the column is to n times the last, or greatest, or n th, term of it in the proportion of 1 to $r + 1$, it is shewn that the sum of $n + 1$ terms of it will be to $n + 1$ times the last, or greatest, or $n + 1$ th, term of it in the same proportion of 1 to $r + 1$.

A demonstration of the 2d, 3d, and 4th foregoing Lemmas, contained in art. 72, 73, and 74, by Mr. John Bernoulli, the author's brother.

75. Many years ago, when I communicated the foregoing propositions concerning the figurate numbers to my brother, Mr. John Bernoulli, he observed to me that the demonstrations of them might be made shorter and more elegant by uniting the three last of the four preceeding lemmas into one, in the manner following.

A LEMMA.

A LEMMA.

If in a table of the figurate numbers (such as the foregoing table of combinations, in page 74, art. 49), it be the property of the terms of any one of the vertical columns that, if we take, 1st, any number of successive terms in it, and, 2dly, the same number of terms, all equal to the last, or greatest, of the said successive terms, the sum of the said successive terms shall be to the sum of the same number of terms, all equal to their last, or greatest, term, in the constant proportion of 1 to a certain number denoted by the letter r ; then it will follow that, if in the next higher vertical column of the said table of figurate numbers we take a number of successive terms greater by an unit than the number of successive terms taken in the former vertical column, the sum of these successive terms in this second column will be to the sum of the same number of terms, all equal to the last, or greatest, of the said successive terms, in the proportion of 1 to $r + 1$.

DEMONSTRATION.

Let the terms of the former of the two vertical columns be a, b, c, d, e , and f , of which the number is 6, or in general, n ; and let the terms of the next higher vertical column be o, g, h, i, l, p, q , of which the number is $n + 1$.

The upper term of this second column is a cypher, o , because every new vertical column of terms must have one more cypher preceeding its significant terms than the column immediately preceeding it.

These two vertical columns of terms will be as follows :

	a	o	
	b	g	
	c	h	
n	d	i	$n + 1$
	e	l	
	f	p	
		q	

If

If these columns are the 1st and 2d columns of the table, the terms a, b, c, d, e , and f will, each of them, be equal to 1, and g, h, i, l, p, q , will be 1, 2, 3, 4, 5, 6. If these columns are the 2d and 3d columns, a will be 0, and b, c, d, e, f will be 1, 2, 3, 4, 5, and g, h, i, l, p, q will be 0, 1, 3, 6, 10, 15. If these columns are the 3d and 4th columns, a will be 0, and b will also be 0, and c, d, e, f will be 1, 3, 6, 10, and g, h, i, k, p, q will be 0, 0, 1, 4, 10, 20. And in like manner more of the upper terms of both these vertical columns will be cyphers, or 0, the farther the columns are taken to the right hand in the table in page 74. But, wherever the columns are taken, the number of terms in them must be so great as to reach below the cyphers, and take in some of the significant terms. These things being premised, the demonstration of this lemma will be as follows.

By the 4th property of the figurate numbers set forth above in art. 54, we shall have $q = a + b + c + d + e + f$,

$$\text{and } p = a + b + c + d + e,$$

$$\text{and } l = a + b + c + d,$$

$$\text{and } i = a + b + c,$$

$$\text{and } b = a + b,$$

$$\text{and } a = a.$$

And, by the supposition, the sum of the n terms a, b, c, d, e, f will be to n times the last, or greatest, term f , as 1 is to r ; and the sum of the $n - 1$ terms a, b, c, d, e will be to $n - 1$ times the last, or greatest, term e , in the same proportion of 1 to r ; and the sum of the $n - 2$ terms a, b, c, d will be to $n - 2$ times the last, or greatest, term d , in the same proportion of 1 to r ; and, in like manner, $a + b + c$ will be to $n - 3$ times c as 1 to r ; and $a + b$ will be to $n - 4$ times b as 1 to r ; and a will be to $n - 5$ times a as 1 to r .

Therefore $a + b + c + d + e + f$ will be $= \frac{nf}{r}$, and $a + b + c + d + e$ will be $= \frac{(n-1) \times e}{r}$, and $a + b + c + d$ will be $= \frac{(n-2) \times d}{r}$, and $a + b + c$ will be $= \frac{(n-3) \times c}{r}$, and

and $a + b$ will be $= \frac{n-4}{r} \times b$, and a will be $= \frac{n-5}{r} \times a$.

Therefore q (which is equal to $a + b + c + d + e + f$) will be $= \frac{nf}{r}$; and p (which is equal to $a + b + c + d + e$) will be $= \frac{n-1}{r} \times e$; and l (which is equal to $a + b + c + d$) will be $= \frac{n-2}{r} \times d$; and i (which is equal to $a + b + c$) will be $= \frac{n-3}{r} \times c$; and b (which is equal to $a + b$) will be $= \frac{n-4}{r} \times b$; and g (which is equal to a) will be $= \frac{n-5}{r} \times a$.

Therefore $q + p + l + i + b + g$ will be $= \frac{nf}{r} + \frac{n-1}{r} \times e + \frac{n-2}{r} \times d + \frac{n-3}{r} \times c + \frac{n-4}{r} \times b + \frac{n-5}{r} \times a = \frac{nf}{r} + \frac{ne-e}{r} + \frac{nd-2d}{r} + \frac{nc-3c}{r} + \frac{nb-4b}{r} + \frac{na-5a}{r} = \frac{nf+ne+nd+nc+nb+na-e-2d-3c-4b-5a}{r} = \frac{n \times f + e + d + c + b + a - e - 2d - 3c - 4b - 5a}{r} = \frac{n \times q - e - 2d - 3c - 4b - 5a}{r} = \frac{n \times q - e - d - c - b - a - d - e - b - a - c - b - a - b - a - a}{r} = \frac{n \times q - p - l - i - b - g}{r}$

Therefore (if we multiply both sides by r) we shall have $r \times q + p + l + i + b + g = nq - p - l - i - b - g$, or $rq + r \times p + l + i + b + g = nq - p - l - i - b - g$. And consequently (adding $p + l + i + b + g$ to both sides) we shall have $rq + r \times p + l + i + b + g + p + l + i + b + g = nq$, or $rq + (r+1) \times p + l + i + b + g = nq$; and (subtracting rq from both sides) $(r+1) \times p + l + i + b + g = nq - rq$.

Therefore

Therefore (dividing both sides by $r+1$) we shall have
 $p+l+i+b+g = \frac{nq-rq}{r+1}$; and (adding q to both sides)
 $q+p+l+i+b+g = \frac{nq-rq}{r+1} + q = \frac{nq-rq}{r+1} + \frac{r+1}{r+1} \times q$
 $= \frac{nq-rq}{r+1} + \frac{rq+q}{r+1} = \frac{nq+q}{r+1} = \frac{n+1}{r+1} \times q$. And consequently
 $q+p+l+i+b+g$, or $g+b+i+l+p+q$, or $o+g+b+i+l+p+q$, will be to $\frac{n+1}{r+1} \times q$ as 1 is to $r+1$,
 or the sum of the $n+1$ successive terms o, g, b, i, l, p, q ,
 of the second vertical column of terms will be to the sum of
 $n+1$ times the last term q , or the same number of terms, all
 equal to the last, or greatest, term q , in the proportion of 1
 to $r+1$ *. Q. E. D.

*The principal Proposition, or the 12th property above-described
 of the figurate numbers, or numbers contained in the foregoing
 table of combinations, is as follows.*

76. The sum of any number of terms in any of the vertical columns contained in the foregoing table of combinations is to the sum of the same number of terms all equal to the last term of them, in the proportion of 1 to the exponent of the said column, or to the number which denotes, or expresses, its place in the said table.

Thus, in the first column, of which the exponent is 1 , the sum of any number of terms of it denoted by n will be to n times the last term of it in the proportion of 1 to 1 , or a proportion of equality. In the second column, of which

* See upon this subject the works of Mr. John Bernoulli himself, published at Laufanne, in Switzerland, in the year 1742, in four volumes, quarto, vol. iii. page 521, in the 47th lecture on the doctrine of the Integration of infinitely small differences, or the Inverse method of differences.

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the exponent is 2, the sum of n terms of it will be to the sum of n terms all equal to the last or greatest term, in the proportion of 1 to 2. In the third column, of which the exponent is 3, the sum of n terms of it will be to the sum of n terms all equal to the last, or greatest, term, in the proportion of 1 to 3. In the fourth column, of which the exponent is 4, the sum of n terms of it will be to the sum of n terms all equal to the last, or greatest, term, in the proportion of 1 to 4. And, in general, in the c th column, or that of which the exponent is c , the sum of n terms of it will be to the sum of n terms all equal to the last, or greatest, or n th, term, in the proportion of 1 to c .

DEMONSTRATION.

77. The truth of this proposition with respect to the 1st vertical column (which consists wholly of units) is shewn above in lemma 1, art. 71, and indeed is almost self-evident. And with respect to the terms of the second vertical column, to wit, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, &c, it may be proved by means of the second and fourth of the foregoing lemmas in the manner following. Since in the first vertical column the sum of any number of terms 1, 1, 1, 1, 1, &c. denoted by n , is to the sum of as many terms, all equal to the last term 1, as 1 is to 1; and in the second vertical column the sum of the two first terms 0 and 1 is to the sum of two terms, both equal to the last term 1, as 1 is to $1+1$, as is shewn in lemma 2, art. 72; it follows from lemma 4, art. 74, that in the same second vertical column the sum of the three first terms 0, 1, and 2, will be to the sum of three terms all equal to the last term 2, in the same proportion of 1 to $1+1$, or 2, and consequently that the sum of the four first terms 0, 1, 2, and 3, will be to the sum of four terms all equal to the last term 3, and the sum of the five first terms 0, 1, 2, 3, and 4, will be to the sum of five terms all equal to the last term 4, and, in general, the sum of any number of its terms denoted by

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by

by n will be to the sum of n terms all equal to the last of them, in the same proportion of 1 to $1 + 1$, or 2.

Q. E. D.

78. This may likewise be proved of the numbers contained in the said second vertical column, independently of the foregoing lemmas, in the manner following.

The numbers contained in the said second vertical column are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Now, if we set down these numbers twice over in two horizontal lines, one under the other, but in contrary orders, so that in the second line the last term of the first line shall be placed first, and the last term but one of the first line shall be placed second, and so on, as in these two lines,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0,

it is evident that the sum of every two numbers standing in the same vertical line, will be equal to 11, or, in general, to the last, or greatest, term of the series, or (if the series consist of n terms, and consequently the last term be $n-1$) to $n-1$. Therefore the sum of both serieses will be equal to a series consisting of the same number of terms, or n terms, all equal to the greatest term $n-1$; and consequently the upper series $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + \&c. \dots n-1$ alone will be equal to only half of the series of n terms all equal to $n-1$, or will be to it in the proportion of 1 to 2. Q. E. D.

79. To demonstrate the said 12th property with respect to the numbers contained in the third and fourth and other following vertical columns of the foregoing table of combinations, we must have recourse to the second and fourth lemmas, as in the first demonstration just now given of the said property with respect to the numbers in the second vertical column. This may be done in the manner following.

By lemma 2d it appears that this property takes place in all the vertical columns, if we continue the terms of each column only till their number is equal to the exponent of
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the column, or so as to take in only the first significant term of the column, which is always an unit. Therefore in the third vertical column, continued only to the three terms 0, 0, 1, the sum of the said three terms is to the sum of three terms all equal to the last term 1, in the proportion of 1 to 3, or 1 to $2 + 1$. But it has been shewn that in the second vertical column 0, 1, 2, 3, 4, 5, 6, 7, &c. it is true universally that, whatever be the number we take of its terms, the sum of the said terms will be to the sum of as many terms all equal to the last term in the proportion of 1 to 2. Here therefore we have the case of lemma 4, to wit, that of two contiguous vertical columns, the second and the third, in the former of which the sum of any number of terms denoted by n is to the sum of the same number of terms all equal to the last term in the proportion of 1 to a certain number, which we there denoted by r , and which here is 2, and in the latter of which the sum of the three first terms 0, 0, 1, is to the sum of three terms all equal to the last term 1 as 1 is to $r + 1$, or $2 + 1$. It follows therefore from lemma 4th, that, if we take the next term 3 of the said latter vertical column, or continue the said column to four terms, the sum of the said four terms 0, 0, 1, 3, will also be to the sum of four terms all equal to the last term 3 in the same proportion of 1 to $2 + 1$. And, for the same reason, the sum of the five first terms of the said third vertical column will be to the sum of five terms all equal to the fifth term of it, and the sum of the six first terms of it will be to the sum of six terms all equal to the sixth term of it, and, in general, the sum of any greater number of its terms, denoted by n , will be to the sum of n terms all equal to the n th term of it, in the same proportion of 1 to $2 + 1$, or of 1 to 3. Q. E. D.

In like manner we may prove that in the 4th vertical column the sum of any number n of its terms is to the sum of n terms all equal to its n th term in the proportion of 1 to $3 + 1$, or 1 to 4. For, since we have proved that the proportion of these two sums in the third column is that of 1 to 3; and by lemma 4 it appears that, if we take only the

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four

four first terms of the 4th column, to wit, 0, 0, 0, 1, the proportion of these sums will be that of 1 to $3 + 1$, or 4; it follows from lemma 4th and its corollary, that, if we take five terms of this fourth column, or six terms of it, or seven terms of it, or, in general, n terms of it, the proportion of the sum of the terms so taken to the sum of the same number of terms all equal to the last term will always be the same proportion of 1 to $3 + 1$, or of 1 to 4. Q. E. D.

And by proceeding to apply lemma 2 and lemma 4 in the same manner to the fifth, and sixth, and seventh, and other following vertical columns, it may be shewn that the proportion of the sum of any number of terms denoted by n to the sum of n terms all equal to the last, or greatest term, will be in the fifth column that of 1 to $4 + 1$, or 5, and in the sixth column that of 1 to $5 + 1$, or 6, and, in the seventh column that of 1 to $6 + 1$, or 7, and, in general, in the c th column that of 1 to c . Q. E. D.

80. COROLL. 1. In each of the aforesaid vertical columns of numbers the sum of any number of the terms beginning with 1, or the first significant term of the column, and not reckoning the cyphers that precede it, as we have hitherto done, will be to the sum of the same number of terms all equal to the next term in the said column after the terms so summed, in the proportion of 1 to c , or the exponent of the column.

Let the terms in the proposed vertical column, whereof we are to sum up the significant terms be $a, b, c, d, \&c, k, \text{ and } l$, including the cyphers, so that some of the first letters $a, b, c, \&c$, shall stand for cyphers, agreeably to the notation in lemma 4; and let the whole number of these terms, including the cyphers, be n , agreeably to the same notation. And let m be the term that comes immediately after l the last term of the set whose sum we are to examine; or, in other words, let m be the $\overline{n+1}$ th term of the proposed vertical column, including the cyphers. Also let c be the exponent of the said column, and r be $c - 1$. Then
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by the first property of the figurate numbers set forth above in art. 51, r will be the number of cyphers in the beginning of the said vertical column, and consequently $n - r$ will be the number of significant terms in the said column, without the cyphers. We are therefore to prove that the sum of all the $n - r$ significant terms of the said column $a, b, c, d, \&c, k$ and l is to the sum of $n - r$ terms all equal to the next term m in the proportion of 1 to c or of 1 to $r + 1$.

Now the sum of all the $n - r$ significant terms of the said vertical column is equal to the sum of all the n terms of the said column, including the cyphers, because the cyphers are all equal to nothing. And it is shewn in the latter part of the demonstration of lemma 4, that the sum of the n terms $a, b, c, d, \&c, k$ and l is to the next term m in the same proportion as $n - r$ is to $r + 1$. Therefore the sum of all the $n - r$ significant terms of the said column will be to the next term m in the same proportion of $n - r$ to $r + 1$. But m is to $\overline{n - r} \times m$ in the same proportion as $r + 1$ is to $\overline{n - r} \times r + 1$. Therefore, *ex æquo*, the sum of all the $n - r$ significant terms of the said column will be to $\overline{n - r} \times m$, or to the sum of $n - r$ terms all equal to the next term m , as $n - r$ is to $\overline{n - r} \times \overline{r + 1}$, and consequently as 1 is to $r + 1$, or as 1 is to c . Q. E. D.

81. COROLL. 2. By the help of the foregoing corollary we may find the sum of any given number of terms in any of the vertical columns of the foregoing table of combinations, without actually adding the terms together, by proceeding in the following manner.

Let the number of terms to which the several vertical columns are continued, be n . Then, as there is one cypher prefixed to the significant terms in the second column, and two cyphers in the third column, and three cyphers in the fourth column, and, in general, $c - 1$ cyphers in the c th column; it is evident that the number of terms in the second column, without the cyphers, will be $n - 1$; and that
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of the terms in the third column, without the cyphers, will be $n - 2$; and that of the terms in the fourth column, without the cyphers, will be $n - 3$; and, in general, that of the terms in the c th column, without the cyphers, will be $n - \overline{c - 1}$, or $n - c + 1$. The sums of the terms in these several columns may therefore be thus determined.

In the first place, the sum of the n significant terms in the first column, which are all units, will be $n \times 1$, or n .

2dly, The sum of the $n - 1$ significant terms in the second column will, by the foregoing corollary, be to $n - 1$ times the next following, or $n + 1$ th, term of the second column as 1 is to 2. But, by the 4th property of the figurate numbers above set forth in art. 54, the $n + 1$ th term of the second column is equal to the sum of the first n terms of the first column, that is, to n . Therefore the sum of the $n - 1$ significant terms of the second column will be to $n - 1$ times n , or to $n \times n - 1$, as 1 is to 2, and consequently will be equal to $\frac{n \times n - 1}{2}$. Q. E. I.

3dly, The sum of the $n - 2$ significant terms in the third column will, by the foregoing corollary, be to $n - 2$ times the next following, or $n + 1$ th, term of the same third column as 1 is to 3. But, by the 4th property of the figurate numbers above set forth, the $n + 1$ th term of the third column is equal to the sum of the first n terms of the second column, including the cyphers, or (which comes to the same thing) to the sum of the first $n - 1$ significant terms of the said second column; which has just now been shewn to be equal to $\frac{n \times n - 1}{2}$. Therefore the sum of the $n - 2$ significant terms in the 3d column will be to $n - 2$ times $\frac{n \times n - 1}{2}$, or to $\frac{n \times n - 1 \times n - 2}{2}$, as 1 is to 3, and consequently will be equal to $\frac{n \times n - 1 \times n - 2}{2 \times 3}$. Q. E. I.

4thly,

4thly, The sum of the $n-3$ significant terms in the fourth column will, by the foregoing corollary, be to $n-3$ times the next following, or $\overline{n+1}$ th, term of the same fourth column as 1 is to 4. But, by the fourth property of the figurate numbers, the $\overline{n+1}$ th term of the fourth column is equal to the sum of the first n terms of the third column, including the cyphers, or of the first $n-2$ significant terms of the said third column: which has just now been shewn to be equal to $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$. Therefore the sum of the $n-3$ significant terms in the 4th column will be to $n-3$ times $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, or to $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, as 1 is to 4, and consequently will be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$. Q. E. T.

5thly, In like manner the sum of the $n-4$ significant terms in the fifth column will, by the foregoing corollary, be to $n-4$ times the next following, or $\overline{n+1}$ th, term of the same column as 1 is to 5. But, by the 4th property of the figurate numbers above set forth, the $\overline{n+1}$ th term of the fifth column is equal to the sum of the first n terms of the fourth column, including the cyphers, or to the first $n-3$ significant terms of it; which has just now been shewn to be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$. Therefore the sum of the $n-4$ significant terms in the fifth column will be to $n-4$ times $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, or to $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$, as 1 is to 5, and consequently will be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$. Q. E. I.

And in like manner it is evident that the sum of the n first terms of the c th column, including the $c-1$ cyphers in the beginning of it, or the sum of the $n-\sqrt{c-1}$, or $n-c+1$, first



first significant terms of the said c th column, will be equal to the fraction $\frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times \&c \times n-c+1}{2 \times 3 \times 4 \times 5 \times \&c \times c}$, in the numerator of which the last factor is $n - \sqrt{c-1}$, or $n-c+1$, and in the denominator of which the last factor is c .

Q. E. I.

82. COROLL. 3. Since, by the 4th property of the figurate numbers above set forth in art. 54, the $n+1$ th terms of the second, third, fourth, and other following vertical columns of numbers in the foregoing table of combinations, are respectively equal to the sums of the n first terms of the first, second, third, and other following vertical columns, which sums have been shewn to be, respectively, equal to n , $\frac{n \times n-1}{2}$, $\frac{n \times n-1 \times n-2}{2 \times 3}$, &c, it follows that the $n+1$ th terms of the second, third, fourth, fifth, sixth, and other following vertical columns will be n , $\frac{n \times n-1}{2}$, $\frac{n \times n-1 \times n-2}{2 \times 3}$, $\frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4}$, and $\frac{n \times n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5}$, &c; and consequently, as every term in the first vertical column is $= 1$, the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns will be 1 , n , $\frac{n \times n-1}{2}$, $\frac{n \times n-1 \times n-2}{2 \times 3}$, $\frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4}$, $\frac{n \times n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5}$, &c, and, in general, the $n+1$ th term of the c th column will be $\frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times \&c \times n-c+2}{2 \times 3 \times 4 \times 5 \times \&c \times c-1}$.

83. COROLL. 4. Since the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of the foregoing table of combinations are 1 , n ,

$$\begin{aligned}
 n, & \frac{n \times n-1}{2}, \frac{n \times n-1 \times n-2}{2 \times 3}, \frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4}, \\
 & \frac{n \times n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5}, \&c, \\
 & \frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times \&c \times n-c+2}{2 \times 3 \times 4 \times 5 \times \&c \times c-1}, \text{ it follows that the } n\text{th} \\
 & \text{terms of the said vertical columns will be such as arise by} \\
 & \text{substituting } n-1 \text{ instead of } n \text{ in the foregoing values of the} \\
 & n+1\text{th terms, and consequently will be as follows, to wit, } 1, \\
 n-1, & \frac{n-1 \times n-2}{2}, \frac{n-1 \times n-2 \times n-3}{2 \times 3}, \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4}, \\
 & \frac{n-1 \times n-2 \times n-3 \times n-4 \times n-5}{2 \times 3 \times 4 \times 5}, \&c, \\
 & \frac{n-1 \times n-2 \times n-3 \times n-4 \times n-5 \times \&c \times n-c+1}{2 \times 3 \times 4 \times 5 \times \&c \times c-1}.
 \end{aligned}$$

An Example of the truth of Coroll. 4.

84. As an example of the truth of this corollary we will derive in this manner the numbers that form the lowest horizontal row of terms in the foregoing table of combinations, or the twelfth terms of the several vertical columns of the said table.

Now in this case n , or the number of terms in the several vertical columns, is = 12. Therefore $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, $n-6$, $n-7$, $n-8$, $n-9$, $n-10$, and $n-11$, are respectively equal to 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, and 1. Therefore $n-1$, or the twelfth term of the second vertical column will be = 11; and $\frac{n-1 \times n-2}{2}$, or the 12th

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term



term of the third vertical column, will be $(= \frac{n-1}{2} \times \frac{n-2}{2} = 11 \times \frac{n-2}{2} = 11 \times \frac{10}{2}) = 11 \times 5 = 55$; and $\frac{n-1 \times n-2 \times n-3}{2 \times 3}$, or the 12th term of the fourth vertical column, will be $(= \frac{n-1 \times n-2}{2} \times \frac{n-3}{3} = 55 \times \frac{n-3}{3} = 55 \times \frac{9}{3} = 55 \times 3) = 165$; and $\frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4}$, or the 12th term of the 5th vertical column, will be $(= \frac{n-1 \times n-2 \times n-3}{2 \times 3} \times \frac{n-4}{4} = 165 \times \frac{n-4}{4} = 165 \times \frac{8}{4} = 165 \times 2) = 330$; and $\frac{n-1 \times n-2 \times n-3 \times n-4 \times n-5}{2 \times 3 \times 4 \times 5}$, or the 12th term of the 6th vertical column, will be $(= \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4} \times \frac{n-5}{5} = 330 \times \frac{n-5}{5} = 330 \times \frac{7}{5} = 66 \times 7) = 462$; and, in like manner, the 12th term of the seventh vertical column will be $(= 462 \times \frac{n-6}{6} = 462 \times \frac{6}{6} = 462 \times 1) = 462$; and the 12th term of the eighth vertical column will be $(= 462 \times \frac{n-7}{7} = 462 \times \frac{5}{7} = 66 \times 5) = 330$; and the 12th term of the ninth vertical column will be $(= 330 \times \frac{n-8}{8} = 330 \times \frac{4}{8} = 330 \times \frac{1}{2}) = 165$; and the 12th term of the tenth vertical column will be $(= 165 \times \frac{n-9}{9} = 165 \times \frac{3}{9} = 165 \times \frac{1}{3} = \frac{165}{3}) = 55$; and the 12th term of the eleventh vertical column will be $(= 55 \times \frac{n-10}{10} = 55 \times \frac{2}{10} = 55 \times \frac{1}{5} = \frac{55}{5}) = 11$; and the 12th term of the twelfth, or last, vertical column will be $(= 11 \times \frac{n-11}{11} = 11 \times \frac{1}{11}) = 1$. Therefore the 12th terms of the said twelve vertical columns will be as follows, to wit, 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, and 1; which are the numbers set down in the foregoing table.

85. COROLL.

85. COROLL. 5. It has been shewn above in art. 60 and 61, that the horizontal rows of numbers in the foregoing table of combinations, beginning with the second row, exhibit the co-efficients of the terms of the several successive powers of a binomial quantity, such as $a+b$, every n th horizontal row of numbers being the co-efficients of the terms of the $n-1$ th power of the said binomial quantity; whence it follows that the numbers contained in every $n+1$ th horizontal row of numbers in the said table will be the co-efficients of the terms of the n th power of the said binomial quantity. But it is evident that the numbers contained in every $n+1$ th horizontal row of terms in the said table are the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of terms in the said table, reckoning the terms from the top of the said table, and including the cyphers at the tops of all the several vertical columns, except the first. Therefore the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of terms in the said table will be the co-efficients of the terms of the n th power of the said binomial quantity. But it has been shewn above in art. 82, coroll. 3, that the $n+1$ th terms of the first, second, third, fourth, fifth, and sixth, and other following vertical columns of terms in the said table are 1, n , $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c. Therefore the co-efficients of the terms of the n th power of the said binomial quantity $a+b$ will also be 1, n , $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c; and consequently the quantity $(a+b)^n$, or the said n th power itself of the said binomial quantity $a+b$, will be equal to the

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series

series $a^n + na^{n-1}b + n \times \frac{n-1}{2} a^{n-2}b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5}b^5 + \&c$, or (if we put A for 1, and B for n and C for $n \times \frac{n-1}{2}$, and D for $n + \frac{n-1}{2} \times \frac{n-2}{3}$, and E for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and F for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and G, H, I, K, L, &c, for the numeral co-efficients of the seventh, eighth, ninth, tenth, eleventh, and other following terms of the series respectively) to the series $a^n + \frac{n}{1} A a^{n-1}b + \frac{n-1}{2} B a^{n-2}b^2 + \frac{n-2}{3} C a^{n-3}b^3 + \frac{n-3}{4} D a^{n-4}b^4 + \frac{n-4}{5} E a^{n-5}b^5 + \frac{n-5}{6} F a^{n-6}b^6 + \frac{n-6}{7} G a^{n-7}b^7 + \frac{n-7}{8} H a^{n-8}b^8 + \frac{n-8}{9} I a^{n-9}b^9 + \frac{n-9}{10} K a^{n-10}b^{10} + \frac{n-10}{11} L a^{n-11}b^{11} + \&c$; which series will continue till the numerator of the generating fraction becomes $n-n$, or 0, and consequently the said fraction itself becomes equal to 0 likewise, and therefore the term in which the said fraction enters as a factor, will also be equal to 0, as will also all the following terms of the series, which would be derived from the said term by continual multiplications. The series therefore will break off, or end with the term next preceeding the said term which is equal to 0*.

We will now proceed to illustrate and confirm the truth of this corollary, by applying the foregoing series to the

* This is the famous binomial theorem invented by Sir Isaac Newton, but of which he has no where given a demonstration. And the demonstration here given of it by Mr. James Bernoulli, is that to which I alluded in the first volume of the Collection of Tracts, in two volumes, quarto, called *Scriptores Logarithmici*, page 349, art. 4, and in the second volume of the same Collection, page 157, art. 9.

computation of the terms of some of the lowest powers of the binomial quantity $a+b$, so as to produce by means of it all the numbers contained in the foregoing table of combinations, in page 74.

Examples of the application of the foregoing series to the computation of the terms of the powers of the binomial quantity $a+b$.

86. In the first place let us suppose n to be $= 1$. Then we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \&c$, ($= a^1 + \frac{1}{1} \times 1 \times a^{1-1}b + \frac{1-1}{2} Ba^{1-2}b^2 + \frac{1-2}{3} \times Ca^{1-3}b^3 + \&c$, $= a^1 + a^0 \times b + \frac{0}{2} \times Ba^{1-2}b^2 + \frac{1-2}{3} \times C \times a^{1-3}b^3 + \&c$, $= a^1 + 1 \times b + 0 + 0 + \&c$), $= a+b$; or the said series is in this case equal to the binomial quantity $a+b$ itself; as it ought to be.

87. Now let n be $= 2$.

Then we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \&c$, ($= a^2 + \frac{2}{1} \times 1 \times a^{2-1}b + \frac{2-1}{2} Ba^{2-2}b^2 + \frac{2-2}{3} Ca^{2-3}b^3 + \&c$, $= a^2 + 2a^1b + \frac{1}{2} \times 2 \times a^0b^2 + \frac{0}{3} \times Ca^{2-3}b^3 + \&c$, $= a^2 + 2ab + \frac{2}{2} \times 1 \times b^2 + 0 + \&c$) $= a^2 + 2ab + b^2$; or the said series will in this case be equal to the trinomial quantity

tity $a^2 + 2ab + b^2$; as it ought to be, because that quantity is the square of the binomial quantity $a + b$.

88. If n is $= 3$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \&c$ ($= a^3 + \frac{3}{1} \times 1 \times a^{3-1}b + \frac{3-1}{2} \times \frac{3}{1} \times 1 \times a^{3-2}b^2 + \frac{3-2}{3} Ca^{3-3}b^3 + \frac{3-3}{4} Da^{3-4}b^4 + \&c$, $= a^3 + 3a^2b + \frac{2}{2} \times 3 \times a^1b^2 + \frac{1}{3} C \times a^0b^3 + \frac{0}{4} D \times a^{3-4}b^4 + \&c = a^3 + 3a^2b + 3ab^2 + \frac{1}{3} \times 3 \times 1 \times b^3 + 0 + \&c$) $= a^3 + 3a^2b + 3ab^2 + b^3$. Therefore $\overline{a+b}^3$, or $\overline{a+b}^3$, or the cube of the binomial quantity $a + b$, will be equal to the quadrinomial quantity $a^3 + 3a^2b + 3ab^2 + b^3$; as it will be found to be upon trial.

89. If n is $= 4$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \&c$ ($= a^4 + \frac{4}{1} \times 1 \times a^{4-1}b + \frac{4-1}{2} Ba^{4-2}b^2 + \frac{4-2}{3} Ca^{4-3}b^3 + \frac{4-3}{4} Da^{4-4}b^4 + \frac{4-4}{5} Ea^{4-5}b^5 + \&c = a^4 + 4a^3b + \frac{3}{2} \times 4 \times a^2b^2 + \frac{2}{3} \times C \times a^1b^3 + \frac{1}{4} D \times a^0 \times b^4 + \frac{0}{5} E \times a^{4-5}b^5 + \&c = a^4 + 4a^3b + 6a^2b^2 + \frac{2}{3} \times 6 \times ab^3 + \frac{1}{4} D \times 1 \times b^4 + 0 + \&c = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + \frac{1}{4} \times 4 \times b^4 + 0 + \&c$) $= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Therefore $\overline{a+b}^4$, or $\overline{a+b}^4$, or the fourth power of the binomial quantity $a + b$, will be equal

equal to the quinquinomial quantity $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; as it will be found to be upon trial.

90. If n is $= 5$, we shall have $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \&c$, ($= a^5 + \frac{5}{1} \times 1 \times a^{5-1}b + \frac{5-1}{2} Ba^{5-2}b^2 + \frac{5-2}{3} Ca^{5-3}b^3 + \frac{5-3}{4} Da^{5-4}b^4 + \frac{5-4}{5} Ea^{5-5}b^5 + \frac{5-5}{6} Fa^{5-6}b^6 + \&c = a^5 + 5a^4b + \frac{5}{2} Ba^3b^2 + \frac{3}{3} Ca^2b^3 + \frac{2}{4} Da^1b^4 + \frac{1}{5} Ea^0b^5 + \frac{0}{6} \times Fa^{5-6}b^6 + \&c$
 $= a^5 + 5a^4b + 2Ba^3b^2 + Ca^2b^3 + \frac{1}{2} Da^1b^4 + \frac{1}{5} E \times 1 \times b^5 + 0 + \&c$) $= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. Therefore $a+b$, or $a+b$, or the fifth power of the binomial quantity $a+b$, will be equal to the sextinomial quantity $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; as, upon trial, it will be found to be.

91. If n is $= 6$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \&c$ ($= a^6 + \frac{6}{1} \times 1 \times a^{6-1}b + \frac{6-1}{2} Ba^{6-2}b^2 + \frac{6-2}{3} Ca^{6-3}b^3 + \frac{6-3}{4} Da^{6-4}b^4 + \frac{6-4}{5} Ea^{6-5}b^5 + \frac{6-5}{6} Fa^{6-6}b^6 + \frac{6-6}{7} Ga^{6-7}b^7 + \&c = a^6 + 6a^5b + \frac{5}{2} Ba^4b^2 + \frac{4}{3} Ca^3b^3 + \frac{3}{4} Da^2b^4 + \frac{2}{5} Ea^1b^5 + \frac{1}{6} Fa^0b^6 + \frac{0}{7} Ga^{6-7}b^7 + \&c$) $= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. Therefore

fore $\overline{a+b}^n$, or $\overline{a+b}^6$, or the sixth power of the binomial quantity $a+b$, will be equal to the septinomial quantity $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. And so, upon trial, it will be found to be.

92. If n is = 7, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 + \&c (= a^7 + \frac{7}{1} \times 1 \times a^6b + \frac{6}{2} Ba^5b^2 + \frac{5}{3} Ca^4b^3 + \frac{4}{4} Da^3b^4 + \frac{3}{5} Ea^2b^5 + \frac{2}{6} Fa^1b^6 + \frac{1}{7} Ga^0b^7 + \frac{0}{8} Ha^{7-8}b^8 + \&c = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + 1 \times b^7 + 0 + \&c) = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$. Therefore $\overline{a+b}^n$, or $\overline{a+b}^7$, or the seventh power of the binomial quantity $a+b$, will be equal to the octinomial quantity $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$. And so, upon trial, it will be found to be.

93. If n is = 8, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \&c (= a^8 + \frac{8}{1} \times 1 \times a^7b + \frac{7}{2} Ba^6b^2 + \frac{6}{3} Ca^5b^3 + \frac{5}{4} Da^4b^4 + \frac{4}{5} Ea^3b^5 + \frac{3}{6} Fa^2b^6 + \frac{2}{7} Ga^1b^7 + \frac{1}{8} Ha^0b^8 + \frac{0}{9} Ia^{8-9}b^9 + \&c = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1 \times b^8 + 0 + \&c) = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

$56a^2b^3 + 28a^3b^2 + 8a^4b + b^4$. Therefore $\overline{a+b}^8$, or $\overline{a+b}^8$, or the eighth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. And so, upon trial, it will be found to be.

94. If n is $= 9$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b$
 $+ \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 +$
 $\frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8}$
 $Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10} + \&c (= a^9$
 $+ \frac{9}{1} \times 1 \times a^8b + \frac{8}{2} Ba^7b^2 + \frac{7}{3} Ca^6b^3 + \frac{6}{4} Da^5b^4 + \frac{5}{5}$
 $Ea^4b^5 + \frac{4}{6} Fa^3b^6 + \frac{3}{7} Ga^2b^7 + \frac{2}{8} Ha^1b^8 + \frac{1}{9} Ia^0b^9 + \frac{0}{10}$
 $Ka^{9-10}b^{10} + \&c = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 +$
 $126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + 1 \times b^9 +$
 $0 + \&c) = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 +$
 $126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. Therefore
 $\overline{a+b}^9$, or $\overline{a+b}^9$, or the ninth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^9 +$
 $9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 +$
 $36a^2b^7 + 9ab^8 + b^9$. And so, upon trial, it will be found to be.

95. If n is $= 10$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b$
 $+ \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 +$
 $\frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8}$
 $Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10} + \frac{n-10}{11}$
 R La^{n-11}

$$La^{n-11}b^{11} + \&c (= a^{10} + \frac{10}{1} \times 1 \times a^9b + \frac{9}{2} Ba^8b^2 + \frac{8}{3} Ca^7b^3 + \frac{7}{4} Da^6b^4 + \frac{6}{5} Ea^5b^5 + \frac{5}{6} Fa^4b^6 + \frac{4}{7} Ga^3b^7 + \frac{3}{8} Ha^2b^8 + \frac{2}{9} Ia^1b^9 + \frac{1}{10} Ka^0b^{10} + \frac{0}{11} La^{10-11}b^{11} + \&c = a^{10}$$

$$+ 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10a^1b^9 + 1 \times b^{10} + 0 + \&c) = a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}.$$
 Therefore $a + b^{10}$, or $a + b^{10}$, or the tenth power of the binomial quantity $a + b$, will be equal to the compound quantity $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$. And so, upon trial, it will be found to be.

96. Lastly, let n be $= 11$. Then will the series $a^n + \frac{n}{1}$

$$Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} D$$

$$a^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7$$

$$+ \frac{n-7}{8} Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10}$$

$$+ \frac{n-10}{11} La^{n-11}b^{11} + \frac{n-11}{12} Ma^{n-12}b^{12} + \&c$$
 be $(= a^{11} + \frac{11}{1} \times 1 \times a^{10}b + \frac{10}{2} Ba^9b^2 + \frac{9}{3} Ca^8b^3 + \frac{8}{4} Da^7b^4 + \frac{7}{5} Ea^6b^5 + \frac{6}{6} Fa^5b^6 + \frac{5}{7} Ga^4b^7 + \frac{4}{8} Ha^3b^8 + \frac{3}{9} Ia^2b^9 + \frac{2}{10} Ka^1b^{10} + \frac{1}{11} La^0b^{11} + \frac{0}{12} Ma^{11-12}b^{12} + \&c = a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + 1 \times b^{11} + 0 + \&c) = a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + b^{11}$. Therefore

fore $\overline{a+b}^n$, or $\overline{a+b}^{11}$, or the eleventh power of the binomial quantity $a+b$, will be equal to the compound quantity $a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + b^{11}$. And so, upon trial, it will be found to be.

97. It appears therefore that the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 + \frac{n-4}{5} E a^{n-5} b^5 + \frac{n-5}{6} F a^{n-6} b^6 + \&c$, which has been obtained in the foregoing 5th corollary for the value of the quantity $\overline{a+b}^n$, or the n th power of the binomial quantity $a+b$, does truly exhibit the value of the said power when the index n is equal to either 1, or 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, or 10, or 11; in which cases the co-efficients of the terms of the said series are equal to the numbers contained in the several successive horizontal rows of terms in the foregoing table of combinations, in page 74, beginning with the second row.

*Additional Corollaries, not contained in the original text of
Mr. James Bernoulli.*

98. To these five corollaries, which are contained in Mr. James Bernoulli's original text, it may not be amiss to add the following corollaries, which are easily deducible from Mr. Bernoulli's propositions, and which will enable us to find a general expression for the terms of any of the verti-

R 2

cal

cal columns in the foregoing table of combinations, or, in other words, for the figurate numbers of any proposed order.

99. Coroll. 6. It has been shewn in art. 82, corol. 3, page 112, that the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of terms in the foregoing table of combinations are $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \&c.$ But the $n+1$ th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of terms in the said table are the first, second, third, fourth, fifth, sixth, and other following terms of the $n+1$ th horizontal row of terms in the said table. Therefore the first, second, third, fourth, fifth, sixth, and other following terms of the $n+1$ th horizontal row of terms in the said table are $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \&c.$

100. Coroll. 7. Since the several terms of the $n+1$ th horizontal row are $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \&c.$ it follows that, if p be any whole number greater than n , as, for example, $n+1$, or $n+2$, or $n+3$, or $n+4$, $\&c.$ the several terms of the $p+1$ th horizontal row will be $1, p, p \times \frac{p-1}{2}, p \times \frac{p-1}{2} \times \frac{p-2}{3}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}, \&c.$ And consequently the third term of the $p+1$ th horizontal row of terms, when p is equal to $n+1$, or the third term of the $n+2$ th horizontal row

row of terms, will be the quantity which arises by substituting $n+1$ instead of p in the third term, $p \times \frac{p-1}{2}$, of the last-mentioned series, that is, $\overline{n+1} \times \frac{n+1-1}{2}$, or $\overline{n+1} \times \frac{n}{2}$, or $n \times \frac{n+1}{2}$; and the fourth term of the $p+1$ th horizontal row of terms when p is $= n+2$, or the fourth term of the $n+3$ th horizontal row of terms will be that which arises by substituting $n+2$ instead of p in the fourth term, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, of the last series, that is, $\overline{n+2} \times \frac{n+2-1}{2} \times \frac{n+2-2}{3}$, or $\overline{n+2} \times \frac{n+1}{2} \times \frac{n}{3}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3}$.

And, in like manner, if we substitute $n+3$ instead of p in the 5th term $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$ of the last series, we shall have $\overline{n+3} \times \frac{n+3-1}{2} \times \frac{n+3-2}{3} \times \frac{n+3-3}{4}$, or $\overline{n+3} \times \frac{n+2}{2} \times \frac{n+1}{3} \times \frac{n}{4}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, for the 5th term of the $n+4$ th horizontal row of terms; and, if we substitute $n+4$ instead of p in the 6th term, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, of the last series, we shall have $\overline{n+4} \times \frac{n+4-1}{2} \times \frac{n+4-2}{3} \times \frac{n+4-3}{4} \times \frac{n+4-4}{5}$, or $\overline{n+4} \times \frac{n+3}{2} \times \frac{n+2}{3} \times \frac{n+1}{4} \times \frac{n}{5}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, for the 6th term of the $n+5$ th horizontal row of terms. So that the 3d term of the $n+2$ th horizontal row of terms, and the 4th term of the $n+3$ th horizontal row, and the 5th term of the $n+4$ th horizontal row, and the 6th term of the $n+5$ th horizontal row, will be $n \times \frac{n+1}{2}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3}$

$\frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, respectively.

101. Coroll. 8. These four terms $n \times \frac{n+1}{2}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, are derived from the number n by the continual multiplication of the fractions $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, and $\frac{n+4}{5}$, the numerators and denominators of which both increase continually by an unit. Therefore, if we put C for the first, D for the second, E for the third, and F for the fourth of these terms, we shall have $C = \frac{n+1}{2} \times n$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$.

And, from the manner in which these four terms were derived from the 3d, 4th, 5th, and 6th terms of the series 1, p , $p \times \frac{p-1}{2}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, &c, in the last corollary, to wit, by substituting $n+1$, $n+2$, $n+3$, and $n+4$, instead of p in the said 3d, 4th, 5th, and 6th terms, respectively, it is easy to see that the 7th term of the $\overline{n+6}$ th horizontal row of terms, and the 8th term of the $\overline{n+7}$ th horizontal row, and the 9th term of the $\overline{n+8}$ th horizontal row, and the 10th term of the $\overline{n+9}$ th horizontal row, and the 11th, 12th, 13th, 14th, 15th, and other following terms of the $\overline{n+10}$ th, $\overline{n+11}$ th, $\overline{n+12}$ th, $\overline{n+13}$ th, $\overline{n+14}$ th, and other following horizontal rows of terms in the said table, respectively, will be equal to $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$,

$\times H, \frac{n+8}{9} \times I, \frac{n+9}{10} \times K, \frac{n+10}{11} \times L, \frac{n+11}{12} \times M, \frac{n+12}{13} \times$
 $N, \frac{n+13}{14} \times O, \&c,$ in which the capital letters G, H, I,
 K, L, M, N, O, &c, denote the 7th, 8th, 9th, 10th, 11th,
 12th, 13th, 14th, 15th, and other next following terms, of
 the $\overline{n+6}$ th, $\overline{n+7}$ th, $\overline{n+8}$ th, $\overline{n+9}$ th, $\overline{n+10}$ th, and
 other following horizontal rows of terms, respectively, as
 they arise, and the generating fractions $\frac{n+5}{6}, \frac{n+6}{7}, \frac{n+7}{8}, \frac{n+8}{9},$
 $\frac{n+9}{10}, \frac{n+10}{11}, \frac{n+11}{12}, \frac{n+12}{13}, \frac{n+13}{14}, \&c,$ are a continuation of
 the foregoing generating fractions $\frac{n+1}{2}, \frac{n+2}{3}, \frac{n+3}{4}, \frac{n+4}{5},$
 and are derived from them by the continual addition of an
 unit to both their numerators and denominators.

102. Coroll. 9. It is shewn above in the 6th property of
 the numbers contained in the foregoing table of combina-
 tions, art. 56 and 57, pages 77 and 78, that in every hori-
 zontal row of terms in the said table of combinations, the
 first and last term are, each of them, an unit, and the terms
 that are equidistant from the first and last terms are equal
 to each other. It follows therefore that the 3d term of the
 $\overline{n+2}$ th horizontal row, reckoned from the end of it, or
 from the right hand to the left, will be equal to the 3d term
 of it reckoned from the beginning, or from the left hand
 to the right; and that the 4th term of the $\overline{n+3}$ th horizon-
 tal row, reckoned from the end of it, or from the right
 hand to the left, will be equal to the 4th term of it reckoned
 from the beginning, or from the left hand to the right; and
 that the 5th term of the $\overline{n+4}$ th horizontal row, reckoned
 from the end of it, or from the right hand to the left, will
 be equal to the 5th term of it, reckoned from the begin-
 ning, or from the left hand to the right; and that the 6th
 term of the $\overline{n+5}$ th horizontal row, reckoned from the end
 of it, or from the right hand to the left, will be equal to the
 6th term of it, reckoned from the beginning, or from
 the left hand to the right; and, in like manner, that the
 5th, 7th,

7th, and 8th, and 9th, and 10th, and other following terms of the $\overline{n+6}$ th, $\overline{n+7}$ th, $\overline{n+8}$ th, $\overline{n+9}$ th, and other following horizontal rows of terms, respectively, reckoned from the ends of the said rows, or from the right hand to the left, will be equal to the 7th, and 8th, and 9th, and 10th, and other following corresponding terms of the same horizontal rows, respectively, reckoned from the beginnings of the said rows, or from the left hand to the right. But it was shewn in corollary 7th, that the 3d term of the $\overline{n+2}$ th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2}$; and that the 4th term of the $\overline{n+3}$ th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$; and that the 5th term of the $\overline{n+4}$ th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$; and that the sixth term of the $\overline{n+5}$ th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$: or that, if the said third term of the $\overline{n+2}$ th horizontal row of terms, reckoned from the beginning of it, be called C, and the said 4th term of the $\overline{n+3}$ th horizontal row of terms, reckoned from the beginning of it, be called D, and the said 5th term of the $\overline{n+4}$ th horizontal row of terms, reckoned from the beginning of it, be called E, and the said 6th term of the $\overline{n+5}$ th horizontal row of terms, reckoned from the beginning of it, be called F, we shall have $C = n \times \frac{n+1}{2}$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$. And it is observed in the last, or 8th, corollary, that, if the 7th term of the $\overline{n+6}$ th horizontal

horizontal row of terms be called G, and the 8th term of the $\overline{n+7}$ th horizontal row be called H, and the 9th term of the $\overline{n+8}$ th horizontal row, and the 10th term of the $\overline{n+9}$ th horizontal row, and the 11th term of the $\overline{n+10}$ th horizontal row, and the 12th term of the $\overline{n+11}$ th horizontal row, and the next following terms of the next following horizontal rows, all reckoned from the beginnings of those several horizontal rows, or from the left hand to the right, be called I, K, L, and M, &c, respectively, we shall have $G = \frac{n+5}{6} \times F$, and $H = \frac{n+6}{7} \times G$, and $I = \frac{n+7}{8} \times H$, and $K = \frac{n+8}{9} \times I$, and $L = \frac{n+9}{10} \times K$, and $M = \frac{n+10}{11} \times L$, and consequently that the 3d term of the $\overline{n+2}$ th horizontal row of terms, and the 4th term of the $\overline{n+3}$ th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}$ th, $\overline{n+5}$ th, $\overline{n+6}$ th, $\overline{n+7}$ th, $\overline{n+8}$ th, $\overline{n+9}$ th, $\overline{n+10}$ th, and $\overline{n+11}$ th, and other next following horizontal rows of terms, reckoned from the beginnings of those several horizontal rows, or from the left hand to the right, will be equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times C$, $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, $\frac{n+9}{10} \times K$, and $\frac{n+10}{11} \times L$, &c, respectively. It follows therefore, that the 3d term of the $\overline{n+2}$ th horizontal row of terms, and the 4th term of the $\overline{n+3}$ th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}$ th, $\overline{n+5}$ th, $\overline{n+6}$ th, $\overline{n+7}$ th, $\overline{n+8}$ th, $\overline{n+9}$ th, $\overline{n+10}$ th, and $\overline{n+11}$ th, and other next following horizontal rows of terms, reckoned from the ends of those several horizontal rows, or from the

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right

right hand to the left, will also be respectively equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times C$, $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, $\frac{n+9}{10} \times K$, and $\frac{n+10}{11} \times L$, &c.

Of the figurate numbers, or the significant terms of the vertical columns of terms in the foregoing table of combinations, page 74.

103. Coroll. 10. We come now to consider the vertical columns of terms in the foregoing table of combinations.

Now it is evident, in the first place, that the first significant term in every vertical column of terms in the said table is an unit, and that the second significant term is the number which is the exponent of the column; as has been observed above in art. 53, page 76. So that, if the whole number n be the exponent of the column, the two first significant terms of the said column, immediately following the cyphers at the top of it, will be 1 and n . It remains that we find the values of the following terms in the said column, after the terms 1 and n . Now this may be done by means of the foregoing corollaries, in the manner following.

In the foregoing table of combinations the number of cyphers at the top of the vertical column of which the exponent is n , is $n-1$; as is observed above in art. 51, page 75. And consequently the first significant term in the said vertical column, to wit, 1, will be the n th term of it, and consequently will be situated in the n th horizontal row of terms in the said table; and the second significant term in the said vertical column of terms, to wit, n , will be situated in

in the $n+1$ th horizontal row of terms; and the 3d significant term in the said vertical column will be situated in the $n+2$ th horizontal row of terms; and the 4th significant figure in the said vertical column will be situated in the $n+3$ th horizontal row of terms; and, in like manner, the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following significant terms in the said n th vertical column will be situated in the $n+4$ th, and $n+5$ th, and $n+6$ th, and $n+7$ th, and $n+8$ th, and $n+9$ th, and other following horizontal rows of terms respectively.

And, further, the first significant term, to wit, 1, in the said n th vertical column of terms, is likewise the first term, reckoned from the right hand to the left, of the horizontal row in which it is situated; and the second significant term in the said n th vertical column of terms, to wit, n , is likewise the second term, reckoned from the right hand to the left, of the horizontal row in which it is situated; and the 3d significant term in the said n th vertical column is likewise the third term, reckoned from the right hand to the left, of the horizontal row in which it is situated; and the 4th significant term of the said n th vertical column is likewise the 4th term, reckoned from the right hand to the left, of the horizontal row in which it is situated; and, in like manner, the 5th, 6th, 7th, 8th, and other following significant terms of the said n th vertical column of terms, are likewise the 5th, 6th, 7th, 8th, and other following terms, reckoned from the right hand to the left, of the several horizontal rows of terms in which they are situated, respectively.

But it has been shewn that the 1st, 2d, 3d, 4th, 5th, 6th, and other following significant terms in the n th vertical column of terms are situated in the n th, $n+1$ th, $n+2$ th, $n+3$ th, $n+4$ th, $n+5$ th, and other next following horizontal rows of terms, respectively.

Therefore the 1st, 2d, 3d, 4th, 5th, 6th, and other following significant terms of the n th vertical column of terms

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are

are likewise the 1st, 2d, 3d, 4th, 5th, 6th, and other next following terms, reckoned from the right hand to the left, of the n th, $\overline{n+1}$ th, $\overline{n+2}$ th, $\overline{n+3}$ th, $\overline{n+4}$ th, $\overline{n+5}$ th, and other next following horizontal rows of terms, respectively.

But it has been shewn in coroll. 9, that the 3d term of the $\overline{n+2}$ th horizontal row of terms, reckoned from the right hand to the left, is equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and that the 4th term of the $\overline{n+3}$ th horizontal row of terms, reckoned from the right hand to the left, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and that the 5th term of the $\overline{n+4}$ th horizontal row of terms, and the 6th term of the $\overline{n+5}$ th horizontal row of terms, and the 7th, 8th, 9th, 10th, and other next following terms of the $\overline{n+6}$ th, $\overline{n+7}$ th, $\overline{n+8}$ th, $\overline{n+9}$ th, and other next following horizontal rows of terms, respectively, all reckoned from the right hand to the left, are equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, &c.

Therefore the 3d significant term of the n th vertical column of terms will be equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and the 4th significant term of the same vertical column will be equal to $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following significant terms of the same vertical column will be equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, &c.; and consequently the whole

whole of the said n th vertical column of terms, including the two first significant terms 1 and n , or, in other words, the whole series of figurate numbers of the n th order, will

be 1, n , or $\frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \frac{n+8}{9} I, \frac{n+9}{10} K, \frac{n+10}{11} L, \frac{n+11}{12} M, \frac{n+12}{13} N, \frac{n+13}{14} O, \frac{n+14}{15} P, \&c, ad infinitum.$

Q. E. I.

Examples of the application of the foregoing series to the computation of the figurate numbers of several successive orders.

104. In the first place we will suppose the letter n to denote 1.

Then we shall have $n + 1 (= 1 + 1) = 2$, and $n + 2 (= 1 + 2) = 3$, and $n + 3 (= 1 + 3) = 4$, and $n + 4 (= 1 + 4) = 5$, and $n + 5, n + 6, n + 7, n + 8, n + 9, n + 10, \&c, = 6, 7, 8, 9, 10, 11, \&c$, respectively.

Therefore the several terms 1, $\frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \frac{n+8}{9} I, \frac{n+9}{10} K, \frac{n+10}{11} L, \&c$, will, in this case, be equal to 1, $\frac{1}{1} A, \frac{2}{2} B, \frac{3}{3} C, \frac{4}{4} D, \frac{5}{5} E, \frac{6}{6} F, \frac{7}{7} G, \frac{8}{8} H, \frac{9}{9} I, \frac{10}{10} K, \frac{11}{11} L, \&c$, respectively,

or to 1, 1 A, 1 B, 1 C, 1 D, 1 E, 1 F, 1 G, 1 H, 1 I, 1 K, 1 L, &c, or to 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c; and therefore the twelve first terms in the first vertical column in the foregoing table of combinations, or the twelve first figurate numbers of the 1st order, obtained by means

means of the foregoing series, will be 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, and 1, or a set of units; as they ought to be, and as they are in the foregoing table of combinations.

105. In the next place we will suppose the exponent n to be equal to 2, in order to obtain, by means of the foregoing series, the significant terms in the second vertical column in the foregoing table of combinations, or the figurate numbers of the second order.

Now, if n is = 2, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \frac{n+8}{9} I, \frac{n+9}{10} K, \&c,$ will be respectively equal to $1, \frac{2}{1} A, \frac{2+1}{2} B, \frac{2+2}{3} C, \frac{2+3}{4} D, \frac{2+4}{5} E, \frac{2+5}{6} F, \frac{2+6}{7} G, \frac{2+7}{8} H, \frac{2+8}{9} I, \frac{2+9}{10} K, \&c,$ or $1, \frac{2}{1} A, \frac{3}{2} B, \frac{4}{3} C, \frac{5}{4} D, \frac{6}{5} E, \frac{7}{6} F, \frac{8}{7} G, \frac{9}{8} H, \frac{10}{9} I, \frac{11}{10} K, \&c,$ or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Therefore the first eleven significant terms in the second vertical column of terms in the foregoing table of combinations, or the first eleven figurate numbers of the second order, obtained by means of the foregoing series, will be the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; as they ought to be.

106. In the 3d place we will suppose the exponent n to be equal to 3, in order to obtain, by means of the foregoing series, the significant terms in the 3d vertical column of terms in the foregoing table of combinations, or the figurate numbers of the third order, or (as they are often called) the *triangular* numbers.

Now, if n is = 3, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \frac{n+8}{9} I, \&c,$

&c, will be respectively equal to $1, \frac{3}{1} A, \frac{3+1}{2} B, \frac{3+2}{3} C, \frac{3+3}{4} D, \frac{3+4}{5} E, \frac{3+5}{6} F, \frac{3+6}{7} G, \frac{3+7}{8} H, \frac{3+8}{9} I, \&c,$ or $1, \frac{3}{1} A, \frac{4}{2} B, \frac{5}{3} C, \frac{6}{4} D, \frac{7}{5} E, \frac{8}{6} F, \frac{9}{7} G, \frac{10}{8} H, \frac{11}{9} I, \&c,$ or $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \&c.$ Therefore the first ten significant terms in the third vertical column of terms in the foregoing table of combinations, or the first ten figurate numbers of the third order, or the first ten triangular numbers, obtained by means of the foregoing series, are $1, 3, 6, 10, 15, 21, 28, 36, 45,$ and 55 ; which are the same numbers with those set down above in page 74, in the third vertical column of the foregoing table of combinations.

107. In the 4th place we will suppose the exponent n to be $= 4$, in order to obtain, by means of the foregoing series, the significant terms in the 4th vertical column in the foregoing table of combinations, or the figurate numbers of the 4th order, or (as they are often called) the *pyramidal* numbers.

Now, if n is $= 4$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \&c,$ will be respectively equal to $1, \frac{4}{1} A, \frac{4+1}{2} B, \frac{4+2}{3} C, \frac{4+3}{4} D, \frac{4+4}{5} E, \frac{4+5}{6} F, \frac{4+6}{7} G, \frac{4+7}{8} H, \&c,$ or $1, \frac{4}{1} A, \frac{5}{2} B, \frac{6}{3} C, \frac{7}{4} D, \frac{8}{5} E, \frac{9}{6} F, \frac{10}{7} G, \frac{11}{8} H, \&c,$ or $1, 4, 10, 20, 35, 56, 84, 120, 165, \&c.$ Therefore the first nine significant terms in the fourth vertical column of terms in the foregoing table of combinations, or the first nine figurate numbers of the 4th order, or the first nine pyramidal numbers, obtained by means of the foregoing series, are $1, 4, 10, 20, 35, 56, 84, 120,$ and 165 ; which are the same numbers

bers with those set down above in page 74, in the fourth vertical column of the foregoing table of combinations.

108. In the 5th place we will suppose the exponent n to be $= 5$, in order to obtain, by means of the foregoing series, the significant terms in the 5th vertical column of terms in the foregoing table of combinations, or the figurate number of the 5th order.

Now, if n is $= 5$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \&c$, will be respectively equal to $1, \frac{5}{1} A, \frac{5+1}{2} B, \frac{5+2}{3} C, \frac{5+3}{4} D, \frac{5+4}{5} E, \frac{5+5}{6} F, \frac{5+6}{7} G, \&c$, or $1, \frac{5}{1} A, \frac{6}{2} B, \frac{7}{3} C, \frac{8}{4} D, \frac{9}{5} E, \frac{10}{6} F, \frac{11}{7} G, \&c$, or $1, 5, 15, 35, 70, 126, 210, 330, \&c$. Therefore the first eight significant terms in the 5th vertical column of terms in the foregoing table of combinations, or the first eight figurate numbers of the 5th order, are $1, 5, 15, 35, 70, 126, 210$, and 330 ; which are the same numbers with those set down above in page 74 in the 5th vertical column of the foregoing table of combinations.

109. In like manner, if the exponent n is $= 6$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \&c$, will be respectively equal to $1, \frac{6}{1} A, \frac{7}{2} B, \frac{8}{3} C, \frac{9}{4} D, \frac{10}{5} E, \frac{11}{6} F, \frac{12}{7} G, \&c$, or $1, 6, 21, 56, 126, 252, 462, \&c$. and consequently the first seven significant terms of the 6th vertical column of terms in the foregoing table, or the first seven figurate numbers of the 6th order, will be $1, 6, 21, 56, 126, 252$, and 462 ; which are the same numbers with those set down above in page 74 in the 6th vertical column of the foregoing table of combinations.

110. And,

110. And, if the exponent n is $= 7$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \&c$, will be respectively equal to $1, \frac{7}{1} A, \frac{8}{2} B, \frac{9}{3} C, \frac{10}{4} D, \frac{11}{5} E, \&c$, or $1, 7, 28, 84, 210, 462, \&c$; and consequently the first six significant terms of the 7th vertical column of terms in the foregoing table of combinations, or the first six figurate numbers of the 7th order, will be $1, 7, 28, 84, 210$, and 462 ; which are the same numbers with those set down above in page 74, in the 7th vertical column of the foregoing table of combinations.

111. And, if the exponent n is $= 8$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \&c$, will be respectively equal to $1, \frac{8}{1} A, \frac{9}{2} B, \frac{10}{3} C, \frac{11}{4} D, \&c$, or $1, 8, 36, 120, 330, \&c$; and consequently the first five significant terms of the 8th vertical column of terms in the foregoing table of combinations, or the first five figurate numbers of the 8th order, will be $1, 8, 36, 120$, and 330 ; which are the same numbers with those set down above in page 74, in the 8th vertical column of the foregoing table of combinations.

112. And, if the exponent n is $= 9$, the terms of the series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \&c$, will be respectively equal to $1, \frac{9}{1} A, \frac{10}{2} B, \frac{11}{3} C, \&c$, or $1, 9, 45, 165, \&c$; and consequently the first four significant terms in the 9th vertical column of terms in the foregoing table of combinations, or the first four figurate numbers of the 9th order, will be $1, 9, 45$, and 165 ; which are the same numbers with those set down above in page 74, in the 9th vertical column of the foregoing table of combinations.

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113. And,

113. And, if the exponent n is $= 10$, the terms $1, \frac{n}{1} A, \frac{n+1}{2} B, \&c$, will be respectively equal to $1, \frac{10}{1} A, \frac{11}{2} B, \&c$, or $1, 10, 55, \&c$; and consequently the three first significant terms of the 10th vertical column of terms in the foregoing table of combinations, or the three first figurate numbers of the 10th order, will be $1, 10$, and 55 ; which are the same numbers with those set down above in page 74, in the 10th vertical column of the said table of combinations.

114. It appears, therefore, that all the numbers set down above in page 74, in the ten first vertical columns of the foregoing table of combinations, may be obtained by the application of the general series $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E, \frac{n+5}{6} F, \frac{n+6}{7} G, \frac{n+7}{8} H, \frac{n+8}{9} I, \frac{n+9}{10} K, \frac{n+10}{11} L, \&c$; which is a confirmation of the truth of the said series, and of the reasonings by which it was obtained.

A general

A general expression of the value of the fraction $\frac{1}{(a+b)^n}$, or the reciprocal of any integral power of the binomial quantity $a + b$, in an infinite series.

115. Coroll. 11. From the foregoing corollary we may derive a general expression for the value of the quantity $\frac{1}{(a+b)^n}$, or $\frac{1}{(a+b)^n}$, in an infinite series of terms, when the index n is any whole number whatsoever.

For the quantity $\frac{1}{(a+b)^n}$ is equal to the series which results from the division of the numerator 1 by the binomial quantity $a + b$ as many times as there are units in the index n . And the quotients that arise from these divisions are a set of infinite serieses consisting of terms marked alternately with the sign — and the sign +, and of which the numeral coefficients will be the figurate numbers of the several successive orders. This will appear by making a few of these divisions; which I shall therefore now proceed to make: but, in order to render the operations somewhat shorter and easier than they otherwise would be, I shall substitute the binomial quantity $1 + x$ instead of the binomial quantity $a + b$, which will make no change whatever in the numeral coefficients of the terms of the several quotients that will result from these divisions: and I shall suppose the quantity x to be less than 1, to the end that the powers of x in the terms of the several quotients may be decreasing quantities.

116. The first of these divisions will be as follows :

Divisor.

Quotient.

$$1+x) (1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+\&c.$$

Dividend.

$$\begin{array}{r}
 1 \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \\
 1+x \\
 \hline
 -x \\
 \hline
 -x-x^2 \\
 \hline
 \quad +x^2 \\
 \hline
 \quad +x^2+x^3 \\
 \hline
 \quad \quad -x^3 \\
 \hline
 \quad \quad -x^3-x^4 \\
 \hline
 \quad \quad \quad +x^4 \\
 \hline
 \quad \quad \quad +x^4+x^5 \\
 \hline
 \quad \quad \quad \quad -x^5 \\
 \hline
 \quad \quad \quad \quad -x^5-x^6 \\
 \hline
 \quad \quad \quad \quad \quad +x^6 \\
 \hline
 \quad \quad \quad \quad \quad +x^6+x^7 \\
 \hline
 \quad \quad \quad \quad \quad \quad -x^7 \\
 \hline
 \quad \quad \quad \quad \quad \quad -x^7-x^8 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad +x^8 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad +x^8+x^9 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad -x^9 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad -x^9-x^{10} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad +x^{10} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad +x^{10}+x^{11} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -x^{11} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -x^{11}-x^{12} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad +x^{12}
 \end{array}$$

By this division it appears that the fraction $\frac{1}{1+x}$ is equal to the infinite series $1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+\&c.$, in which the second, fourth, sixth, eighth, tenth, and twelfth terms are marked with the sign $-$, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, and eleventh terms are marked with the sign $+$, or are to be added to the first term 1. And it is easy to see, from the manner of making this division, that, if the operation was to be continued to any greater

greater number of terms whatsoever, the 14th, and 16th, and 18th terms, and all the following even terms in the quotient would also be marked with the sign —; and that the 13th, and 15th, and 17th terms, and all the following odd terms in the quotient would be marked with the sign +. And the numeral co-efficients of all the terms in this quotient are units, or are the terms of the first vertical column of terms in the foregoing table of combinations, or the figurate numbers of the first order; agreeably to what has been just now asserted.

117. The next division will be as follows :

Divisor.

Quotient.

$$1+x) (1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11} + \&c.$$

Dividend.

$$1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11} + \&c.$$

$$\begin{array}{r}
 1+x \\
 \hline
 * -2x+x^2 \\
 \hline
 \quad -2x-2x^2 \\
 \hline
 \quad \quad +3x^2-x^3 \\
 \hline
 \quad \quad +3x^2+3x^3 \\
 \hline
 \quad \quad \quad -4x^3+x^4 \\
 \hline
 \quad \quad \quad -4x^3-4x^4 \\
 \hline
 \quad \quad \quad \quad +5x^4-x^5 \\
 \hline
 \quad \quad \quad \quad +5x^4+5x^5 \\
 \hline
 \quad \quad \quad \quad \quad -6x^5+x^6 \\
 \hline
 \quad \quad \quad \quad \quad -6x^5-6x^6 \\
 \hline
 \quad \quad \quad \quad \quad \quad +7x^6-x^7 \\
 \hline
 \quad \quad \quad \quad \quad \quad +7x^6+7x^7 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad -8x^7+x^8 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad -8x^7-8x^8 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad +9x^8-x^9 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad +9x^8+9x^9 \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad -10x^9+x^{10} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad -10x^9-10x^{10} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad +11x^{10}-x^{11} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad +11x^{10}+11x^{11} \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -12x^{11}+\&c. \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -12x^{11} \quad \&c. \\
 \hline
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad *
 \end{array}$$

By

By this division it appears that the fraction $\frac{1}{1+x}$ is equal to the infinite series $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + 9x^8 - 10x^9 + 11x^{10} - 12x^{11} + \&c$, in which, as in the former quotient, the second, fourth, sixth, eighth, tenth, twelfth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, eleventh, and other following odd terms are marked with the sign +, or are to be added to the said first term. And the numeral coefficients of the several terms of this quotient are the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, or the terms of the second vertical column of terms in the foregoing table of combinations, or the figurate numbers of the second order.

118. The third division will be as follows :

Divisor.

Quotient.

$$1+x) (1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-55x^9+66x^{10}-78x^{11}+\&c.$$

Dividend.

$$\begin{array}{r} 1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11}+\&c. \\ 1+x \\ \hline * -3x+3x^2 \\ \hline -3x-3x^2 \\ \hline * +6x^2-4x^3 \\ \hline +6x^2+6x^3 \\ \hline * -10x^3+5x^4 \\ \hline -10x^3-10x^4 \\ \hline * +15x^4-6x^5 \\ \hline +15x^4+15x^5 \\ \hline * -21x^5+7x^6 \\ \hline -21x^5-21x^6 \\ \hline * +28x^6-8x^7 \\ \hline +28x^6+28x^7 \\ \hline * -36x^7+9x^8 \\ \hline -36x^7-36x^8 \\ \hline * +45x^8-10x^9 \\ \hline +45x^8+45x^9 \\ \hline * -55x^9+11x^{10} \\ \hline -55x^9-55x^{10} \\ \hline * +66x^{10}-12x^{11} \\ \hline +66x^{10}+66x^{11} \\ \hline * -78x^{11}+\&c. \\ \hline -78x^{11}-\&c. \\ \hline * + \end{array}$$

By this division it appears that the fraction $\frac{1}{1+x^3}$ is equal

to the infinite series $1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + 28x^6 - 36x^7 + 45x^8 - 55x^9 + 66x^{10} - 78x^{11} + \&c.$, in which, as in the two former quotients, the second, fourth, sixth, eighth, tenth, twelfth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, eleventh, and other following odd terms have the sign + prefixed to them,

them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the third vertical column in the aforesaid table of combinations, or the figurate numbers of the third order, or the *triangular* numbers.

119. The fourth division will be as follows :

Divisor.

Quotient.

$$1+x) (1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+165x^8-\&c.$$

Dividend.

$$\begin{array}{r}
 1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-\&c. \\
 1+x \\
 \hline
 -4x+6x^2 \\
 -4x-4x^2 \\
 \hline
 * +10x^2-10x^3 \\
 +10x^2+10x^3 \\
 \hline
 * -20x^3+15x^4 \\
 -20x^3-20x^4 \\
 \hline
 * +35x^4-21x^5 \\
 +35x^4+35x^5 \\
 \hline
 * -56x^5+28x^6 \\
 -56x^5-56x^6 \\
 \hline
 * +84x^6-36x^7 \\
 +84x^6+84x^7 \\
 \hline
 * -120x^7+45x^8 \\
 -120x^7-120x^8 \\
 \hline
 * +165x^8-\&c. \\
 +165x^8+\&c. \\
 \hline
 * -\&c.
 \end{array}$$

By this division it appears that the fraction $\frac{1}{1+x}$ is equal to the infinite series $1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+165x^8-\&c.$ *ad infinitum*, in which, as in the three former quotients, the second, fourth, sixth, and eighth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, and other following odd terms have the sign + prefixed to them, or are to be added

added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the fourth vertical column in the aforefaid table of combinations, or the figurate numbers of the fourth order, or the *pyramidal* numbers.

120. The fifth division will be as follows :

Divisor.

Quotient.

$$1+x) (1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-\&c.$$

Dividend.

$$1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+\&c.$$

$$1+x$$

$$\begin{array}{r} -5x+10x^2 \\ -5x-5x^2 \end{array}$$

$$\begin{array}{r} * +15x^2-20x^3 \\ +15x^2+15x^3 \end{array}$$

$$\begin{array}{r} * -35x^3+35x^4 \\ -35x^3-35x^4 \end{array}$$

$$\begin{array}{r} * +70x^4-56x^5 \\ +70x^4+70x^5 \end{array}$$

$$\begin{array}{r} * -126x^5+84x^6 \\ -126x^5-126x^6 \end{array}$$

$$\begin{array}{r} * +210x^6-\&c. \\ +210x^6+\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

$$\begin{array}{r} * -\&c. \end{array}$$

By this division it appears that the fraction $\frac{1}{1+x}$ is equal

to the infinite series $1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-\&c.$ *ad infinitum*, in which, as in the four preceding quotients, the second, fourth, sixth, and other following even terms have the sign $-$ prefixed to them, or are to be subtracted from the first term 1 ; and the third, fifth, seventh, and other following odd terms have the sign $+$ prefixed to them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the fifth vertical column in the foregoing table of combinations, or the figurate numbers of the fifth order.

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121.

121. The sixth division will be as follows :

Divisor.	Quotient.
$1+x$	$(1-6x+21x^2-56x^3+126x^4-252x^5+\&c.$

Dividend.

$$\begin{array}{r}
 1-5x+15x^2-35x^3+70x^4-126x^5+\&c. \\
 \underline{1+x} \\
 * -6x+15x^2 \\
 \underline{-6x-6x^2} \\
 * +21x^2-35x^3 \\
 \underline{+21x^2+21x^3} \\
 * -56x^3+70x^4 \\
 \underline{-56x^3-56x^4} \\
 * +126x^4-126x^5 \\
 \underline{+126x^4+126x^5} \\
 * -252x^5+\&c. \\
 \underline{-252x^5-\&c.} \\
 * +\&c.
 \end{array}$$

By this division it appears that the fraction $\frac{1}{1+x}$ is equal

to the infinite series $1-6x+21x^2-56x^3+126x^4-252x^5+\&c$; in which, as in the five former quotients, the second, fourth, sixth, and other following even terms have the sign $-$ prefixed to them, or are to be subtracted from the first term 1; and the third, and fifth, and other following odd terms have the sign $+$ prefixed to them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the sixth vertical column in the foregoing table of combinations, or the figurate numbers of the sixth order.

Conclusions

Conclusions from the foregoing Operations of Division.

122. From the operations of the foregoing six divisions with the same divisor $1 + x$, I presume that it will be evident to the reader, that, if we were to continue each of the foregoing quotients to any number of terms, how great soever, the said terms would continue to be marked with the signs $+$ and $-$ alternately, and that the co-efficients of the following terms after those that have been above computed, would be the following numbers of the same order of figurate numbers to which the co-efficients of the terms above computed in the said quotients, belonged, respectively. And I likewise presume that it will be evident to him, that, if we were to divide the last, or sixth, quotient by $1 + x$, and the next, or seventh, quotient, by the same quantity $1 + x$, and the several next following, or the eighth, ninth, tenth, and eleventh, &c, quotients, continued to any number whatsoever, by the same quantity $1 + x$ (whereby we should obtain several infinite serieses that would be equal to

the fractions $\frac{1}{1+x}$, $\frac{1}{1+x^2}$, $\frac{1}{1+x^3}$, $\frac{1}{1+x^4}$, $\frac{1}{1+x^5}$, &c), and

were to continue each of the said divisions till we had obtained any number of terms in the quotient, how great soever, the several even terms in every quotient would be marked with the sign $-$, or subtracted from the first term 1 ; and the third, fifth, seventh, and other following odd terms in every quotient would be marked with the sign $+$, or added to the said first term; and also, that the co-efficients of the terms of the said 7th quotient (that would be equal to the fraction $\frac{1}{1+x^7}$) would be the figurate numbers of the

7th order, and the co-efficients of the terms of the said 8th
U 2
quotient

quotient (that would be equal to the fraction $\frac{1}{1+x}$) would be the figurate numbers of the 8th order, and that the coefficients of the terms of the 9th, 10th, 11th, and other following quotients (which would be equal to the fractions $\frac{1}{1+x}$, $\frac{1}{1+x^2}$, $\frac{1}{1+x^3}$, &c) would be the figurate numbers of the 9th, 10th, 11th, and other following orders, respectively.

Observations on the foregoing Operations of Division, tending to establish the foregoing Conclusions.

123. The foregoing conclusions may be derived from the following observations, which cannot but occur to every person who shall go through the foregoing operations of algebraïck division with attention, namely,

1st, That in every separate operation of division, by which a new term in the quotient is to be obtained, the dividend will always consist of two terms which will have different signs + and — prefixed to them; so that, when the first of the two has the sign + prefixed to it, the second will be marked with the sign —; and when the first has the sign — prefixed to it, the second will be marked with the sign +.

2dly, That the subtrahend, or quantity which is to be subtracted from the said dividend, will always consist of two terms, which will be both marked with the same sign + or —, which sign will also be the same with that of the first of the two terms of the dividend from which the said subtrahend is to be subtracted; and therefore the sign which is prefixed

prefixed to the second term of the said subtrahend will be contrary to that which is prefixed to the second term of the said dividend, from which it is to be subtracted; whence it follows that when, in order to subtract the said second term of the subtrahend from the second term of the dividend, which is placed just above it, we shall (according to the rules of algebraick subtraction) have changed its sign into the contrary sign, and have added it, with its sign so changed, to the second term of the dividend, the residue thence resulting (which will be the first term of the next dividend) will have the same sign prefixed to it as is prefixed to the second term of the former dividend, or the contrary sign to that which is prefixed to the first term of the former dividend; so that the first terms of every two contiguous dividends throughout the whole division will be marked with contrary signs, and consequently every two contiguous terms in the quotient (which have always the same signs with the first terms of the two dividends from which they are derived) will also be marked with contrary signs.

3dly, Since the two terms of the divisor $1 + x$, to wit, 1 and x , have the same numeral co-efficient 1 , and every new subtrahend is produced by multiplying the divisor $1 + x$ into the last-found term of the quotient, it follows that the numeral co-efficient of the second term of every new subtrahend must be the same with the numeral co-efficient of the first term of the same subtrahend. And consequently, when the sign of the second term of the subtrahend is changed, and thereby become the same with the sign of the second term of the dividend, which is just above it, and it is added, with its sign so changed, to the said second term of the dividend, the co-efficient of the quantity resulting from this addition, or algebraical subtraction (which is evidently the sum of the co-efficient of the second term of the subtrahend and of the co-efficient of the second term of the dividend) will also be the sum of the co-efficient of the first term of the subtrahend and of the co-efficient of the second term of the dividend, and consequently (because the first term of the subtrahend is always equal to, or the same

same with, the first term of the dividend) will also be the sum of the co-efficient of the first term of the dividend and the co-efficient of the second term of the dividend; that is, the co-efficient of the first term of every new dividend will be the sum of the co-efficients of the first and second terms of the next preceeding dividend. And consequently the co-efficient of every new term in the quotient (which is the same with the co-efficient of the first term of the dividend from which it is derived) will be the sum of the two co-efficients of the two terms of the next preceeding dividend. But the second term of the next preceeding dividend is a term of the last preceeding series, or quotient obtained by the division by $1 + x$; and the co-efficient of the first term of the said next preceeding dividend is equal to the sum of the co-efficients of all the preceeding terms of the said last preceeding series, or quotient obtained by the division by $1 + x$. Therefore the co-efficient of every new term in the quotient arising from the present division by $1 + x$ will be equal to the sum of all the co-efficients of the terms in the foregoing series, or quotient, as far as the corresponding term, or term involving the same power of x , and including the said term. Thus, if the former series, or quotient, be called A , and the present quotient, now arising from the division of the series A by $1 + x$, be called B , and m be a whole number denoting the place of any term in the quotient B , the co-efficient of the m th term of the series, or quotient, B , will be equal to the sum of the co-efficients of all the terms of the preceeding series, or quotient, A , as far as the m th term of the said series, and including the said m th term.

And therefore, 4thly, that the co-efficients of the terms of the several serieses, or quotients, arising by the continual division of 1 by the binomial quantity $1 + x$ will be the several orders of figurate numbers, or the terms of the several vertical columns of terms in the foregoing table of combinations; since both the said co-efficients of the terms of the said serieses, or quotients, and the said figurate numbers, or terms of the several vertical columns of terms in the said

said table of combinations, arise in the same manner from a series of units, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, to wit, by the continual addition of them to each other, and by the like continual addition of the terms of every series generated from them to each other.

Application of the foregoing reasonings to the finding of a general expression of the value of the fraction $\frac{1}{a+b}^n$ in an infinite series of simple terms.

124. It having been now proved that the terms of the several serieses, or quotients, that are equal to the fractions $\frac{1}{1+x}$, $\frac{1}{1+x}^2$, $\frac{1}{1+x}^3$, $\frac{1}{1+x}^4$, $\frac{1}{1+x}^5$, $\frac{1}{1+x}^6$, &c, *ad infinitum*, (beginning with the second term in each series), are to be marked with the sign — and the sign + alternately, and that the co-efficients of the terms of the said serieses will be the figurate numbers of the corresponding orders; and it having been proved above in coroll. 10, that the figurate numbers of the n th order, or the significant terms of the n th vertical column of terms in the foregoing table of combinations (n being put for any whole number whatsoever) are equal to the terms of the following series, to wit, 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, &c, *ad infinitum*; it follows that the fraction $\frac{1}{1+x}^n$ will be equal to the infinite series $1 - \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 - \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 - \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 - \frac{n+6}{7} Gx^7 + \frac{n+7}{8} Hx^8 - \frac{n+8}{9} Ix^9 + \frac{n+9}{10} Kx^{10} - \frac{n+10}{11} Lx^{11} + \dots$

$$+ \frac{n+1}{2} B x^2 - \sqrt{\frac{n+2}{3}} C x^3 + \frac{n+3}{4} D x^4 - \sqrt{\frac{n+4}{5}} E x^5 + \frac{n+5}{6} F x^6 - \sqrt{\frac{n+6}{7}} G x^7 + \frac{n+7}{8} H x^8 - \sqrt{\frac{n+8}{9}} I x^9 + \frac{n+9}{10} K x^{10} - \sqrt{\frac{n+10}{11}} L x^{11} + \&c, \text{ ad infinitum.}$$

125. Now let $\frac{b}{a}$ be substituted instead of x . And the fraction $\frac{1}{1+x}$ will then be $= \frac{1}{1+\frac{b}{a}}$, and the series $1 - \frac{n}{1} A x$

$$+ \frac{n+1}{2} B x^2 - \sqrt{\frac{n+2}{3}} C x^3 + \frac{n+3}{4} D x^4 - \sqrt{\frac{n+4}{5}} E x^5 + \frac{n+5}{6} F x^6 - \sqrt{\frac{n+6}{7}} G x^7 + \frac{n+7}{8} H x^8 - \sqrt{\frac{n+8}{9}} I x^9 + \frac{n+9}{10} K x^{10} - \sqrt{\frac{n+10}{11}} L x^{11} + \&c, \text{ ad infinitum, will be } = 1 - \frac{n}{1} A \frac{b}{a} + \frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} - \sqrt{\frac{n+2}{3}} C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \sqrt{\frac{n+4}{5}} E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} - \sqrt{\frac{n+6}{7}} G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \sqrt{\frac{n+8}{9}} I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} - \sqrt{\frac{n+10}{11}} L \frac{b^{11}}{a^{11}} + \&c, \text{ ad infinitum. Therefore the fraction } \frac{1}{1+\frac{b}{a}} \text{ will be equal to the series } 1 - \frac{n}{1} A \frac{b}{a} +$$

$$\frac{n+1}{2} B \frac{b^2}{a^2} - \sqrt{\frac{n+2}{3}} C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \sqrt{\frac{n+4}{5}} E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} - \sqrt{\frac{n+6}{7}} G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \sqrt{\frac{n+8}{9}} I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} - \sqrt{\frac{n+10}{11}} L \frac{b^{11}}{a^{11}} + \&c; \text{ and consequently (dividing both$$

sides of this equation by a^n) we shall have $\left(\frac{1 - \frac{n}{1} A \frac{b}{a} + \frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} - \sqrt{\frac{n+2}{3}} C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \sqrt{\frac{n+4}{5}} E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} - \sqrt{\frac{n+6}{7}} G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \sqrt{\frac{n+8}{9}} I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} - \sqrt{\frac{n+10}{11}} L \frac{b^{11}}{a^{11}} + \&c}{a^n} \right)$

or

$$\text{or } \frac{1}{1 + \frac{b}{a} \times a^n}, \text{ or } \frac{1}{1 + \frac{b}{a}} \times a^n, \text{ or } \frac{1}{a+b} = \frac{1}{a^n} - \frac{n}{1} A$$

$$\frac{b}{a^{n+1}} + \frac{n+1}{2} B \frac{b^2}{a^{n+2}} - \frac{n+2}{3} C \frac{b^3}{a^{n+3}} + \frac{n+3}{4} D \frac{b^4}{a^{n+4}} -$$

$$\frac{n+4}{5} E \frac{b^5}{a^{n+5}} + \frac{n+5}{6} F \frac{b^6}{a^{n+6}} - \frac{n+6}{7} G \frac{b^7}{a^{n+7}} + \frac{n+7}{8} H$$

$$\frac{b^8}{a^{n+8}} - \frac{n+8}{9} I \frac{b^9}{a^{n+9}} + \frac{n+9}{10} K \frac{b^{10}}{a^{n+10}} - \frac{n+10}{11} L \frac{b^{11}}{a^{n+11}}$$

+ &c, *ad infinitum*; or, according to Sir Isaac Newton's notation with negative indexes of powers, we shall have

$$\overline{a+b}^{-n} = \text{the series } a^{-n} - \frac{n}{1} A a^{-n-1} b + \frac{n+1}{2}$$

$$B a^{-n-2} b^2 - \frac{n+2}{3} C a^{-n-3} b^3 + \frac{n+3}{4} D a^{-n-4} b^4 - \frac{n+4}{5}$$

$$E a^{-n-5} b^5 + \frac{n+5}{6} F a^{-n-6} b^6 - \frac{n+6}{7} G a^{-n-7} b^7 + \frac{n+7}{8}$$

$$H a^{-n-8} b^8 - \frac{n+8}{9} I a^{-n-9} b^9 + \frac{n+9}{10} K a^{-n-10} b^{10} -$$

$$\frac{n+10}{11} L a^{-n-11} b^{11} + \&c, \text{ ad infinitum.} \quad Q. E. D.$$

126. This last series is the same with that which would result from Sir Isaac Newton's original series for expressing the value of the quantity $\overline{a+b}^n$, or the n th power of the binomial quantity $a+b$, to wit, the series $a^n + \frac{n}{1} a^{n-1} b +$
 $\frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 + \frac{n}{1} \times$
 $\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times$
 $\frac{n-4}{5} a^{n-5} b^5 + \&c, \text{ by supposing the index } n \text{ of the said}$
 power to be negative, or by substituting $-n$ instead of n
 in the terms of the said series.

X

For,

For, if this substitution be made in the terms of the said series, it will become equal to $a^{-n-\frac{n}{1}} a^{-n-1} b^1 - \frac{n}{1} \times -\frac{n-1}{2} a^{-n-2} b^2 - \frac{n}{1} \times -\frac{n-1}{2} \times -\frac{n-2}{3} a^{-n-3} b^3 - \frac{n}{1} + -\frac{n-1}{2} \times -\frac{n-2}{3} \times -\frac{n-3}{4} \times a^{-n-4} b^4 - \frac{n}{1} \times -\frac{n-1}{2} \times -\frac{n-2}{3} \times -\frac{n-3}{4} \times -\frac{n-4}{5} \times a^{-n-5} b^5 - \&c$, and consequently, (because $-\frac{n}{1} \times -\frac{n-1}{2}$ is $= +\frac{n}{1} \times +\frac{n+1}{2}$, and $-\frac{n-2}{3} \times -\frac{n-3}{4}$ is $= +\frac{n+2}{3} \times +\frac{n+3}{4}$), equal to $a^{-n-\frac{n}{1}} a^{-n-1} b^1 + \frac{n}{1} \times +\frac{n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times +\frac{n+1}{2} \times -\frac{n-2}{3} a^{-n-3} b^3 + \frac{n}{1} \times +\frac{n+1}{2} \times +\frac{n+2}{3} \times +\frac{n+3}{4} a^{-n-4} b^4 + \frac{n}{1} \times +\frac{n+1}{2} \times +\frac{n+2}{3} \times +\frac{n+3}{4} \times -\frac{n-4}{5} a^{-n-5} b^5 - \&c$, $= a^{-n-\frac{n}{1}} a^{-n-1} b^1 + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{n+1}{2} \times -\sqrt{\frac{n+2}{3}} a^{-n-3} b^3 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^4 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times -\sqrt{\frac{n+4}{5}} a^{-n-5} b^5 + \&c$, $= a^{-n-\frac{n}{1}} A a^{-n-1} b^1 + \frac{n+1}{2} B a^{-n-2} b^2 - \sqrt{\frac{n+2}{3}} C a^{-n-3} b^3 + \frac{n+3}{4} D a^{-n-4} b^4 - \sqrt{\frac{n+4}{5}} E a^{-n-5} b^5 + \&c$; which is the series we just now derived from Mr. James Bernoulli's doctrine of combinations for the value of the quantity $\overline{a+b}^{-n}$, or $\frac{1}{a+b}^n$.

Q. E. D.

A general

A general expression of the value of the fraction $\frac{1}{a-b)^n}$, or of the reciprocal of any integral power of the residual quantity $a-b$, in an infinite series.

127. Coroll. 12. The fraction $\frac{1}{a-b)^n}$ will be equal to the series $\frac{1}{a^n} + \frac{n}{1} A \frac{b}{a^{n+1}} + \frac{n+1}{2} B \frac{b^2}{a^{n+2}} + \frac{n+2}{3} C \frac{b^3}{a^{n+3}} + \frac{n+3}{4} D \frac{b^4}{a^{n+4}} + \frac{n+4}{5} E \frac{b^5}{a^{n+5}} + \frac{n+5}{6} F \frac{b^6}{a^{n+6}} + \frac{n+6}{7} G \frac{b^7}{a^{n+7}} + \frac{n+7}{8} H \frac{b^8}{a^{n+8}} + \frac{n+8}{9} I \frac{b^9}{a^{n+9}} + \frac{n+9}{10} K \frac{b^{10}}{a^{n+10}} + \frac{n+10}{11} L \frac{b^{11}}{a^{n+11}} + \&c, ad infinitum$, which consists of the very same terms as the series obtained in the foregoing corollary for the value of the fraction $\frac{1}{a+b)^n}$, but with the sign $+$ prefixed to all the terms after the first term $\frac{1}{a^n}$, instead of being prefixed only to the 3d, 5th, 7th, 9th, and other following odd terms of it, as in the former series.

128. This will appear by dividing 1 two, or three times following by the residual quantity $1-x$ instead of the binomial quantity $1+x$. For we shall easily perceive that all the terms, after the first terms, in the quotients arising from those divisions will be marked with the sign $+$, or must be added to the first terms. The three first of these divisions will be as follows:

X 2

The

The first Division.

Divisor,

Quotient.

$$1-x) (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.$$

Dividend.

$$\begin{array}{r}
 1 \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \\
 \begin{array}{r}
 1-x \\
 \hline
 +x \\
 \hline
 +x-x^2 \\
 \hline
 +x^2 \\
 \hline
 +x^2-x^3 \\
 \hline
 +x^3 \\
 \hline
 +x^3-x^4 \\
 \hline
 +x^4 \\
 \hline
 +x^4-x^5 \\
 \hline
 +x^5 \\
 \hline
 +x^5-x^6 \\
 \hline
 +x^6 \\
 \hline
 +x^6-x^7 \\
 \hline
 +x^7 \\
 \hline
 +x^7-x^8 \\
 \hline
 +x^8 \\
 \hline
 +x^8-x^9 \\
 \hline
 +x^9 \\
 \hline
 +x^9-x^{10} \\
 \hline
 +x^{10}
 \end{array}
 \end{array}$$

The second Division.

Divisor.

Quotient.

$$1-x) (1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+10x^9+\&c.$$

Dividend.

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.$$

$$\begin{array}{r} 1-x \\ \hline \end{array}$$

$$+2x+x^2$$

$$+2x-2x^2$$

$$\begin{array}{r} * +3x^2+x^3 \\ \hline \end{array}$$

$$+3x^2-3x^3$$

$$\begin{array}{r} * +4x^3+x^4 \\ \hline \end{array}$$

$$+4x^3-4x^4$$

$$\begin{array}{r} * +5x^4+x^5 \\ \hline \end{array}$$

$$+5x^4-5x^5$$

$$\begin{array}{r} * +6x^5+x^6 \\ \hline \end{array}$$

$$+6x^5-6x^6$$

$$\begin{array}{r} * +7x^6+x^7 \\ \hline \end{array}$$

$$+7x^6-7x^7$$

$$\begin{array}{r} * +8x^7+x^8 \\ \hline \end{array}$$

$$+8x^7-8x^8$$

$$\begin{array}{r} * +9x^8 \\ \hline \end{array}$$

$$+9x^8$$

$$*$$

The

The third Division.

Divisor.

Quotient.

$$1-x) (1+3x+6x^2+10x^3+15x^4+21x^5+36x^6+\&c.$$

Dividend.

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+\&c.$$

$$\begin{array}{r} 1-x \\ \hline \end{array}$$

$$+3x+3x^2$$

$$+3x-3x^2$$

$$\hline$$

$$* +6x^2+4x^3$$

$$+6x^2-6x^3$$

$$\hline$$

$$* +10x^3+5x^4$$

$$+10x^3-10x^4$$

$$\hline$$

$$* +15x^4+6x^5$$

$$+15x^4-15x^5$$

$$\hline$$

$$* +21x^5+7x^6$$

$$+21x^5-21x^6$$

$$\hline$$

$$* +28x^6+8x^7$$

$$+28x^6-28x^7$$

$$\hline$$

$$* +36x^7$$

129. It is easy to see that, both in these three divisions, and in all the following divisions that may be made of the last quotient hereby obtained, by the same divisor $1-x$, all the terms of the several quotients, after the first terms, will be marked with the sign $+$, or be added to the first terms, and that the co-efficients of the several terms will be the very same numbers as the co-efficients of the corresponding terms in the former quotients which resulted from the divisions by the binomial quantity $1+x$. It follows therefore that the fraction $\frac{1}{1-x}x$ will be equal to the infinite series $1 + \frac{x}{1}$

$$Ax + \frac{x+1}{2} Bx^2 + \frac{x+2}{3} Cx^3 + \frac{x+3}{4} Dx^4 + \frac{x+4}{5} Ex^5 \\ + \frac{x+5}{6} Fx^6 + \frac{x+6}{7} Gx^7 + \frac{x+7}{8} Hx^8 + \frac{x+8}{9} Ix^9 + \\ \frac{x+9}{10} Kx^{10} + \frac{x+10}{11} Lx^{11} + \&c, \text{ ad infinitum, and consequently}$$

quently (substituting $\frac{b}{a}$ instead of x in this equation) that the fraction $\frac{1}{1 - \frac{b}{a}}$ will be equal to the infinite series $1 + \frac{x}{1}$

$$A \frac{b}{a} + \frac{x+1}{2} B \frac{b^2}{a^2} + \frac{x+2}{3} C \frac{b^3}{a^3} + \frac{x+3}{4} D \frac{b^4}{a^4} + \frac{x+4}{5} E \frac{b^5}{a^5} + \frac{x+5}{6} F \frac{b^6}{a^6} + \frac{x+6}{7} G \frac{b^7}{a^7} + \frac{x+7}{8} H \frac{b^8}{a^8} + \frac{x+8}{9} I \frac{b^9}{a^9} + \frac{x+9}{10} K \frac{b^{10}}{a^{10}} + \frac{x+10}{11} L \frac{b^{11}}{a^{11}} + \&c, \text{ ad infinitum, and conse-}$$

quently (dividing both sides of the last equation by a^n) that the fraction $(\frac{1}{1 - \frac{b}{a}})^n$, or $\frac{1}{a^n \times 1 - \frac{b}{a}}$, or $\frac{1}{a \times 1 - \frac{b}{a}}$, or

$$\frac{1}{a - b}$$

will be equal to the infinite series $\frac{1}{a^n} + \frac{x}{1} A \frac{b}{a^{n+1}} + \frac{x+1}{2} B \frac{b^2}{a^{n+2}} + \frac{x+2}{3} C \frac{b^3}{a^{n+3}} + \frac{x+3}{4} D \frac{b^4}{a^{n+4}} + \frac{x+4}{5} E \frac{b^5}{a^{n+5}} + \frac{x+5}{6} F \frac{b^6}{a^{n+6}} + \frac{x+6}{7} G \frac{b^7}{a^{n+7}} + \frac{x+7}{8} H \frac{b^8}{a^{n+8}} + \frac{x+8}{9} I \frac{b^9}{a^{n+9}} + \frac{x+9}{10} K \frac{b^{10}}{a^{n+10}} + \frac{x+10}{11} L \frac{b^{11}}{a^{n+11}} + \&c, \text{ ad infinitum. Q. E. D.}$

130. If we make use of Sir Isaac Newton's notation with negative indexes of powers, the last equation will be as follows, to wit, $a - b^{-x} =$ the infinite series $a^{-x} + \frac{x}{1} A a^{-x-1} b + \frac{x+1}{2} B a^{-x-2} b^2 + \frac{x+2}{3} C a^{-x-3} b^3 + \frac{x+3}{4} D a^{-x-4} b^4 + \frac{x+4}{5} E a^{-x-5} b^5 + \frac{x+5}{6} F a^{-x-6} b^6 + \frac{x+6}{7} G a$

$$G a^{-n-7} b^7 + \frac{n+7}{8} H a^{-n-8} b^8 + \frac{n+8}{9} I a^{-n-9} b^9 + \frac{n+9}{10} K a^{-n-10} b^{10} + \frac{n+10}{11} L a^{-n-11} b^{11} + \&c, \text{ ad infinitum.}$$

But the other way of expressing this equation seems to be clearer and more natural than this way, and, for ordinary purposes, preferable to it.

131. This last series is the same with that which would result from Sir Isaac Newton's original series for expressing the value of the quantity $a-b$, or the n th power of the residual quantity $a-b$, to wit, the series $a^n - \frac{n}{1} a^{n-1} b^1 + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5} b^5 + \&c$, by supposing the index n of the said power to be negative, or by substituting $-n$ instead of n in the terms of the said series.

For, if this substitution be made in the terms of the said series, it will become equal to $a^{-n} + \frac{n}{1} a^{-n-1} b^1 - \frac{n}{1} \times \frac{n-1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{-n-3} b^3 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{-n-4} b^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{-n-5} b^5 + \&c, = a^{-n} + \frac{n}{1} a^{-n-1} b^1 + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} a^{-n-3} b^3 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^4 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} a^{-n-5} b^5 + \&c, = a^{-n} + \frac{n}{1} a^{-n-1}$

$$\begin{aligned}
 & a^{-n-1} b^1 + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \\
 & a^{-n-3} b^3 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^4 + \frac{n}{1} \\
 & \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} a^{-n-5} b^5 + \&c, = a^{-n} \\
 & + \frac{n}{1} A a^{-n-1} b^1 + \frac{n+1}{2} B a^{-n-2} b^2 + \frac{n+2}{3} C a^{-n-3} b^3 \\
 & + \frac{n+3}{4} D a^{-n-4} b^4 + \frac{n+4}{5} E a^{-n-5} b^5 + \&c; \text{ which is the}
 \end{aligned}$$

series we just now derived in art. 130, from Mr. James Bernoulli's doctrine of combinations for the value of the quantity

$$\overline{a-b}^{-n}, \text{ or } \frac{1}{a-b)^n}. \quad \text{Q. E. D.}$$

132. We have now seen how from Mr. James Bernoulli's doctrine of combinations, and his explanation of the properties of the figurate numbers derived from it, may be deduced just and regular demonstrations of Sir Isaac Newton's famous binomial and residual theorems in the case of the integral and negative powers of a binomial and a residual quantity, or of the reciprocals of their integral and affirmative powers, as well as in the case of their integral and affirmative powers themselves, in which Mr. Bernoulli himself has demonstrated them above in coroll. 5. And I doubt whether any other method of demonstrating these two famous theorems in the case of the integral and negative powers of a binomial and a residual quantity has yet been found out, that is equally clear and satisfactory.

A Difficulty that may occur concerning the foregoing Theorem relating to the integral and negative Powers of a residual Quantity, as $2 - b$, or $1 - x$.

133. Before we conclude this subject of the binomial and residual theorems in the case of integral and negative powers, I will endeavour to clear up a difficulty which may, perhaps, occur to the reader's mind concerning the latter of the said theorems.

It has been shewn in art. 129, that $(1-x)^{-n}$, or $\frac{1}{1-x}^n$, is equal to the infinite series $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \frac{n+6}{7} Gx^7 +$ &c, *ad infinitum*, in which all the terms following the first term 1 are marked with the sign +, or are to be added to the said first term. And the co-efficients of the terms in this series continually increase, when n is of any magnitude greater than 1. Thus, if n is = 2, we shall have $\frac{n}{1}$ ($= \frac{2}{1}$) = 2, and $\frac{n+1}{2}$ ($= \frac{2+1}{2}$) = $\frac{3}{2}$, and $\frac{n+2}{3}$ ($= \frac{2+2}{3}$) = $\frac{4}{3}$, and $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{5}{4}$, $\frac{6}{5}$, $\frac{7}{6}$, $\frac{8}{7}$, &c, respectively; in all which fractions the numerators exceed the denominators by an unit; and, if n is = 3, we shall have $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{3}{1}$, $\frac{4}{2}$, $\frac{5}{3}$, $\frac{6}{4}$, $\frac{7}{5}$, $\frac{8}{6}$, $\frac{9}{7}$, &c, respectively; in all which fractions the numerators exceed the denominators by 2. And the like excess of the numerators above the denomi-

nators will take place in a still higher degree in the said generating fractions $\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \frac{n+3}{4}, \frac{n+4}{5}, \frac{n+5}{6}, \frac{n+6}{7}, \&c,$ when the index n is equal to 4, or 5, or 6, or any higher number. And consequently the co-efficients B, C, D, E, F, G, &c, which are derived from the first term 1, or A, by the continual multiplication of the said generating frac-

tions $\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \frac{n+3}{4}, \frac{n+4}{5}, \frac{n+5}{6}, \frac{n+6}{7}, \&c,$ must continually increase in all these serieses. And accordingly we find that the figurate numbers of every order, or the several significant terms in every vertical column of terms in the foregoing table of combinations, page 74, (which are equal to the co-efficients of the terms of the foregoing series

$1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \&c, \text{ ad infinitum}$), increase continually.

And hence it may happen that, if x is but little less than 1, the whole terms at the beginning of the series $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \&c, \text{ ad infinitum}$, may (by means of this increase of their co-efficients) be increasing quantities.

Now, from this circumstance it may, perhaps, be apprehended, that all the terms of this series will in some cases diverge, or increase, continually, to what number of terms soever the said series may be continued, and consequently that the said series (consisting of an infinite number of terms that are every one greater than that next before it) will be infinite in magnitude as well as in the number of its terms, and therefore cannot be equal to the finite quantity

$\frac{1}{1-x}^n$. This is a difficulty that seems naturally to arise upon this subject. But it may be removed by the following considerations.

An Explanation of the foregoing Difficulty.

134. The proportion of the numerators of the several generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, to their denominators (though it is always a proportion of majority, when n is greater than 1) approaches continually nearer and nearer to a ratio of equality, as its limit; so that, if n be ever so great a number, we may, by continuing the series of these generating fractions to a great number of terms, come to one in which the ratio of the numerator to the denominator shall be less than any proposed ratio of majority. Thus, for example, if n is = 1000, and the ratio of majority that is proposed, or given, and with which the ratios of the numerators of these generating fractions to their denominators is to be compared, is that of 1 to 0.99999, or of 100,000 to 99,999, it will be possible, by continuing the series of the said generating fractions, to assign one in which the ratio of the numerator to the denominator shall be less than the ratio of 1 to 0.99999, or of 100,000 to 99,999. This may be shewn in the manner following. Let z be the general representative of the several numbers added to the index n in the numerators of these successive generating fractions; so that the said numerators shall be equal to the several successive values of $n+z$, or, on the present supposition that n is = 1000, to the several successive values of $1000+z$. Then will the denominators of the said successive generating fractions be denoted by the successive values of $z+1$, and the said generating fractions themselves will be equal to the several successive values of the fraction $\frac{1000+z}{z+1}$, or $\frac{n+1000}{z+1}$. Now it is evident that, by continually increasing the number z , the proportion of the numerator $z+1000$ to the denominator $z+1$ may be made to approach

proach as near as we please to the proportion of z to z , or the proportion of equality. The number z may therefore be increased till the said proportion of $z + 1000$ to $z + 1$ shall be nearer to a proportion of equality, or shall be a less ratio of majority, than the proposed ratio of 1 to 0.99999, or of 100,000 to 99,999. And the same thing might be done if the proposed ratio, instead of being that of 1 to 0.99999, or of 100,000 to 99,999, had been that of 1 to 0.999,999, or of 1,000,000 to 999,999, or that of 1 to 0.999,999,9, or of 10,000,000 to 9,999,999, or any other ratio of majority, how small soever. Therefore, however nearly the quantity x may approach to an equality with 1 (than which it is always supposed to be somewhat less) it will always be possible to increase the number z till the proportion of $z + x$ to $z + 1$, or of $n + z$ to $z + 1$, becomes less than the proportion of 1 to x , or till the fraction $\frac{n+z}{z+1}$ becomes less than the

fraction $\frac{1}{x}$. And, as the number z increases with the num-

ber of terms of the series $1 + \frac{n}{1} A x + \frac{n+1}{2} B x^2 + \frac{n+2}{3}$

$C x^3 + \frac{n+3}{4} D x^4 + \frac{n+4}{5} E x^5 + \frac{n+5}{6} F x^6 + \frac{n+6}{7} G x^7$

+ &c (being always less by 2 than the number of the terms from the beginning of the series to the term in which it occurs, including the said term), it is evident, that, by continuing the terms of the series, we must always come to

a term in which the generating fraction $\frac{n+z}{z+1}$ shall be less than

the fraction $\frac{1}{x}$. And when we are arrived at this term, the

next term of the series will be less increased by being multiplied into the next generating fraction (which will be less than the fraction $\frac{1}{x}$) than it will be diminished by being mul-

tiplied into the fraction $\frac{n}{1}$, or the reciprocal of the fraction

$\frac{1}{x}$; and consequently it will be less than the last preceeding

term of the series from which it is derived. And therefore,
when

when we are come to this term, all the following terms of the said series (which have hitherto been increasing quantities) will decrease continually, and in a greater and greater proportion continually, as the series advances. And consequently the said series will in all cases be of a finite magnitude, however nearly the quantity x may approach to an equality with 1. Q. E. D.

End of the Additional Corollaries not contained in the original text of Mr. James Bernoulli, which began in page 123.

A SCHOLIUM.

135. We may here take occasion to observe, that, though many writers on mathematical subjects (as for example, Faulhaber and Remmelin of the city of Ulm in Germany, and Dr. John Wallis of Oxford, Mr. Nicholas Mercator in his *Logarithmotechnia* *, and Monsieur Prestet, a learned French mathematician) have made the properties of the figurate numbers the subject of their consideration, yet no one has hitherto given the publick a general and scientific demonstration of the foregoing important 12th property of them. At least I may say, that no such demonstration has ever come to my knowledge. Dr. Wallis, indeed, in that part of his learned treatise on the arithmetick of infinites, in which he establishes the foundations of his method, has investigated by arguments of induction the proportions which a series of the squares of a given number of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and a series of their cubes, and a series of their fourth powers, and serieses

* See Vol. I. of the Collection of Tracts, in quarto, intitled, *Scriptores Logarithmici*, pages 192, 193.

of their following higher powers, would bear to a series consisting of the same number of terms all equal to the last, or greatest, term of the former series; and, after performing these investigations, has, in his 176th proposition, made a transition to the contemplation of the trigonal, or triangular, and the pyramidal, and trigono-pyramidal, or triangulo-pyramidal, numbers, and other following orders of the figurate numbers. But I apprehend he would have acted more judiciously and more agreeably to the nature of the subject he was considering, if he had taken the contrary course, and begun with the investigation of the properties of the figurate numbers, and then, after having discovered those properties, and given a just and general demonstration of them, had proceeded to investigate the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. For, besides the objections that may be justly made to his method of making these investigations by inductions from particular examples, as being by no means scientifick or satisfactory to a mind accustomed to more accurate modes of reasoning, and likewise as being more prolix and tedious than need be, on account of the necessity of having a separate investigation for every new series of powers;—I say, besides these objections to his method of treating this subject, it may be considered as inelegant and unnatural on another account, namely, because it treats of the more abstruse parts of the subject, to wit, the investigation of the sums of powers, before the more simple and easy parts of it, or the doctrine of the figurate numbers. For these numbers may be justly esteemed to be more simple and easy to be understood than the powers of the natural numbers, partly, because the several orders of them are generated one from another by the easy operation of addition; whereas, the powers of numbers are produced by the more complicated operation of multiplication; and partly and especially, because the sums of the several orders of figurate numbers (reckoning from the beginning of the foregoing table of them, or including the several cyphers prefixed to the significant terms of the several vertical columns of the said table) are (as we have seen) exact aliquot parts of the serieses that consist of the
same

same numbers of terms all equal to their last, or greatest, terms, respectively; whereas the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, never are exact aliquot parts of the sums of equal numbers of terms equal to the last, or greatest, of them, respectively, but always exceed, or fall short of, such aliquot parts by some small finite quantity, how great soever the number of the terms of such serieses may be supposed to be, and what number of cyphers soever we may prefix to the said serieses consisting of the powers of the natural numbers. Nor can it be alledged, that it was necessary for Dr. Wallis to begin by investigating the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, on account of the difficulty of deducing the values of those sums from the doctrine of the sums of the figurate numbers. For, it is full as easy to deduce the sums of the said powers from the sums of the several orders of figurate numbers, as to deduce the latter from the sums of the powers of numbers in the manner adopted by Dr. Wallis: as I shall now proceed to shew by deducing the sums of the said powers from the sums of the several orders of figurate numbers, which we have already investigated.

An investigation of the sum of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to any given number of terms, and of the sums of their squares, and of their cubes, and of their fourth powers, and other higher powers, continued to the same number of terms.

136. If the letter x be made to denote the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to the n th term, which, it is evident, will be n ,
the

$= 72 + 6 = 78$. And so we shall find it to be by actually adding up the terms.

$$\begin{array}{r}
 1 \\
 2 \\
 3 \\
 4 \\
 5 \\
 6 \\
 7 \\
 8 \\
 9 \\
 10 \\
 11 \\
 12 \\
 \hline
 78
 \end{array}$$

Of the sum of the squares of the natural numbers, 1, 2, 3, 4, 5, &c, continued to any given number n.

137. Let it now be required to find the sum of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to n , or the sum of the numbers 1, 4, 9, 16, 25, 36, 49, &c, continued to the n th term, which will be nn . This may be done in the manner following.

Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n . Then, since by coroll. 4, of the foregoing proposition, art. 83, the n th term of the third vertical column of the foregoing table of combinations is $= \frac{n-1 \times n-2}{2} = \frac{nn-3n+2}{2}$, it follows,

lows, that every x th term of the same vertical column will be.

$= \frac{xx-3x+2}{2}$, or that, if x be made successively equal to 1, 2, 3, 4, 5, 6, &c. the successive values of the fraction $\frac{xx-3x+2}{2}$, will produce the first, second, third, fourth, fifth,

sixth, and other following terms of the same vertical column, which are 0, 0, 1, 3, 6, 10, &c. Thus, for example, if x is = 1, we shall have $\frac{xx-3x+2}{2} = \frac{1-3+2}{2} = \frac{0}{2}$

= 0; and, if x is = 2, we shall have $\frac{xx-3x+2}{2} = \frac{4-6+2}{2}$

= $\frac{0}{2}$ = 0; and, if x is = 3, we shall have $\frac{xx-3x+2}{2} = \frac{9-9+2}{2} = \frac{2}{2} = 1$; and, if x is = 4, we shall have

$\frac{xx-3x+2}{2} = \frac{16-12+2}{2} = \frac{6}{2} = 3$; and, if x is = 5, we shall

have $\frac{xx-3x+2}{2} = \frac{25-15+2}{2} = \frac{12}{2} = 6$; and, if x is = 6,

we shall have $\frac{xx-3x+2}{2} = \frac{36-18+2}{2} = \frac{20}{2} = 10$; which

numbers 0, 0, 1, 3, 6, and 10, are the first six terms of the said third vertical column. And the same thing will be found to be true in any greater number of its terms. But, by the second corollary of the foregoing proposition, art. 81, the sum of all the $n-2$ significant terms, or, including the two cyphers at the beginning of it, the sum of all the n terms, of the said third vertical column is = $\frac{n \times n-1 \times n-2}{2 \times 3}$

= $\frac{n \times n-3n+2}{2 \times 3} = \frac{n^2-3nn+2n}{6}$. Therefore the sum of all the n successive values of the fraction $\frac{xx-3x+2}{2}$ will be =

$\frac{n^2-3nn+2n}{6}$. But the sum of all the n successive values of

$\frac{xx-3x+2}{2}$ is evidently equal to the sum of all the n successive

values of $\frac{xx}{2}$, together with the sum of all the n successive values of $\frac{2}{2}$, or 1, diminished by the sum of all the n successive values of $\frac{3x}{2}$, or (according to our former notation)

$S. \sqrt{\frac{xx-3x+2}{2}}$ is $= S. \frac{xx}{2} + S. \frac{2}{2} - S. \frac{3x}{2} = S. \frac{xx}{2} + S. 1 - S. \frac{3x}{2} = S. \frac{xx}{2} + n - S. \frac{3x}{2}$. Therefore $S. \frac{xx}{2} + n - S. \frac{3x}{2}$ will be $= \frac{n^3-3nn+2n}{6}$. But $S. \frac{3x}{2}$ is evidently $= \frac{3}{2} \times S. x$.

Therefore $S. \frac{xx}{2} + n - \frac{3}{2} \times S. x$ will be $= S. \frac{xx}{2} + n - S. \frac{3x}{2}$, and consequently will be $= \frac{n^3-3nn+2n}{6}$. But it has

been shewn in art. 136, that $S. x$ is $= \frac{nn}{2} + \frac{n}{2}$. Therefore

$\frac{3}{2} \times S. x$ will be $= \frac{3}{2} \times \left[\frac{nn}{2} + \frac{n}{2} \right] = \frac{3nn}{4} + \frac{3n}{4}$. Therefore

$S. \frac{xx}{2} + n - \frac{3nn}{4} - \frac{3n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or $S. \frac{xx}{2} -$

$\frac{3nn}{4} + \frac{n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or $S. \frac{xx}{2} - \frac{9nn}{12} + \frac{3n}{12}$ will be

$= \frac{n^3}{6} - \frac{6nn}{12} + \frac{4n}{12}$. Therefore (adding $\frac{9nn}{12}$ to both sides) we

shall have $S. \frac{xx}{2} + \frac{3n}{12} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{4n}{12}$; and (subtracting $\frac{3n}{12}$

from both sides) we shall have $S. \frac{xx}{2} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{n}{12} =$

$\frac{n^3}{6} + \frac{nn}{4} + \frac{n}{12}$, and consequently $S. xx (= 2 \times S. \frac{xx}{2}) = \frac{n^3}{3}$

$+ \frac{nn}{2} + \frac{n}{6}$; that is, the sum of all the n squares 1, 4, 9,

16, 25, 36, 49, &c, of the n first natural numbers 1, 2,

3, 4, 5, 6, 7, &c n will be $= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$, or one-

third of the cube of the greatest number n , together with
half

half the square of the said number, and a sixth part of the said number itself. Q. E. I.

Thus, for example, if n is $= 12$, we shall have $n = 144$, and $n^2 = 1728$, and $\frac{n^2}{3} + \frac{n^2}{2} + \frac{n}{6} (= \frac{1728}{3} + \frac{144}{2} + \frac{12}{6} = 576 + 72 + 2) = 650$. Therefore the sum of the following twelve numbers, to wit, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, and 144, (which are the squares of the twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12) will be equal to 650. And so, upon adding them up together, we shall find them to be

$$\begin{array}{r}
 1 \\
 4 \\
 9 \\
 16 \\
 25 \\
 36 \\
 49 \\
 64 \\
 81 \\
 100 \\
 121 \\
 144 \\
 \hline
 650
 \end{array}$$

Of the sum of the cubes of the natural numbers 1, 2, 3, 4, 5, &c, continued to any given number n.

138. Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n . Then, since by coroll. 4, of the foregoing proposition, art. 83, the n th term of the fourth vertical column of the foregoing table of combinations is $= \frac{n-1 \times n-2 \times n-3}{2 \times 3} =$

$$\frac{n^2-3n+2}{2} \times \frac{n-3}{3} = \frac{n^3-6n^2+11n-6}{6};$$

it follows, that every x th term of the same fourth column will be $= \frac{x^3-6xx+11x-6}{6},$

or that, if x be made successively equal to 1, 2, 3, 4, 5, 6, &c. the successive values of the fraction $\frac{x^3-6xx+11x-6}{6}$ will

produce the first, second, third, fourth, fifth, sixth, and other following terms of the same vertical column, which are 0, 0, 0, 1, 4, 10, 20, 35, &c. But, by the 2d corollary of the foregoing proposition, art. 81, the sum of all the terms of the said fourth vertical column (the number of which, including the three cyphers at the beginning of it, is n) is

$$= \frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4} = \frac{n^4-6n^3+11n^2-6n}{24}.$$

Therefore the sum of the n successive values of the fraction $\frac{x^3-6xx+11x-6}{6}$

will be $= \frac{n^4-6n^3+11n^2-6n}{24},$ or, according to our former nota-

tion, $S. \sqrt{\frac{x^3-6xx+11x-6}{6}}$ will be $= \frac{n^4-6n^3+11n^2-6n}{24}.$ But $S.$

$$\sqrt{\frac{x^3-6xx+11x-6}{6}} \text{ is } = S. \frac{x^3}{6} - S. \frac{6xx}{6} + S. \frac{11x}{6} - S. \frac{6}{6} = S. \frac{x^3}{6}$$

$\frac{x^3}{6} - S. xx + \frac{11}{6} \times S: x - S. 1 =$ (by the two foregoing articles 136 and 137) $S. \frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11}{6} \times \sqrt{\frac{nn}{2} + \frac{n}{2}}$
 $- n = S. \frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11nn}{12} + \frac{11n}{12} - n = S. \frac{x^3}{6}$
 $- \frac{n^3}{3} - \frac{6nn}{12} - \frac{2n}{12} + \frac{11nn}{12} + \frac{11n}{12} - \frac{12n}{12} = S. \frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12}$
 $- \frac{3n}{12} = S. \frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{n}{4}$. Therefore $S. \frac{x^3}{6} - \frac{n^3}{3} +$
 $\frac{5nn}{12} - \frac{n}{4}$ will be $= \frac{n^4 - 6n^3 + 11nn - 6n}{24}$; and consequently
 (adding $\frac{n^3}{3} + \frac{n}{4}$ to both sides) $S. \frac{x^3}{6} + \frac{5nn}{12}$ will be $=$
 $\frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{n^3}{3} + \frac{n}{4} = \frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{8n^3}{24} + \frac{6n}{24}$
 $= \frac{n^4 + 2n^3 + 11nn}{24}$; and (subtracting $\frac{5nn}{12}$ from both sides) $S. \frac{x^3}{6}$
 will be $= \frac{n^4 + 2n^3 + 11nn}{24} - \frac{5nn}{12} = \frac{n^4 + 2n^3 + 11nn}{24} - \frac{10nn}{24}$
 $= \frac{n^4 + 2n^3 + nn}{24}$. Therefore $S. x^3$ will be $= 6 \times \frac{n^4 + 2n^3 + nn}{24}$
 $= \frac{n^4 + 2n^3 + nn}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$, or the sum of the n succes-
 five values of x^3 , or of the several cube numbers 1, 8, 27,
 64, 125, 216, 343, 512, 729, 1000, 1331, &c, continued
 to n^3 , will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Q. E. I.

Thus, for example, if n is $= 12$, we shall have $nn =$
 144, and $n^3 = 1728$, and $n^4 = 20,736$, and consequently
 $\frac{n^4}{4} (= \frac{20736}{4}) = 5184$, and $\frac{n^3}{2} (= \frac{1728}{2}) = 864$, and $\frac{nn}{4} (=$
 $\frac{144}{4} = 36$, and $\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4} (= 5184 + 864 + 36) =$
 6084. Therefore the sum of the twelve numbers 1, 8, 27,
 64, 125, 216, 343, 512, 729, 1000, 1331, and 1728,
 (which are the cubes of the first twelve natural numbers 1,

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 6084.
And so we shall find the said sum to be, if we actually add
up together the said twelve cube numbers.

$$\begin{array}{r}
 1 \\
 8 \\
 27 \\
 64 \\
 125 \\
 216 \\
 343 \\
 512 \\
 729 \\
 1000 \\
 1331 \\
 1728 \\
 \hline
 6084
 \end{array}$$

Of the sum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, &c, continued to any given number n.

139. Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n .

Then, since by coroll. 4, of the foregoing proposition, art. 83, the n th term of the fifth vertical column of the foregoing table of combinations is $\frac{x-1 \times x-2 \times x-3 \times x-4}{2 \times 3 \times 4}$
 $= \frac{x^3 - 6xx + 11x - 6}{6} \times \frac{x-4}{4} = \frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24}$, it follows
that

that every x th term of the same fifth column will be $= \frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24}$, or that, if n be made successively

equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, the successive values of the fraction $\frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24}$ will produce the

first, second, third, fourth, fifth, sixth, and other following terms of the same vertical column, which are 0, 0, 0, 0, 1, 5, 15, 35, 70, &c. But, by the second corollary of the foregoing proposition, art. 81, the sum of all the terms of the said fifth vertical column (the number of which, including the four cyphers at the beginning of it, is n) is

$$= \frac{n \times n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} = \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{120}.$$

Therefore the sum of all the n successive values of the fraction $\frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24}$ will be $= \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{120}$;

or, according to our former notation,

$$S. \frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24} \text{ will be } = \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{120}.$$

$$\text{But } S. \frac{x^4 - 10x^3 + 35x^2 - 50x + 24}{24} \text{ is } = S. \frac{x^4}{24} - S. \frac{10x^3}{24} +$$

$$S. \frac{35x^2}{24} - S. \frac{50x}{24} + S. \frac{24}{24} = S. \frac{x^4}{24} - \frac{10}{24} \times S. x^3 + \frac{35}{24}$$

$$\times S. xx - \frac{50}{24} \times S. x + S. 1 = S. \frac{x^4}{24} - \frac{10}{24} \times \left[\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4} \right]$$

$$+ \frac{35}{24} \times \left[\frac{n^3}{3} + \frac{nn}{2} + \frac{n^2}{6} \right] - \frac{50}{24} \times \left[\frac{nn}{2} + \frac{n}{2} + n^1 \right] = S. \frac{x^4}{24} -$$

$$\frac{10n^4}{96} - \frac{10n^3}{48} - \frac{10nn}{96} + \frac{35n^3}{72} + \frac{35nn}{48} + \frac{35n}{144} - \frac{50nn}{48} - \frac{50n}{48} + n$$

$$= S. \frac{x^4}{24} - \frac{5n^4}{48} - \frac{30n^3}{144} - \frac{10nn}{96} + \frac{70n^3}{144} + \frac{70nn}{96} + \frac{35n}{144} - \frac{190nn}{96}$$

$$- \frac{150n}{144} + \frac{144n}{144} = S. \frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{70n}{144}.$$

2 A

There-

Therefore $S. \frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{29n}{144}$ will be = $\frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}$; and consequently (adding $\frac{5n^4}{48} + \frac{40nn}{96}$ to both sides) we shall have $S. \frac{x^4}{24} + \frac{40n^3}{144} + \frac{29n}{144}$ $(= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120} + \frac{5n^4}{48} + \frac{40nn}{96} = \frac{n^5}{120} - \frac{n^4}{12} + \frac{7n^3}{24} - \frac{5nn}{12} + \frac{n}{5} + \frac{5n^4}{48} + \frac{5nn}{12}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5}$, and subtracting $\frac{40n^3}{144} + \frac{29n}{144}$ from both sides) $S. \frac{x^4}{24} (= \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5} - \frac{40n^3}{144} - \frac{29n}{144} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{42n^3}{144} + \frac{141n}{720} - \frac{40n^3}{144} - \frac{145n}{720} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{2n^3}{144} - \frac{n}{720}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{n^3}{72} - \frac{n}{720}$; and consequently $S. x^4$ will be $(= 24 \times \frac{n^5}{120} + 24 \times \frac{n^4}{48} + 24 \times \frac{n^3}{72} - 24 \times \frac{n}{720}) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$; or the sum of the successive values of x^4 , or of the several fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n , will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Q. E. I.

Thus, for example, if n is = 12, we shall have $n^5 = 1728$, and $n^4 = 20,736$, and $n^3 (= n^4 \times n = 20,736 \times 12) = 248,832$, and consequently $\frac{n^5}{5} = \frac{248,832}{5}$, and $\frac{n^4}{2} (= \frac{20,736}{2} = 10,368$, and $\frac{n^3}{3} (= \frac{1728}{3}) 576$, and $\frac{n}{30} (= \frac{12}{30}) = \frac{2}{5}$, and $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} (= \frac{248,832}{5} + 10,368 + 576 - \frac{2}{5} = \frac{248,830}{5} + 10,368 + 576 = 49,766 + 10,368 + 576) = 60,710$. Therefore the sum of the twelve numbers

bers 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, and 20,736, (which are the fourth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 60,710. And so we shall find the said sum to be, if we actually add up together the said twelve numbers, or fourth powers of the first twelve natural numbers; which may be done as follows.

$$\begin{array}{r}
 1 \\
 16 \\
 81 \\
 256 \\
 625 \\
 1,296 \\
 2,401 \\
 4,096 \\
 6,561 \\
 10,000 \\
 14,641 \\
 20,736 \\
 \hline
 60,710.
 \end{array}$$

140. The foregoing examples are, I presume, sufficient to shew how the sums of the several powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to any number n , may be deduced from the sums of the several corresponding orders of figurate numbers contained in the foregoing table of combinations. I shall not therefore add the investigations of the sums of any higher powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, than the foregoing ones, but shall only set down the results of the like investigations which I have made for my own satisfaction with respect to the sums of the six next higher powers of those numbers, to wit, the fifth powers, the sixth powers, the seventh powers, the eighth powers, the ninth powers, and the tenth powers of them. These results are as follows.

2 A 2

The

The sum of all the fifth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n , is =

$$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{nn}{12}.$$

The sum of all the sixth powers of the same numbers is =

$$\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

The sum of all the seventh powers of the same numbers is =

$$\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^5}{24} + \frac{nn}{12}.$$

The sum of all the eighth powers of the same numbers is =

$$\frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} + \frac{n}{30}.$$

The sum of all the ninth powers of the same numbers is =

$$\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3nn}{20}.*$$

And the sum of all the tenth powers of the same numbers is =

$$\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}.$$

* N. B. In computing the last term of this expression (which is equal to the sum of all the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, continued to n) the author has fallen into a mistake, having made the said last term to be $\frac{nn}{12}$, instead of $\frac{3nn}{20}$. I have therefore set down $\frac{3nn}{20}$

instead of $\frac{nn}{12}$ in this translation. I had not discovered this mistake when the sheet containing it in the original text of the author, page 32, was printing; or I should have set it right before.

Examples

Examples of the summation of the fifth, sixth, seventh, eighth, ninth, and tenth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, by means of the foregoing expressions.

Let n be = 12.

Then for the sum of the fifth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which are 1; 32; 243; 1,024; 3,125; 7,776; 16,807; 32,768; 59,049; 100,000; 161,051; and 248,832;) we shall have $\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{nn}{12}$, or $\frac{12^6}{6} + \frac{12^5}{2} + \frac{5 \times 12^4}{12} - \frac{12^5}{12}$ ($= \frac{12 \times 12^5}{6} + \frac{12^5}{2} + 5 \times 12^3 - 12 = 2 \times 12^5 + \frac{12^5}{2} + 5 \times 1728 - 12 = 2 \times 248,832 + \frac{248,832}{2} + 8640 - 12 = 497,664 + 124,416 + 8628 = 630,708$). And so we shall find the said sum to be, if we add up together the said twelve numbers, or fifth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

$$\begin{array}{r}
 1 \\
 32 \\
 243 \\
 1,024 \\
 3,125 \\
 7,776 \\
 16,807 \\
 32,768 \\
 59,049 \\
 100,000 \\
 161,051 \\
 248,832 \\
 \hline
 630,708.
 \end{array}$$

And

And for the sum of the sixth powers of the first twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 64; 729; 4,096; 15,625; 46,656; 117,649; 262,144; 531,441; 1,000,000; 1,771,561; and 2,985,984), we shall have $\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$, or $\frac{1^7}{7} + \frac{12^6}{2} + \frac{12^5}{2} - \frac{12^3}{6} + \frac{12}{42} = \frac{12^6 \times 12}{7} + \frac{12^6}{2} + \frac{12^5}{2} - \frac{12^3}{6} + \frac{2}{7} = \frac{2,985,984 \times 12}{7} + \frac{2,985,984}{2} + \frac{248,832}{2} - \frac{1728}{6} + \frac{2}{7} = \frac{35,831,808}{7} + 1,492,992 + 124,416 - 288 + \frac{2}{7} = \frac{35,831,810}{7} + 1,492,992 + 124,416 - 288 = 5,118,830 + 1,492,992 + 124,416 - 288 = 5,118,830 + 1,492,992 + 124,128 = 6,735,950. And so we shall find the said sum to be, if we add up together the said twelve numbers, or sixth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.$

$$\begin{array}{r}
 1 \\
 64 \\
 729 \\
 4,096 \\
 15,625 \\
 46,656 \\
 117,649 \\
 262,144 \\
 531,441 \\
 1,000,000 \\
 1,771,561 \\
 2,985,984 \\
 \hline
 6,735,950.
 \end{array}$$

And

And for the sum of the seventh powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 128; 2,187; 16,384; 78,125; 279,936; 823,543; 2,097,152; 4,782,969; 10,000,000; 19,487,171; and 35,831,808), we shall

have $\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{nn}{12}$, or $\frac{12^8}{8} + \frac{12^7}{2} + \frac{7 \times 12^6}{12} - \frac{7 \times 12^4}{24} + \frac{12^3}{12}$ ($= \frac{12 \times 12^7}{8} + \frac{12^7}{2} + 7 \times 12^5 - \frac{7 \times 12^3}{2} + 12$)
 $= \frac{3 \times 12^7}{2} + \frac{12^7}{2} + 7 \times 12^5 - \frac{7 \times 12^3}{2} + 12 = \frac{3 \times 35,831,808}{2}$
 $+ \frac{35,831,808}{2} + 7 \times 248,832 - \frac{7 \times 1728}{2} + 12 = 3 \times$
 $17,915,904 + 17,915,904 + 1,741,824 - 7 \times 864 + 12$
 $= 4 \times 17,915,904 + 1,741,824 - 6048 + 12 =$
 $71,663,616 + 1,741,824 - 6048 + 12 = 73,405,452 -$
 $6048 = 73,399,404$. And so we shall find the said sum to be, if we add up together the said twelve numbers, or seventh powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

1
128
2,187
16,384
78,125
279,936
823,543
2,097,152
4,782,969
10,000,000
19,487,171
35,831,808
73,399,404

And

And for the sum of the eighth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 256; 6,561; 65,536; 390,625; 1,679,616; 5,764,801; 16,777,216; 43,046,721; 100,000,000; 214,358,881; and 429,981,696)

we shall have $\frac{n^9}{9} + \frac{n^8}{3} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}$, or $\frac{12^9}{9} + \frac{12^8}{3} + \frac{2 \times 12^7}{3} - \frac{7 \times 12^5}{15} + \frac{2 \times 12^3}{9} - \frac{12}{30} (= \frac{12 \times 12^8}{9} + \frac{12^8}{3} + \frac{2 \times 12^7}{3} - \frac{7 \times 12^5}{15} + \frac{2 \times 12^3}{9} - \frac{2}{5} = \frac{4 \times 12^8}{3} + \frac{12^8}{2} + \frac{2 \times 12^7}{3} - \frac{7 \times 12^5}{15} + \frac{2 \times 1728}{9} - \frac{2}{5} = \frac{4 \times 429,981,696}{3} + \frac{429,981,696}{2} + \frac{2 \times 15,831,808}{3} - \frac{7 \times 248,832}{15} + \frac{3456}{9} - \frac{2}{5} = \frac{1,719,926,784}{3} + 214,990,848 + \frac{71,661,616}{3} - \frac{1,741,824}{15} + 384 - \frac{2}{5} = 573,308,928 + 214,990,848 + 23,887,872 - \frac{580,608}{5} + 384 - \frac{2}{5} = 812,188,032 - \frac{580,610}{5} = 812,188,032 - 116,12) = 812,071,910. And so we shall find the said sum to be, if we add up together the said twelve numbers, or eighth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.$

$$\begin{array}{r}
 1 \\
 256 \\
 6,561 \\
 65,536 \\
 390,625 \\
 1,679,616 \\
 5,764,801 \\
 16,777,216 \\
 43,046,721 \\
 100,000,000 \\
 214,358,881 \\
 429,981,696 \\
 \hline
 812,071,910
 \end{array}$$

And for the sum of the ninth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, (which powers are 1; 512; 19,683; 262,144; 1,953,125; 10,077,696; 40,353,607; 134,217,728; 387,420,489; 1,000,000,000; 2,357,947,691; and 5,159,780,352;) we shall have

$$\begin{aligned}
 & \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} \\
 & - \frac{3n^2}{20}, \text{ or } \frac{12^{10}}{10} + \frac{12^9}{2} + \frac{3 \times 12^8}{4} - \frac{7 \times 12^6}{10} + \frac{12^4}{2} - \frac{3 \times 12^2}{20} \\
 & (= \frac{12 \times 12^9}{10} + \frac{12^9}{2} + \frac{3 \times 12^8}{4} - \frac{7 \times 12^6}{10} + \frac{12^4}{2} - \frac{3 \times 12^2}{20} = \\
 & \frac{12 \times 5,159,780,352}{10} + \frac{5,159,780,352}{2} + \frac{3 \times 429,981,696}{4} - \\
 & \frac{7 \times 2,985,984}{10} + \frac{20,736}{2} - \frac{3 \times 144}{20} = \frac{61,917,364,224}{10} + 2,579,890, \\
 & 176 + \frac{1,289,945,088}{4} - \frac{20,901,888}{10} + 10,368 - \frac{3 \times 72}{10} = \\
 & \frac{61,917,364,224}{10} + 2,579,890,176 + 322,486,272 - \frac{20,901,888}{10} \\
 & + 10,368 - \frac{216}{10} = \frac{61,917,364,224}{10} - \frac{20,902,104}{10} + 2,579,890,
 \end{aligned}$$

2 B

176

$176 + 322,486,272 + 10,368 = \frac{61,896,462,120}{10} + 2,902,$
 $386,816 = 6,189,646,212 + 2,902,386,816 = 9,092,$
 $033,028.$ And so we shall find the said sum to be, if we
 add up together the said twelve numbers, or ninth powers
 of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9,
 10, 11, and 12. This addition will be as follows:

$$\begin{array}{r}
 1 \\
 512 \\
 19,683 \\
 262,144 \\
 1,953,125 \\
 10,077,696 \\
 40,353,607 \\
 134,217,728 \\
 387,420,489 \\
 1,000,000,000 \\
 2,357,947,691 \\
 5,159,780,352 \\
 \hline
 9,092,033,028.
 \end{array}$$

And for the sum of the tenth
 powers of the first twelve natural numbers, 1, 2, 3, 4, 5, 6,
 7, 8, 9, 10, 11 and 12, (which powers are 1; 1,024;
 59,049; 1,048,576; 9,765,625; 60,466,176; 282,475,
 249; 1,073,741,824; 3,486,784,401; 10,000,000,000;
 25,937,424,601; and 61,917,364,224), we shall have $\frac{n^{12}}{11} +$
 $\frac{n^{10}}{2} + \frac{5n^8}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$, or $\frac{12^{11}}{11} + \frac{12^{10}}{2} + \frac{5 \times 12^8}{6}$
 $- 12^7 + 12^5 - \frac{12^3}{2} + \frac{5 \times 12}{66}$ ($= \frac{12 \times 12^{10}}{11} + \frac{12^{10}}{2} + \frac{5 \times 12^8}{6}$
 $- 12^7 + 12^5 - \frac{12^3}{2} + \frac{10}{11} = \frac{12 \times 61,917,364,224}{11} +$
 $\frac{61,917,364,224}{2}$

$$\frac{61,917,364,224}{2} + \frac{5 \times 5,150,780,352}{6} - 35,831,808 + 248,832$$

$$- \frac{1728}{2} + \frac{10}{11} = \frac{743,008,370,688}{11} + 30,958,682,112 + 5 \times$$

$$859,963,392 - 35,831,808 + 248,832 - 864 + \frac{10}{11} =$$

$$\frac{743,008,370,698}{11} + 30,958,682,112 + 4,299,816,960 -$$

$$35,831,808 + 248,832 - 864 = 67,546,215,518 +$$

$$30,958,682,112 + 4,299,816,960 - 35,831,808 + 248,$$

$$832 - 864 = 102,804,963,422 - 35,832,672 = 102,$$

$$769,130,750. \text{ And so we shall find the said sum to be, if}$$

we add up together the said twelve numbers, or tenth powers of the first twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows:

1
1,024
59,049
1,048,576
9,765,625
60,466,176
282,475,249
1,073,741,824
3,486,784,401
10,000,000,000
25,937,424,601
61,917,364,224
102,769,130,750.

141. If the foregoing expressions of the values of the sums of these several sets of powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n , be set down one under the other in a regular table, the said table will be as follows:

2 B 2

A Table

A Table of the values of the sums of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms, and of the squares, and the cubes, and the fourth powers, and other following powers of the said numbers, as far as the tenth powers, inclusively.

The sum of the first n terms of the said series of natural numbers is equal to

$$\frac{nn}{2} + \frac{n}{2}.$$

The sum of the squares of the said n terms is

$$= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}.$$

The sum of the cubes of the said n terms is

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}.$$

The sum of the fourth powers of the said n terms is

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30}.$$

The sum of the fifth powers of the said n terms is

$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{nn}{12}.$$

The sum of the sixth powers of the said n terms is

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42}.$$

The sum of the seventh powers of the said n terms is

$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12}.$$

The sum of the eighth powers of the said n terms is

$$= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30}.$$

The sum of the ninth powers of the said n terms is

$$= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}.$$

And the sum of the tenth powers of the said n terms is

$$= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} * - n^7 * + n^5 * - \frac{n^3}{2} * + \frac{5n}{66}.$$

The

The law of the generation, or derivation, of the terms of the several expressions set down in the foregoing table, one from the other.

142. By an attentive consideration of the foregoing table we may discover the law by which the terms of the several expressions of which it consists, may be derived one from the other; after which we shall be able to continue the said table to the sums of the eleventh and twelfth and other higher powers of the numbers 1, 2, 3, 4, 5, 6, &c, without entering into the long trains of reasoning and making the laborious substitutions of the sums already known in the expression of the value of the new sum, which have been used in obtaining the foregoing sums. This law will be found to be as follows.

Let the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, be supposed to be continued to any number n ; and let it be required to find the sum of the n th powers of the said n terms, or the value of the series $1^n + 2^n + 3^n + 4^n + 5^n + 6^n + 7^n + 8^n + 9^n + 10^n + 11^n + \&c$, continued to n^n . Let the capital letters A, B, C, D, &c, be put for the co-efficients of the last terms of the sums of the squares, and the fourth powers, and the sixth powers, and the eighth powers, and the other following even powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, already computed, with the same signs + or - prefixed to them, as are prefixed to the said last terms, of which they are the co-efficients. Thus, because the last term of the sum of the squares of the numbers 1, 2, 3, 4, 5, 6, 7, &c, is $+\frac{n}{6}$, or $+\frac{1}{6} \times n$, A will be $= +\frac{1}{6}$; and, because

the

the last term of the sum of the fourth powers of the said numbers is $-\frac{n}{30}$, or $-\frac{1}{30} \times n$, B will be $= -\frac{1}{30}$; and, because the last term of the sum of the sixth powers of the said numbers is $+\frac{n}{42}$, or $+\frac{1}{42} \times n$, C will be $= +\frac{1}{42}$; and, because the last term of the sum of the eighth powers of the said numbers is $-\frac{n}{30}$, or $-\frac{1}{30} \times n$, D will be $= -\frac{1}{30}$; and, because the last term of the sum of the tenth powers of the said numbers is $+\frac{5n}{66}$, or $+\frac{5}{66} \times n$, E will be $= +\frac{5}{66}$. These being the several values of the capital letters A, B, C, D, E, &c, the series $1^c + 2^c + 3^c + 4^c + 5^c + 6^c + 7^c + 8^c + 9^c + 10^c + 11^c + \&c, + n^c$ will be equal to the series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7} + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6 \times c-7 \times c-8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \times E n^{c-9} + \&c$; in which the indexes of the powers of n , after the third term $\frac{c}{2} \times A n^{c-1}$, decrease continually by 2, till we come at last to n or nn , and the co-efficients of $A n^{c-1}$, $B n^{c-3}$, $C n^{c-5}$, $D n^{c-7}$, $E n^{c-9}$, &c, are formed by the continual

continual multiplication of 1 into the fractions $\frac{c}{2}$, $\frac{c-1}{3} \times \frac{c-2}{4}$, $\frac{c-3}{5} \times \frac{c-4}{6}$, $\frac{c-5}{7} \times \frac{c-6}{8}$, $\frac{c-7}{9} \times \frac{c-8}{10}$, $\frac{c-9}{11} \times \frac{c-10}{12}$, &c, till we come to $\frac{c-\sqrt{c-1}}{c+1} \times \frac{c-c}{c+2}$, or $\frac{c-c+1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{0}{c+2}$, which will be = 0. And thus

we shall determine the powers of n in all the terms of the said series, and also the co-efficients of the said powers of n in all the terms of the said series, except the last term. And this last co-efficient may be derived from the co-efficients of the preceeding terms, by an easy addition or subtraction, being always the quantity which is necessary to be added to, or subtracted from, the result of all the preceeding co-efficients, in order to make such result become equal to 1.

Thus, in the first sum in the foregoing table, to wit, $\frac{n^2}{2} + \frac{n}{2}$, the co-efficient of the last term $\frac{n}{2}$ is $\frac{1}{2}$, which is the quantity which must be added to $\frac{1}{2}$, the co-efficient of the first term $\frac{n^2}{2}$, in order to make it equal to 1; and, in the second sum, $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, the co-efficient of the last term $\frac{n}{6}$ is $\frac{1}{6}$, which is the quantity that must be added to $\frac{1}{3} + \frac{1}{2}$, or the sum of the co-efficients of the first and second terms, $\frac{n^3}{3} + \frac{n^2}{2}$, in order to make it equal to 1; and in the third sum, $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$, the co-efficient of the last term is $\frac{1}{4}$, which is the quantity that must be added to $\frac{1}{4} + \frac{1}{2}$, or the sum of the co-efficients of the first and second terms, $\frac{n^4}{4} + \frac{n^3}{2}$, in order

order to make it equal to 1; and in the fourth sum, $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$, the co-efficient of the last term is $\frac{1}{30}$, which is the quantity which must be subtracted from $\frac{1}{5} + \frac{1}{2} + \frac{1}{3}$, or the sum of the co-efficients of the three first terms $\frac{n^3}{5} + \frac{n^4}{2} + \frac{n^3}{3}$, in order to make it equal to 1. And the same thing is true with respect to the co-efficients of the last terms of all the following sums*; and consequently the said co-efficients, and the signs + or —, to be prefixed to them, may always be determined, by means of the co-efficients of the preceding terms of the sums to which they belong. And thus all the terms of the series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times B n^{c-3} + \frac{c}{2} \times \frac{c-1}{3}$

* And hence it will appear that the co-efficient of the last term of the expression that is equal to the sum of the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n , must be — $\frac{3}{20}$, and not — $\frac{1}{12}$; agreeably to what was observed above in the note at the bottom of page 180. For, as the several terms of that expression preceding the last term are $\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2}$, of which the co-efficients are $\frac{1}{10} + \frac{1}{2} + \frac{3}{4} - \frac{7}{10} + \frac{1}{2}$, which are $= \frac{2}{20} + \frac{10}{20} + \frac{15}{20} - \frac{14}{20} + \frac{10}{20}$ or $\frac{22}{20} + \frac{1}{20}$, or $\frac{23}{20}$, or $\frac{20}{20} + \frac{3}{20}$, or $1 + \frac{3}{20}$ (from which it is necessary to subtract the fraction $\frac{3}{20}$ in order to make it become equal to 1), it follows, according to the rule here laid down by the author, that the co-efficient of the last term of the said expression must be — $\frac{3}{20}$, and consequently that the last term of the said expression must be — $\frac{3nn}{20}$.

Q. E. D.

X

$$\begin{aligned} & \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times C n^{c-5} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \\ & \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times D n^{c-7} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \\ & \frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times \frac{c-7}{9} \times \frac{c-8}{10} \times E n^{c-9} + \&c, \end{aligned}$$

may be determined, and consequently the value of the series $1^c + 2^c + 3^c + 4^c + 5^c + 6^c + 7^c + 8^c + 9^c + 10^c + 11^c + \&c + n^c$ (to which the said series is equal) may be assigned, without the help of the long reasonings and laborious substitutions that were employed for this purpose in the foregoing articles. The method of doing this will appear more clearly from the following example.

An example of the computation of the expression that is equal to the sum of certain powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to n, by means of the foregoing series,

143. As an example to the foregoing series, let it be required to find the value of the series $1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10} + 6^{10} + 7^{10} + 8^{10} + 9^{10} + 10^{10} + 11^{10} + \&c + n^{10}$, or the sum of the 10th powers of the several natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms; which sum, as set down in the foregoing table, is $= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$.

In this case the index c is $= 10$, and consequently $c + 1$ is $= 11$, and $c - 1$ is $= 9$, and $c - 2$, $c - 3$, $c - 4$, $c - 5$,
 $\frac{c}{2}$ C $\frac{c-1}{3}$ $\frac{c-2}{4}$ $\frac{c-3}{5}$ $\frac{c-4}{6}$

$c-6$, $c-7$, $c-8$, $c-9$, and $c-10$ are, respectively, equal to 8, 7, 6, 5, 4, 3, 2, 1, and 0. We shall therefore have

$$\frac{c}{2} (= \frac{10}{2}) = 5,$$

$$\text{and } \frac{c-1}{3} \times \frac{c-2}{4} (= \frac{9}{3} \times \frac{8}{4} = 3 \times 2) = 6,$$

$$\text{and } \frac{c-3}{5} \times \frac{c-4}{6} (= \frac{7}{5} \times \frac{6}{6}) = \frac{7}{5},$$

$$\text{and } \frac{c-5}{7} \times \frac{c-6}{8} (= \frac{5}{7} \times \frac{4}{8} = \frac{5}{7} \times \frac{1}{2}) = \frac{5}{14},$$

$$\text{and } \frac{c-7}{9} \times \frac{c-8}{10} (= \frac{3}{9} \times \frac{2}{10} = \frac{1}{3} \times \frac{1}{5}) = \frac{1}{15},$$

$$\text{and } \frac{c-9}{11} \times \frac{c-10}{12} (= \frac{1}{11} \times \frac{0}{12}) = 0.$$

Therefore the series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times$
 $A n^{c-1} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times B n^{c-3} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4}$
 $\times \frac{c-3}{5} \times \frac{c-4}{6} \times C n^{c-5} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6}$
 $\times \frac{c-5}{7} \times \frac{c-6}{8} \times D n^{c-7} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6}$
 $\times \frac{c-5}{7} \times \frac{c-6}{8} \times \frac{c-7}{9} \times \frac{c-8}{10} \times E n^{c-9}$ will be $= \frac{1}{11} \times n^{11}$
 $+ \frac{1}{2} \times n^{10} + 5 \times A n^9 + 5 \times 6 \times B n^7 + 5 \times 6 \times \frac{7}{5}$
 $\times C n^5 + 5 \times 6 \times \frac{7}{5} \times \frac{5}{14} \times D n^3 + 5 \times 6 \times \frac{7}{5} \times \frac{5}{14}$
 $\times \frac{1}{15} \times E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 A n^9 + 30 B n^7 + 42 C n^5 + 42$
 $\times \frac{5}{14} \times D n^3 + 42 \times \frac{5}{14} \times \frac{1}{15} \times E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 A n^9$
 $+ 30 B n^7 + 42 C n^5 + 15 D n^3 + E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 \times$
 $+ \frac{1}{6} \times n^9 + 30 \times -\frac{1}{30} \times n^7 + 42 \times + \frac{1}{42} \times n^5 + 15$
 $\times$

$\times - \frac{1}{30} \times n^3 - En = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2}$
 $+ En$; of which expression all the terms, except the last term En , are known quantities. And this last term En may be found in the following manner. The co-efficients of all the preceeding terms are $\frac{1}{11} + \frac{1}{2} + \frac{5}{6} - 1 + 1 - \frac{1}{2}$, which are equal to $\frac{1}{11} + \frac{5}{6} = \frac{6}{66} + \frac{55}{66} = \frac{61}{66}$; to which it is necessary to add $\frac{5}{66}$, in order to make the result equal to 1. Therefore E , or the co-efficient of the last term En , will be $= + \frac{5}{6}$; and consequently the compleat value of the foregoing series in this case of the 10th powers of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms, will be $\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{6}$; which is the value set down for the sum of the said 10th powers in the foregoing table.

A numerical example of the computation of the sum of the tenth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to 1000, by means of the foregoing expression.

144. If n is = 1000, we shall have

$$n^3 (= 1000^3)$$

$$= 1,000,000,000,$$

$$\text{and } n^4 (= 1000^4)$$

$$= 1,000,000,000,000,000,$$

$$\text{and } n^7 (= 1000^7)$$

$$= 1,000,000,000,000,000,000,000,$$

$$\text{and } n^9 (= 1000^9)$$

$$= 1,000,000,000,000,000,000,000,000,$$

$$\text{and } n^{10} (= 1000^{10})$$

$$= 1,000,000,000,000,000,000,000,000,000,$$

$$\text{and } n^{11} (= 1000^{11})$$

$$= 1,000,000,000,000,000,000,000,000,000,000,$$

and consequently

$$\frac{11}{11} (= \frac{1,000,000,000,000,000,000,000,000,000,000}{11}) =$$

$$90,909,090,909,090,909,090,909,090,909,090,$$

$$\text{and } \frac{n^{10}}{2} (= \frac{1,000,000,000,000,000,000,000,000,000}{2}) =$$

$$500,000,000,000,000,000,000,000,000,$$

$$\text{and } \frac{5n^9}{6} (= \frac{5}{6} \times 1,000,000,000,000,000,000,000,000,$$

$$= \frac{5,000,000,000,000,000,000,000,000}{6}) =$$

$$833,333,333,333,333,333,333,333,333,333, \&c.$$

$$\text{and } \frac{n^3}{2} (= \frac{1,000,000,000}{2} = 500,000,000;$$

and

$$\text{and } \frac{5^n}{6} (= \frac{5 \times 1000}{66} = \frac{5000}{66}) = 75.757,575, \&c;$$

$$\text{and } \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$$

$$= \left\{ \begin{array}{l} 90,909,090,909,090,909,090,909,090,909,090,909,090, \&c \\ + 500,000,000,000,000,000,000,000,000,000,000,000, \&c \\ + 833,333,333,333,333,333,333,333,333,333,333,333, \&c \\ - 1,000,000,000,000,000,000,000,000,000,000,000, \&c \\ + 1,000,000,000,000,000,000,000,000,000,000, \&c \\ - 500,000,000,000,000,000,000,000,000,000, \&c \\ + 75.757,575, \&c \end{array} \right.$$

$$= \left\{ \begin{array}{l} 91,409,924,241,424,243,424,241,924,242,499,999,998, \&c \\ - 1,000,000,000,000,000,500,000,000,000,000, \&c \end{array} \right.$$

$$= 91,409,924,241,424,243,424,241,924,242,499,999,998,$$

$$\text{or } 91,409,924,241,424,243,424,241,924,242,500.$$

Therefore the sum of all the tenth powers of the first thousand natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, ... 1000, is 91,409,924,241,424,243,424,241,924,242,500, or more than 91 quintillions, or 91 times the fifth power of a million.

145. I cannot but observe on this occasion, that the learned *Ismael Bullialdus*, or *Bouillaud*, has been rather unfortunate in his manner of treating this subject, in his Treatise on the Arithmetick of Infinites^{*}; since the whole of the folio volume which he has written upon it does nothing more than enable us to find the sums of the first six powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to any given number *n*; which is only a part of what we have here accomplished in the compass of a dozen pages.

* See an account of this book of Monsieur Bouillaud in Dr. Wallis's Algebra, chapter lxxx. pages 310, 311.

A computation of all the other expressions given above in the Table set down in art. 141, page 188, for the values of the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , by means of the foregoing general series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c +$

$$\begin{aligned} & \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6 \times c-7 \times c-8}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10} \times E n^{c-9} \\ & + \&c. \end{aligned}$$

146. If the foregoing series be applied in the same manner to the computation of the sums of the preceeding powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , as it has here been applied to the computation of the sum of their tenth powers, it will be found to produce the several expressions set down above in the table in art. 141, for the values of the sums of those powers; to wit, for the value of the sum of the first, or simple, powers of the said natural numbers, continued to the number n , or for the sum of the said natural numbers themselves, continued to the number n , the expression

$$\frac{nn}{2} + \frac{n}{2};$$

And for the sum of the squares of the said n terms the expression

$$\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6};$$

And

And for the sum of the cubes of the said n terms the expression

$$\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4};$$

And for the sum of the fourth powers of the said n terms the expression

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30};$$

And for the sum of the fifth powers of the said n terms the expression

$$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{nn}{12};$$

And for the sum of the sixth powers of the said n terms the expression

$$\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42};$$

And for the sum of the seventh powers of the said n terms the expression

$$\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12};$$

And for the sum of the eighth powers of the said n terms the expression

$$\frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30};$$

And for the sum of the ninth powers of the said n terms the expression

$$\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20};$$

This may be done in the manner following.

In applying this series to the first case, or the sum of the first, or simple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n , or to the sum of the said natural numbers themselves, it is evident that we must compute only the two first terms of the said series,

to

to wit, the terms $\frac{1}{c+1} \times n^{c+1}$ and $\frac{1}{2} \times n^c$; because the following terms involve in them the numbers A, B, C, D, E, &c, which are derived from the values of the sums of the squares, and the fourth powers, and the sixth powers, and the eighth powers, and the tenth powers, and the other following even powers of the said natural numbers, with which several sums we have as yet nothing to do.

Now, because c is in this case $= 1$, the two first terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c$ of the foregoing series will be $= \frac{1}{2+1} \times n^{1+1} + \frac{1}{2} \times n^1 (= \frac{1}{2} \times n^2 + \frac{1}{2} \times n) = \frac{n^2}{2} + \frac{n}{2}$. Therefore the sum of the first, or simple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n , or the sum of the said natural numbers themselves, will be $= \frac{n^2}{2} + \frac{n}{2}$. Q. E. I.

Secondly, When c is $= 2$, and the sum of the squares of the said natural numbers is to be investigated by means of the foregoing series, we must compute only the three first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1}$; because the following terms involve in them the numbers B, C, D, E, &c, which relate to the sums of the fourth, and the sixth, and the eighth, and the tenth, and the other following even powers of the said natural numbers, with which sums we have as yet nothing to do.

Now, when c is $= 2$, the three terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1}$ will be $(= \frac{1}{2+1} \times n^{2+1} + \frac{1}{2} \times n^2 + \frac{2}{2} \times A n^{2-1} = \frac{1}{3} \times n^3 + \frac{1}{2} \times n^2 + 1 \times A n^1)$

$\dfrac{1}{3} \times n^3 + \dfrac{1}{2} \times n^2 + An$; of which expression the two first terms are known quantities, and only the third, or last, term An remains to be investigated. Now this last term An is to be found in the following manner. The co-efficients of the two preceeding terms are $\dfrac{1}{3} + \dfrac{1}{2}$, which are equal to $\dfrac{2}{6} + \dfrac{3}{6} = \dfrac{5}{6}$; to which it is necessary to add $\dfrac{1}{6}$, in order to make the result equal to 1. Therefore A , or the co-efficient of the last term An , will be $= + \dfrac{1}{6}$; and consequently the compleat value of the three first terms of the foregoing series in this case of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, continued to the number n , will be $\dfrac{1}{3} \times n^3 + \dfrac{1}{2} \times n^2 + \dfrac{1}{6} \times n$, or $\dfrac{n^3}{3} + \dfrac{n^2}{2} + \dfrac{n}{6}$. Q. E. I.

3dly, When c is $= 3$, and the sum of the cubes of the said natural numbers is to be investigated by means of the foregoing series, we must (as in the last case) compute only the three first terms of the said series, to wit, the terms $\dfrac{1}{c+1} \times n^{c+1} + \dfrac{1}{2} \times n^c + \dfrac{c}{2} \times An^{c-1}$; because the following terms involve in them the numbers B, C; D, E, &c, which relate to the sums of the fourth, and the sixth, and the eighth, and the tenth, and the other following even powers of the said natural numbers, with which sums we have as yet nothing to do.

Now, when c is $= 3$, the three terms $\dfrac{1}{c+1} \times n^{c+1} + \dfrac{1}{2} \times n^c + \dfrac{c}{2} \times An^{c-1}$ will be $= \dfrac{1}{3+1} \times n^{3+1} + \dfrac{1}{2} \times n^3 + \dfrac{3}{2} \times An^{3-1} (= \dfrac{1}{4} \times n^4 + \dfrac{1}{2} \times n^3 + \dfrac{3}{2} \times An^2)$

2 D =

$$= \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{3}{2} \times n^2 + \frac{1}{6} \times n = \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{1}{4} \times n^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}. \text{ Therefore the sum}$$

of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$. Q. E. I.

4thly, When c is $= 4$, and the sum of the fourth powers of the said natural numbers is to be investigated by means of the foregoing series, we must only compute the four first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3}$; because the following terms involve in them the numbers C, D, E, &c, which relate to the sums of the sixth, and the eighth, and the tenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $= 4$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3}$ will be $= \frac{1}{4+1} \times n^{4+1} + \frac{1}{2} \times n^4 + \frac{4}{2} \times A n^{4-1} + \frac{4 \times \overline{4-1} \times \overline{4-2}}{2 \times 3 \times 4} \times B n^{4-3} (= \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times A n^3 + \frac{4 \times 3 \times 2}{2 \times 3 \times 4} \times B n = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times A n^3 + B n) = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 + B n$; of which expression the last term $B n$ is to be determined in the manner following.

ing. The co-efficients of the three first terms $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3$ are $\frac{1}{5} + \frac{1}{2} + \frac{1}{3} (= \frac{6}{30} + \frac{15}{30} + \frac{10}{30} = \frac{31}{30}) = 1 + \frac{1}{30}$; from which it is necessary to subtract $\frac{1}{30}$, in order to make the result equal to 1. Therefore B, or the co-efficient of the last term Bn , will be $= -\frac{1}{30}$, and consequently the compleat value of the four first terms of the said series in this case will be $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 - \frac{1}{30} \times n$, or $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Therefore the sum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Q. E. I.

5thly, When c is $= 5$, and the sum of the fifth powers of the said natural numbers is to be investigated by means of the foregoing series, we must, as in the last case, compute only the four first terms of the said series, to wit, the terms

$$\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3};$$

because the following terms involve in them the numbers C, D, E, &c, which relate to the sums of the sixth, and the eighth, and the tenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $= 5$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3}$ will be $= \frac{1}{5+1} \times n^{5+1} + \frac{1}{2} \times n^5 + \frac{5}{2} \times A n^{5-1}$

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†

$$\begin{aligned}
& + \frac{5 \times \overline{5-1} \times \overline{5-2}}{2 \times 3 \times 4} \times B n^{5-3} (= \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times \\
& \times A n^4 + \frac{5 \times 4 \times 3}{2 \times 3 \times 4} \times B n^3 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times \\
& A n^4 + \frac{5}{2} \times B n^3 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times + \frac{1}{6} \times \\
& n^4 + \frac{5}{2} \times - \frac{1}{30} \times n^3 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{12} \times n^4 \\
& - \frac{5}{12} \times n^3) = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{5nn}{12}. \text{ Therefore the sum} \\
& \text{of the fifth powers of the natural numbers 1, 2, 3, 4, 5,} \\
& 6, 7, 8, 9, 10, 11, 12, \&c, \text{ continued to } n, \text{ will be } = \\
& \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{5nn}{12}. \quad Q. E. I.
\end{aligned}$$

6thly, When c is = 6, and the sum of the sixth powers of the said natural numbers is to be investigated by means of the foregoing series, we must compute only the five first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1}$

$$\begin{aligned}
& + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3} \\
& + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5}; \text{ because the fol-} \\
& \text{lowing terms involve in them the numbers D, E, F, G,} \\
& \&c, \text{ which relate to the sums of the eighth, and the tenth,} \\
& \text{and the twelfth, and the fourteenth, and the other following} \\
& \text{even powers of the said natural numbers, with which sums} \\
& \text{we have hitherto nothing to do.}
\end{aligned}$$

$$\begin{aligned}
& \text{Now, when } c \text{ is } = 6, \text{ the five terms } \frac{1}{c+1} \times n^{c+1} \\
& + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3} \\
& + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} \text{ will be } = \frac{1}{6+1} \\
& \times n^{6+1}
\end{aligned}$$

$$\times n^{6+1} + \frac{1}{2} \times n^6 + \frac{6}{2} \times A n^{6-1} + \frac{6 \times \overline{6-1} \times \overline{6-2}}{2 \times 3 \times 4} \times$$

$$B n^{6-3} + \frac{6 \times \overline{6-1} \times \overline{6-2} \times \overline{6-3} \times \overline{6-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{6-5} (= \frac{1}{7}$$

$$\times n^7 + \frac{1}{2} \times n^6 + 3 \times A n^5 + \frac{6 \times 5 \times 4}{2 \times 3 \times 4} \times B n^3 +$$

$$\frac{6 \times 5 \times 4 \times 3 \times 2}{2 \times 3 \times 4 \times 5 \times 6} \times C n = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + 3 \times A n^5$$

$$+ 5 \times B n^3 + C n = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + 3 \times + \frac{1}{6}$$

$$\times n^5 + 5 \times -\frac{1}{30} \times n^3 + C n) = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 +$$

$$\frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + C n; \text{ of which expression the last}$$

term $C n$ is to be determined in the manner following. The

co-efficients of the four first terms of the expression $\frac{1}{7} \times n^7$

$$+ \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + C n \text{ are } \frac{1}{7} + \frac{1}{2} + \frac{1}{2}$$

$$- \frac{1}{6} (= \frac{6}{42} + \frac{21}{42} + \frac{21}{42} - \frac{7}{42} = \frac{6}{42} + \frac{42}{42} - \frac{7}{42} = \frac{6}{42} + 1$$

$$- \frac{7}{42}) = 1 - \frac{1}{42}; \text{ to which it is necessary to add } \frac{1}{42}, \text{ in or-}$$

der to make the result be equal to 1. Therefore C , or the

co-efficient of the last term $C n$, will be $+\frac{1}{42}$, and conse-

quently the whole expression $\frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5$

$$- \frac{1}{6} \times n^3 + C n \text{ will be } = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5$$

$$- \frac{1}{6} \times n^3 + \frac{1}{42} \times n, \text{ or } \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

Therefore the sum of the sixth powers of the natural num-

bers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued

to the number n , will be $= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$

Q. E. I.
7thly,

7thly, When c is $= 7$, and the sum of the seventh powers of the said natural numbers is to be investigated by means of the foregoing series, we must, as in the last case, compute only the five first terms of the said series, to wit, the terms

$$\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \\ \times B n^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5}; \text{ because}$$

the following terms involve the numbers D, E, F, G, &c, which relate to the sums of the eighth, and the tenth, and the twelfth, and the fourteenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

$$\begin{aligned} \text{Now, when } c \text{ is } = 7, \text{ the said five terms } & \frac{1}{c+1} \times n^{c+1} \\ & + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3} \\ & + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} \text{ will be } = \frac{1}{7+1} \times n^{7+1} \\ & + \frac{1}{2} \times n^7 + \frac{7}{2} \times A n^{7-1} + \frac{7 \times \overline{7-1} \times \overline{7-2}}{2 \times 3 \times 4} \times B n^{7-3} \\ & + \frac{7 \times \overline{7-1} \times \overline{7-2} \times \overline{7-3} \times \overline{7-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{7-5} \left(= \frac{1}{8} \times n^8 + \frac{1}{2} \right. \\ & \times n^7 + \frac{7}{2} \times A n^6 + \frac{7 \times 6 \times 5}{2 \times 3 \times 4} \times B n^4 + \frac{7 \times 6 \times 5 \times 4 \times 3}{2 \times 3 \times 4 \times 5 \times 6} \times C n^2 \\ & = \frac{1}{8} \times n^8 + \frac{1}{2} \times n^7 + \frac{7}{2} \times A n^6 + \frac{7 \times 5}{4} \times B n^4 + \frac{7}{2} \\ & \times C n^2 = \frac{1}{8} \times n^8 + \frac{1}{2} \times n^7 + \frac{7}{2} \times A + \frac{1}{6} \times n^6 + \frac{7 \times 5}{4} \\ & \times -\frac{1}{30} \times n^4 + \frac{7}{2} \times B + \frac{1}{42} \times n^2 = \frac{1}{8} \times n^8 + \frac{1}{2} \times n^7 \\ & + \frac{7}{12} \times n^6 - \frac{7}{24} \times n^4 + \frac{1}{12} \times n^2) = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \\ & \frac{7n^4}{24} + \frac{nn}{12}. \end{aligned}$$

Therefore the sum of the seventh powers of the
natural

natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , will be $= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12}$. Q. E. I.

8thly, When c is $= 8$, and the sum of the eighth powers of the said natural numbers is to be investigated by means of the foregoing series, we must compute only the six first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1}$

$$+ \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} \\ + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} \\ + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7}; \text{ because}$$

the following terms involve in them the numbers E, F, G, &c, which relate to the sums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $= 8$, the six terms $\frac{r}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7}$ will be $=$

$$\frac{1}{8+1} \times n^{8+1} + \frac{1}{2} \times n^8 + \frac{8}{2} \times A n^{8-1} + \frac{8 \times 8-1 \times 8-2}{2 \times 3 \times 4} \times B n^{8-3} + \frac{8 \times 8-1 \times 8-2 \times 8-3 \times 8-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{8-5} \\ + \frac{8 \times 8-1 \times 8-2 \times 8-3 \times 8-4 \times 8-5 \times 8-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{8-7}$$

4

(=

$$\begin{aligned}
 & \left(= \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + 4 \times A n^7 + \frac{8 \times 7 \times 6}{2 \times 3 \times 4} \times B n^6 + \right. \\
 & \left. \frac{8 \times 7 \times 6 \times 5 \times 4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^5 + \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^4 = \frac{1}{9} \times n^9 \right. \\
 & \left. + \frac{1}{2} \times n^8 + 4 \times A n^7 + 7 \times 2 \times B n^6 + \frac{7 \times 4}{3} \times C n^5 \right. \\
 & \left. + D n = \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + 4 \times + \frac{1}{6} \times n^7 + 7 \times 2 \times \right. \\
 & \left. - \frac{1}{30} \times n^6 + \frac{7 \times 4}{3} \times + \frac{1}{42} \times n^5 + D n \right) = \frac{1}{9} \times n^9 + \frac{1}{2} \times \\
 & \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^6 + \frac{2}{9} \times n^5 + D n; \text{ of which} \\
 & \text{expression the last term } D n \text{ is to be determined in the man-} \\
 & \text{ner following. The co-efficients of the five first terms of the} \\
 & \text{expression } \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^6 + \frac{2}{9} \\
 & \times n^5 + D n \text{ are } \frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}, \text{ which are } (= \\
 & \frac{10}{90} + \frac{45}{90} + \frac{60}{90} - \frac{42}{90} + \frac{20}{90} = \frac{135}{90} - \frac{42}{90} = \frac{93}{90} = \frac{90}{90} + \frac{3}{90} = \\
 & 1 + \frac{3}{90}) = 1 + \frac{1}{30}; \text{ from which it is necessary to subtract} \\
 & \frac{1}{30}, \text{ in order to make the result be equal to 1. Therefore} \\
 & D, \text{ or the co-efficient of the last term } D n, \text{ will be } = -\frac{1}{30}, \\
 & \text{and consequently the whole expression } \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \\
 & \frac{2}{3} \times n^7 - \frac{7}{15} \times n^6 + \frac{2}{9} \times n^5 + D n \text{ will be } = \frac{1}{9} \times n^9 \\
 & + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^6 + \frac{2}{9} \times n^5 - \frac{1}{30} \times n = \\
 & \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^6}{15} + \frac{2n^5}{9} - \frac{n}{30}. \text{ Therefore the} \\
 & \text{sum of the eighth powers of the natural numbers 1, 2, 3, 4,} \\
 & 5, 6, 7, 8, 9, 10, 11, 12, \&c, \text{ continued to the number } n, \\
 & \text{will be } = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^6}{15} + \frac{2n^5}{9} - \frac{n}{30}. \quad \text{Q. E. D.}
 \end{aligned}$$

And,

And, 9thly, When c is = 9, and the sum of the ninth powers of the said natural numbers is to be investigated by means of the foregoing series, we must, as in the last case, compute only the six first terms of the said series, to wit,

$$\begin{aligned} & \text{the terms } \frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} \\ & + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5} \times C n^{c-5} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6} \times D n^{c-7}; \text{ because} \end{aligned}$$

the following terms involve the numbers E, F, G, &c. which relate to the sums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

$$\begin{aligned} & \text{Now, when } c \text{ is } = 9, \text{ the said six terms } \frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \\ & \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times B n^{c-3} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5} \\ & + \frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7} \text{ will be } = \\ & \frac{1}{9+1} \times n^{9+1} + \frac{1}{2} \times n^9 + \frac{9}{2} \times A n^{9-1} + \frac{9 \times 9-1 \times 9-2}{2 \times 3 \times 4} \\ & \times B n^{9-3} + \frac{9 \times 9-1 \times 9-2 \times 9-3 \times 9-4}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{9-5} \\ & + \frac{9 \times 9-1 \times 9-2 \times 9-3 \times 9-4 \times 9-5 \times 9-6}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{9-7} \\ & (= \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{9}{2} \times A n^8 + \frac{9 \times 8 \times 7}{2 \times 3 \times 4} \times B n^6 + \\ & \frac{9 \times 8 \times 7 \times 6 \times 5}{2 \times 3 \times 4 \times 5 \times 6} \times C n^4 + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^2 = \frac{1}{10} \times n^{10} \\ & \qquad \qquad \qquad 2 \text{ E} \qquad \qquad \qquad + \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \times n^9 + \frac{19}{2} \times A n^8 + 7 \times 3 \times B n^7 + 7 \times 3 \times C n^6 \\
& + \frac{9}{2} \times D n^5 = \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{9}{2} \times n^8 + \frac{1}{6} \times n^7 \\
& + 7 \times 3 \times n^6 - \frac{1}{30} \times n^5 + 7 \times 3 \times n^4 + \frac{1}{42} \times n^3 + \frac{9}{2} \times n^2 \\
& - \frac{1}{30} \times n^1 (= \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{3}{4} \times n^8 - \frac{7}{10} \times n^7 \\
& + \frac{1}{2} \times n^6 - \frac{3}{20} \times n^5) = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^7}{10} + \\
& \frac{n^6}{2} - \frac{3n^5}{20}. \text{ Therefore the sum of the ninth powers of the} \\
& \text{natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \&c,} \\
& \text{continued to the number } n, \text{ will be } = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} \\
& - \frac{7n^7}{10} + \frac{n^6}{2} - \frac{3n^5}{20}. \quad \text{Q. E. I.}
\end{aligned}$$

These several expressions of the values of the sums of the first nine powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n , are the same with those set down above in the table in art. 141, page 188. And it is evident that this way of obtaining them, by means of the foregoing general series, is much less laborious than the former method of obtaining them, set forth above in art. 136, 137, 138, and 139.

*End of the applications of the foregoing general series $\frac{1}{c+1}$
 $\times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c \times c-1 \times c-2}{2 \times 3 \times 4} \times n^{c-3}$
 + &c, to the investigation of the expressions set down above in
 art. 141, page 188.*

of

Of certain serieses consisting of numbers analogous to the figurate numbers.

147. Before we conclude this chapter, it will not be amiss to shew how certain other serieses, which bear a great resemblance to the serieses formed by the figurate numbers, may be reduced to, or compared with, the corresponding serieses of those numbers, and how their values, or the sums of their terms, and likewise the values of their last terms, may be thereby determined. The serieses I here speak of, and which I call *analogous to the figurate numbers*, are such as have the differences of their terms, or the differences of those differences, or the differences of those second differences, or the differences of the differences of some remoter order, equal to each other, and which therefore are generated by the continual addition of a set of equal quantities. Let $d, d, d, d, d, d, \&c.$ be a set of equal quantities, by the continual addition of which to another quantity c we obtain the quantities $c, c + d, c + 2d, c + 3d, c + 4d, c + 5d, \&c.$ And let the terms of this second series $c, c + d, c + 2d, c + 3d, c + 4d, \&c.$ be continually added to each other, and to a third quantity b , whereby we shall obtain a third series of terms, which will be $b, b + c, b + 2c + d, b + 3c + 3d, b + 4c + 6d, b + 5c + 10d, \&c.$; and let the terms of this third series be continually added to each other, and to a fourth quantity a , whereby we shall obtain a fourth series of terms, which will be $a, a + b, a + 2b + c, a + 3b + 3c + d, a + 4b + 6c + 4d, a + 5b + 10c + 10d, \&c.$ And let the first series $d, d, d, d, d, d, \&c.$ be called D; the second series $c, c + d, c + 2d, c + 3d, c + 4d, c + 5d, \&c.$ be called C; the third series $b, b + c, b + 2c + d, b + 3c + 3d, b + 4c + 6d, b + 5c + 10d, \&c.$ be called B; and the fourth, or last, series $a, a + b, a + 2b + c, a + 3b + 3c + d, a + 4b + 6c + 4d, a + 5b + 10c + 10d, \&c.$ be called A. This last series A (the first differences of the

the terms of which constitute the terms of the third series B, and the second differences of the terms of which constitute the terms of the second series C, and the third differences of the terms of which constitute the terms of the first series D, which consists of the equal quantities d, d, d, d, d, d , &c), may, I think, with propriety be called a series *analogous to the figurate numbers*. The generation of the terms of this series will, perhaps, appear more clearly from the following table.

D	C	B	A
d	c	b	a
d	$c + d$	$b + c$	$a + b$
d	$c + 2d$	$b + 2c + d$	$a + 2b + c$
d	$c + 3d$	$b + 3c + 3d$	$a + 3b + 3c + d$
d	$c + 4d$	$b + 4c + 6d$	$a + 4b + 6c + 4d$
d	$c + 5d$	$b + 5c + 10d$	$a + 5b + 10c + 10d$

148. Now in the last series A, it is obvious that the co-efficients of the letters a , which are the first members of the several terms $a, a + b, a + 2b + c, a + 3b + 3c + d, a + 4b + 6c + 4d$, and $a + 5b + 10c + 10d$, are a set of units, or the first order of the figurate numbers; and that the co-efficients of the several letters b in the second members of the said terms are the lateral, or natural, numbers 1, 2, 3, 4, 5, &c, or the second order of the figurate numbers; and that the co-efficients of the several letters c in the third members of the said terms are the trigonal, or triangular, numbers 1, 3, 6, 10, &c, or the third order of the figurate numbers; and that the co-efficients of the several letters d in the fourth members of the said terms are the pyramidal numbers 1, 4, 10, &c, or the fourth order of the figurate numbers. And therefore, as we have above shewn how the sums of the figurate numbers of the several successive orders, and likewise the values of the last terms in them, may be determined, when the number of terms contained in them is known; it will be easy to find both the sum of all the terms of the series A, by multiplying the sums of the successive columns of figurate numbers, into the letters

a,

a , b , c , and d , respectively, and adding the products so obtained into one sum, and likewise to find the value of the last term of the said series, by multiplying the last terms of the several columns of co-efficients, or figurate numbers, into the letters a , b , c , and d , respectively, and adding the said products into one sum. For, if the number of terms in the series A be denoted by the letter n , it follows from coroll. 2, art. 81, pages 109, 110, 111, that the sum of the co-efficients of the letter a will be n ; and the sum of the co-efficients of the letter b will be $n \times \frac{n-1}{2}$; and the sum of the co-efficients of c will be $\frac{n \times n-1 \times n-2}{2 \times 3}$; and the sum of the co-efficients of d will be $\frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4}$; and consequently the sum of all the n terms of the said series A will be $= n \times a + \frac{n \times n-1}{2} \times b + n \times \frac{n \times n-1 \times n-2}{2 \times 3} \times c + \frac{n \times n-1 \times n-2 \times n-3}{2 \times 3 \times 4} \times d$. And it follows from coroll. 4, art. 83, pages 112, 113, that the co-efficients of the letters a , b , c , and d , in the last, or n th, term of the series A will be 1, $n-1$, $\frac{n-1 \times n-2}{2}$, and $\frac{n-1 \times n-2 \times n-3}{2 \times 3}$, respectively; and consequently that the said last, or n th, term will be $= a + n-1 \times b + \frac{n-1 \times n-2}{2} \times c + \frac{n-1 \times n-2 \times n-3}{2 \times 3} \times d$.

Q. E. I. *

* See upon this subject Mr. Thomas Simpson's *Essays on several curious and useful subjects in speculative and mixed mathematics*, published in the year 1740, pages 98, 99, 100, 101, 102, 103, 104, and 105; and likewise his *Algebra*, 6th Edition, published in the year 1790, Sections XIV and XV, pages 201, 202, &c. — — — 228.

End of the Translation of the foregoing Extract from Mr. James Bernoulli's Treatise De Arte Conjectandi.

A

NEW AND GENERAL METHOD

OF FINDING THE

SUM OF ANY SERIES OF POWERS

OF A SET OF

QUANTITIES THAT ARE IN ARITHMETICAL PROGRESSION;

BEING THE TENTH OF THE LATE LEARNED

MR. THOMAS SIMPSON'S MATHEMATICAL ESSAYS,

PUBLISHED IN THE YEAR 1740.

Mr. SIMPSON'S^T Tenth Mathematical Essay*.

PROPOSITION.

To find the sum of any Series of Powers whose roots are in arithmetical progression, as $\overline{m+d}^n + \overline{m+2d}^n + \overline{m+3d}^n \dots$
 $\dots x^n$, m , d , and n , being any numbers whatsoever.

LET $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3} + Fx^{n-4}$, &c. — K , if possible, be always equal to $\overline{m+d}^n + \overline{m+2d}^n \dots x^n$, and A , B , C , &c, determinate quantities. Then, if any other number in the progression $m+d$, $m+2d$, $m+3d \dots x+d$, $x+2d$, $x+3d$, &c, as $x+d$, be substituted instead of x , the equality will still continue; and we shall have

$A \times \overline{x+d}^{n+1} + B \times \overline{x+d}^n + C \times \overline{x+d}^{n-1} + D \times \overline{x+d}^{n-2}$ &c. — K equal $\overline{m+d}^n + \overline{m+2d}^n \dots \overline{x+d}^n$; from which if we take the former equation, there will remain $A \times \overline{x+d}^{n+1} - x^{n+1} + B \times \overline{x+d}^n - x^n + C \times \overline{x+d}^{n-1} - x^{n-1}$, &c. = $\overline{x+d}^n$, shewing how much each side is increased by augmenting the number of terms in the given series by unity; where, by transposing $\overline{x+d}^n$, and throwing the several powers of $x+d$ into serieses, we shall have

* This Essay of Mr. Simpson's is the part of his Essays alluded to in the Note at the bottom of page 213. As it is so nearly connected with the subject of the latter part of the foregoing Extract from Mr. James Bernoulli's Treatise *De Arte Conjectandi*, relating to the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, and is not very long, I thought it would be agreeable to the Reader to see it here immediately after the said Extract, and therefore I have caused it to be re-printed. F. M.

Thomas Simpson English Mathematician was born —
 1710. & died A.D. 1761.

From which, by equating the homologous terms, A will come out $= \frac{1}{n+1 \times d}$, $B = \frac{1}{2}$, $C = \frac{nd}{3 \cdot 4}$, $D = 0$, $E = -\frac{n \times n-1 \times n-2 \times d^3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, $F = 0$, $G = \frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times d^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 6}$

$H = 0$, &c. wherefore the values of A, B, C, &c, being so assigned, the whole expression, or its equal $-\overline{x+d}^n + A \times \overline{x+d}^{n+1} - x^{n+1} + B \times \overline{x+d}^n - x^n$, &c, must

be equal 0, and consequently $A \times \overline{x+d}^{n+1} - x^{n+1} + B \times \overline{x+d}^n - x^n$, &c, $= \overline{x+d}$; that is, let x and n be

what they will, the foresaid increments of $A x^{n-1} + B x^n + C x^{n+1}$, &c, $-\overline{K}$ and $\overline{m+d}^n + \overline{m+2d}^n$, &c. will, under the above assigned values of A, B, &c, be equal to one another: Therefore, if K be taken equal $A m^{n+1} +$

$B m^n + C m^{n-1}$, &c, so that when x equal m , or the proposed series is equal to nothing, $A x^{n+1} + B x^n$, &c, $-\overline{K}$ may be also $= 0$, it is manifest, that these two expressions, as they are increased alike, will, in all other circum-

stances, be equal; that is, let x be what it will, $A x^{n+1} + B x^n + C x^{n-1} + D x^{n-2}$, &c, $-\overline{A m^{n+1} - B m^n - C m^{n-1} - D m^{n-2}}$, &c, under the said values of A, B, C, &c, will be always equal to $\overline{m+d}^n + \overline{m+2d}^n + \overline{m+3d}^n \dots x^n$; which values being therefore substituted,

there will be $\frac{x^{n+1}}{n+1 \times d} + \frac{x^n}{2} + \frac{dx^{n-1}}{3 \cdot 4} - \frac{n \times n-1 \times n-2 \times d^3 x^{n-3}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times d^5 x^{n-5}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 6}$

$$\begin{aligned}
& - \frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times n-5 \times n-6}{2.3.4.5.6.7.8.5.6} d^7 x^{n-7} \\
& + \frac{n \times n-1 \times n-2 \times n-3 \times n-4 \times n-5 \times n-6 \times n-7 \times n-8}{2.3.4.5.6.7.8.9.11.12} d^8 \\
& x^{n-9}, \text{ \&c, } - \frac{m^{n+1}}{n+1 \times d} - \frac{m^n}{2} - \frac{ndm^{n-1}}{3.4} + \frac{n \times n-1 \times n-2}{2.3.4.5.6} \times d^3 \\
& m^{n-3}, \text{ \&c, } = \overline{m+d}^n + \overline{m+2d}^n + \overline{m+3d}^n \dots x^n.
\end{aligned}$$

Q. E. I,

C O R O L. I.

Hence, if n be a whole positive number, and m be taken equal 0; then all the terms in the second series $-\frac{m^{n+1}}{n+1 \times d} - \frac{m^n}{2} + \frac{ndm^{n-1}}{3.4}$, &c, vanishing when n is even, and all but that where the exponent of m is nothing, when odd, we shall, in this case, have $d^n + 2d^n + 3d^n + 4d^n \dots x^n$ barely equal to $\frac{x^{n+1}}{n+1 \times d} + \frac{x^n}{2} + \frac{ndx^{n-1}}{3.4} - \frac{n \times n-1 \times n-2 d^3 x^{n-3}}{2.3.4.5.6}$, &c, the first series continued 'till it terminates, provided that the last term, when n is an odd number, be rejected.

C O R O L. II.

Wherefore, by taking d equal to 1, and n equal to 2, 3, 4, 5, &c, successively, we have

$$1 + 2$$

$$\begin{aligned}
 1 + 2 + 3 + 4 + 5 \dots + x &= \frac{x^2}{2} + \frac{x}{2} \\
 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \dots + x^2 &= \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} \\
 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \dots + x^3 &= \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4} \\
 1^4 + 2^4 + 3^4 + 4^4 + 5^4 \dots + x^4 &= \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} \\
 1^5 + 2^5 + 3^5 + 4^5 + 5^5 \dots + x^5 &= \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^3}{12} \\
 1^6 + 2^6 + 3^6 + 4^6 + 5^6 \dots + x^6 &= \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42} \\
 \&c, & \&c.
 \end{aligned}$$

C O R O L. III.

Moreover, if d be taken equal to 1, and m equal to 1, our general equation will become $2^n + 3^n + 4^n \dots + x^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3 \cdot 4}, \&c, - \frac{1}{n+1} - \frac{1}{2} - \frac{n}{3 \cdot 4} + \frac{n \times n-1 \times n-2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \&c,$ each side of which being increased by unity, and the whole multiplied by d^n , gives $d^n + \overline{2d}^n + \overline{3d}^n + \overline{4d}^n \dots + \overline{dx}^n = d^n$ into $\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3 \cdot 4} - \frac{n \times n-1 \times n-2 \times x^{n-3}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \&c, - \frac{1}{n+1} + \frac{1}{2} - \frac{n}{3 \cdot 4} + \frac{n \times n-1 \times n-2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \&c.$

EXAMPLE

EXAMPLE I.

Let it be required to find the sum of a series, consisting of 100 cube numbers, whose roots are, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, &c.

Here d , the common difference of the roots, being equal $\frac{1}{2}$, $n = 3$, and $x = 0$, let these values be substituted in the equation in Cor. II. and it will become $(\frac{1}{2})^3$ in, $\frac{(100)^4}{4} + \frac{(100)^3}{2} + \frac{(100)^2}{4} = 3187812.5$, the number that was to be found.

EXAMPLE II.

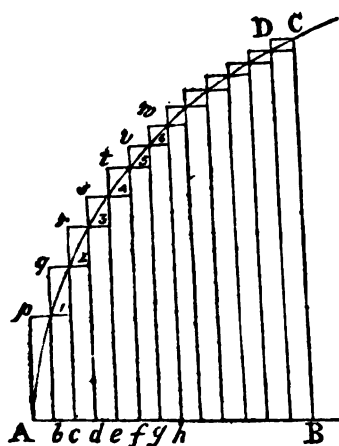
Let $n = \frac{1}{2}$, $d = \frac{1}{4}$. Then the equation in the last Corollary will become $\sqrt[1]{\frac{1}{4}} + \sqrt[2]{\frac{2}{4}} + \sqrt[3]{\frac{3}{4}} \dots + \sqrt[x]{\frac{x}{4}} = \sqrt[1]{\frac{1}{4}}$
 $\times \frac{2 \times x^{\frac{1}{2}}}{3} + \frac{x^{\frac{1}{2}}}{2} + \frac{1}{24 \times x^{\frac{1}{2}}}$, &c, $= \frac{339}{1920}$ very nearly; so that, taking x equal 4, it will be $\sqrt[1]{\frac{1}{4}} + \sqrt[2]{\frac{2}{4}} + \sqrt[3]{\frac{3}{4}} + 1 = 3.0731$; which differs from the true value by less than $\frac{1}{100000}$; and if more terms had been used, the answer would still have been more exact; but never can come *accurately* true, when n is negative or a fraction, because then both serieses run on *ad infinitum*.

SCHOLIUM.

SCHOLIUM.

The theorems, above found, are not only useful in finding the sum of a Series of Powers, but may be of service also in the Quadrature of Curves, &c, especially as the conclusions will be accurately true, and the reasoning thereupon scientific.

This I shall endeavour to shew by the following instance; wherein AC, being supposed a curve, whose equation is $y = z^n$ (AB being equal z , and CB equal y). the area ABC is required.



Let AB be divided into any number, x , of equal parts, as Ab, bc, cd , &c, and from the points of division let perpendiculars be raised, cutting the curve in the points, 1, 2, 3, &c, and having made $p1, q2, r3, s4$, &c, parallel to AB, let the base Ab, bc, cd , &c, of each of the rectangles pb, qc, rd , &c, be represented by d : Then $b1, c2, d3$, &c, the heights of those rectangles,

being ordinates to the curve, will be $d^n, 2d^n, 3d^n$, &c, respectively, each of which $\therefore$ being multiplied by d , the common base, and the sum of all the products taken, will give d into $d^n + 2d^n + 3d^n \dots xd^n$, ($= Ap1q2r$, &c, CBA) for the area of the whole circumscribing polygon; and this series, according to the above said Theorem

(Cor. III.) is equal to d^{n+1} in, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$, &c, $= \frac{dx^{n+1}}{n+1}$

+

$+ \frac{d \times d^{\frac{n}{2}}}{2}$, &c, or, because $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dz^{\frac{n}{2}}}{2}$, &c. Now, if from this the difference of the inscribed and circumscribed polygons, or the rectangle $BD = dz^{\frac{n}{2}}$ be taken; there will remain $\frac{z^{n+1}}{n+1} - \frac{dz^{\frac{n}{2}}}{2}$, for the area of the inscribed polygon. Hence, it is manifest, that, let d be what it will, the inscribed polygon can never be so great, nor the circumscribed so small, as $\frac{z^{n+1}}{n+1} (= \frac{AB \times BC}{n+1})$: And therefore this expression must be *accurately* equal to the required curvilinear area ACB .

A CON-

A N
I N V E S T I G A T I O N

A N D
D E M O N S T R A T I O N

O F
S I R I S A A C N E W T O N ' S B I N O M I A L T H E O R E M ,

I N T H E C A S E O F
I N T E G R A L A N D A F F I R M A T I V E P O W E R S ,

I N W H I C H

The Law of the generation of the numeral co-efficients of the Series which is equal to the quantity $\overline{a+b^m}$, is discovered by a conjecture grounded on the observation of some particular instances; but, when so discovered, is shewn to be true universally in all other Integral and Affirmative Powers whatsoever, by a strict and accurate Demonstration,

A General Statement, or Expression, of the Binomial Theorem.

Art. 1. **S**IR ISAAC NEWTON's Binomial Theorem is a Proposition affirming that, if m be any number whatsoever, either integral or fractional, affirmative or negative, the quantity $\sqrt[m]{a+b}$, or the m th power of the binomial quantity $a+b$, will be equal to the series $a^m + \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m-3}{4} a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c$, or (if we put A for 1, or the co-efficient of the first term a^m , and B for $\frac{m}{1}$, or the co-efficient of the second term $\frac{m}{1} \times a^{m-1} b$, and C for $\frac{m}{1} \times \frac{m-1}{2}$, or the co-efficient of the third term $\frac{m}{1} \times \frac{m-1}{2} \times a^{m-2} b^2$, and D, E, F, G, H, I, K, &c, for the co-efficients of the fourth, fifth, sixth, seventh, eighth, ninth, tenth, and other following terms, respectively), to the series $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \frac{m-5}{6} F a^{m-6} b^6 +$

2 G 2

+

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+ $\frac{m-6}{7} G a^{m-7} b^7$ + $\frac{m-7}{8} H a^{m-8} b^8$ + $\frac{m-8}{9} I a^{m-9} b^9$ +
 &c, in which series the powers of a , to wit, $a^m, a^{m-1},$
 $a^{m-2}, a^{m-3}, a^{m-4}, a^{m-5}, a^{m-6}, a^{m-7}, a^{m-8}, a^{m-9},$ &c,
 are produced from each other by a continual division by a ,
 and the powers of b , to wit, $b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9,$
 &c, are produced from each other by a continual multipli-
 cation by b , and the numeral co-efficients B, C, D, E,
 F, G, H, I, K, &c, of the second, third, fourth, fifth,
 sixth, seventh, eighth, ninth, tenth, and other following
 terms are derived, or generated, from 1, or A, the numeral
 co efficient of the first term a^m , by a continual multiplication
 of it into the fractions $\frac{m}{1}$, or $\frac{m-0}{1}, \frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \frac{m-4}{5},$
 $\frac{m-5}{6}, \frac{m-6}{7}, \frac{m-7}{8}, \frac{m-8}{9},$ &c, continued *ad infinitum*, or to
 the end of the series when the number of its terms is
 finite.

Of the Invention of the said Theorem.

2. Sir Isaac Newton was the first person that expressed
 this important Theorem in the foregoing short and convenient
 Algebraick notation, and likewise the first person that
 discovered that it would be true, not only when the index
 m of the power to which the binomial quantity is to be raised
 is a whole number, as 2, 3, 4, 5, 6, &c, but also when
 it is a fraction of any kind, as $\frac{1}{2}, \frac{1}{3}, \frac{1}{17},$ or $\frac{2}{3}, \frac{3}{5}, \frac{4}{17},$ or
 $\frac{3}{2},$

$\frac{3}{2}$, $\frac{5}{3}$, $\frac{17}{4}$, &c, or even a negative quantity, as -2 , -3 ,
 -5 , -17 , or $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{17}$, or $-\frac{2}{3}$, $-\frac{3}{5}$, $-\frac{4}{17}$,
 or $-\frac{3}{2}$, $-\frac{5}{3}$, $-\frac{17}{4}$, &c. But he was not the first person
 that discovered it to be true in the first, or simplest, case,
 or when the index m is equal to an integral and affirmative
 number. For in that case it was known to Mr. Henry Briggs,
 the celebrated improver and computer of Logarithms, above
 40 years before it was discovered by Sir Isaac Newton; and
 it was published by Mr. Briggs, in his learned Treatise on
 Logarithms, intitled, *Aritbmetica Logarithmica*, in the year
 1624; as has been clearly shewn by the learned Dr. Hutton,
 of Woolwich Academy, in his very curious, historical, In-
 troduction to the new edition of Sherwin's Mathematical
 Tables, published in the year 1784.

3. But, though Mr. Briggs had published this famous
 Theorem, in this first case of it, in his *Aritbmetica Logarith-
 mica*, in the year 1624, yet it seems to have been but little
 known to Mathematicians till about 60 years after. For
 even the famous Dr. John Wallis, of Oxford, (who was a
 very extensive reader of Mathematical Works, as well as a
 great improver of the Science,) appears to have been igno-
 rant of it till a little before the year 1685, in which he pub-
 lished his learned, historical, Treatise of Algebra, at which
 time he was about 69 years old. For he there tells us, in
 page 319, that he had formerly sought to discover the law by
 which the numeral co-efficients of the terms of the series
 which is equal to $\sqrt[n]{a + b}$ are generated from each other, but
 had not been able to find it; and that he had lately been
 made acquainted with it by the perusal of a very learned
 letter of Mr. Isaac Newton, the Professor of Mathematicks
 in the University of Cambridge, to Mr. Oldenburgh, Se-
 cretary to the Royal Society, written in the year 1676.
 His words are these, after speaking of some other excellent
 inventions in the Mathematicks contained in the said letter—

“ He

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“ He [Mr. Newton] then observes (what I had formerly sought after, but unsuccessfully), that the following numbers are, from the two first, to be found by continual multiplication of this series $1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$

“ &c.” From this passage of Dr. Wallis’s Algebra, I am inclined to think that this famous theorem was never generally known to Mathematicians till this publication of it in that work. And from its having thus been communicated to the learned world as a discovery of Mr. Newton (who was afterwards better known by the title of Sir Isaac Newton), it has usually been called *his* Theorem.

4. This Theorem had been discovered by Sir Isaac Newton about the year 1665, as appears from his letters to Mr. Oldenburgh in the year 1676, copies of which were sent to Mr. Leibnitz by Mr. Newton’s direction. But these letters do not appear to have been known to the Mathematical world in general, till the year 1712, when they were printed in the *Commercium Epistolicum* by the order of the Royal Society. And no part of them seems to have been published before the year 1685, when the foregoing account of the generation of the numeral co-efficients of the terms of the series that is equal to the m th power of a binomial quantity, and a few more curious discoveries contained in them, were inserted by Dr. Wallis, in his Treatise on Algebra.

5. It has been observed above, that Mr. Briggs, and not Sir Isaac Newton, was *the first inventor* of this Theorem in the first and simplest case of it, or when the index m is an affirmative whole number. Yet I am inclined to think that Sir Isaac Newton was likewise *an inventor* of it even in that case, though *not the first inventor*. For it is well known that he was not an extensive reader of Mathematical Works; and he appears to have applied himself principally in his younger years to the study of Des Cartes’s Geometry, with Schooten’s Commentary on it, and the other Tracts published by Schooten with it, and of Dr. Wallis’s *Aritmetica Infinitorum*, and his other works on mathematical subjects then

then published; in none of which books is any mention made of this useful Theorem that had been discovered so many years before by Mr. Briggs. And, as these were the books to which Mr. Newton is known to have given the greatest part of his attention at that time, he may very well be supposed not to have seen Mr. Briggs's *Arithmetica Logarithmica*, in which this Theorem is contained, at the time of his discovering it himself, which was about the year 1665, or when he was only 23 years old. And, if he had seen that book, and observed this discovery to be contained in it, I can hardly conceive that, when he was speaking of this Theorem, and setting forth its great utility in mathematical investigations, he would have omitted to make mention of the name of Mr. Briggs, and to acknowledge that what he had delivered upon the subject in his *Arithmetica Logarithmica* contained the substance of the said Binomial Theorem in the case of Integral Powers, though not expressed in Algebraic Symbols. For these reasons I am inclined to think that Sir Isaac Newton had not seen Mr. Briggs's *Arithmetica Logarithmica* when he invented the Binomial Theorem, and consequently that he was truly *an inventor* of it even in the case of Integral Powers, though *not the first inventor*.

6. But it seems more surprising that Dr. Wallis, who was a much more copious reader of Mathematical Works than Sir Isaac Newton, and who actually had seen and read Mr. Briggs's *Arithmetica Logarithmica*, and makes mention of it in his Algebra, chapter XII, page 60, should not have attended to the contents of that ingenious Treatise enough to have observed that it contained this most useful Theorem. Yet this appears to have been the fact, from what the Doctor tells us in the 85th chapter of his Algebra, page 319, in the passage that has been already cited in art. 3, where he mentions the law of the generation of the co-efficients of the terms of the series that is equal to the m th power of a binomial quantity, as a discovery that had recently come to his knowledge by the perusal of Mr. Newton's letter to Mr. Oldenburgh. For, surely, it must be concluded from this declaration, that, though he had seen Mr. Briggs's *Arithmetica*

metica Logarithmica, he had not read it with sufficient attention to discover that this method of generating the co-efficients of the terms of the series that is equal to the m th power of a binomial quantity, when m was equal to any whole number whatsoever, was contained in it: though it seems indeed unaccountably strange that he should not have taken notice of it.

7. We may therefore, upon the whole matter, consider the Binomial Theorem, in the case of Integral and Affirmative Powers, as having been first invented by Mr. Briggs before the year 1624, and published by him in that year in his *Arithmetica Logarithmica*, but in such a manner, and in such expressions, as did not much engage the attention of Mathematicians towards it; since it does not seem to have been generally known amongst Mathematicians till it was afterwards published in the year 1685, as an invention of Mr. Isaac Newton, by Dr. Wallis in his Algebra. And we may consider it as having been invented a second time by Mr. Newton about the year 1665, and extended by him at the same time to the other cases of Fractional and Negative Powers; and also expressed in the very short and convenient Algebraick notation, in which it is set forth above in art. 1, and which has contributed so much to give it currency amongst Mathematicians. And, lastly, we may consider it as having been communicated by Mr. Newton to Mr. Oldenburgh and Mr. Leibnitz, and probably also to his friend and patron Dr. Isaac Barrow, the Master of Trinity College, Cambridge, and a few more of his Mathematical friends, in the year 1676, in the letter above-mentioned; and as having afterwards been communicated to the world at large in the aforesaid extracts from the said letter to Mr. Oldenburgh, which Dr. Wallis published in his Algebra in the year 1685.

Of

Of Mr. James Bernoulli's¹ demonstration of the said Theorem, in
his Treatise on the Doctrine of Chances, intituled,
De Arte Conjectandi.

8. But, by what steps, or what train of reasoning, Sir Isaac Newton discovered this law of the said co-efficients to be such as he described it, is not known; nor is any demonstration of it, even in the easiest case of it (or when the index m of the power to which the binomial quantity is to be raised, is a whole number), any where to be found in all his works. Nor has Dr. Wallis attempted to supply this defect, nor, as I believe, any other mathematical author whatsoever in the last century, from the year 1685 (when the Theorem was first published by Dr. Wallis) to the end of it; nor do I know of any demonstration of it given in the beginning of the present century before the year 1713, when the learned and sagacious Mr. James Bernoulli's excellent Treatise on the Doctrine of Chances, intituled, *De Arte Conjectandi*, was published at Basil, or Basle, in Switzerland. But there we find an excellent demonstration of it, in the case of Integral Powers, derived from the doctrine of Permutations and Combinations, and the properties of the Figurate numbers, which are the true principles to which it ought to be referred. This demonstration is contained in the 3d chapter of the second part of that valuable Treatise, and may be perfectly understood by a careful perusal of the three first chapters of that second part, without the help of the first part of the Treatise. For the doctrine of Permutations and Combinations is explained from its first principles in the two first chapters of that second part of the Treatise, without any reference to the first part; and the properties of the Figurate numbers are derived from that doctrine in a most strict and scientific manner, in the third chapter of the same second part; and amongst these proper-

¹He was born 1654 and died 1705.² H

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ties of the Figurate numbers, so derived, is the Binomial Theorem, in the case of Integral and Affirmative Powers, or the law of the generation of the numeral co-efficients of the terms of a series that is equal to any integral and affirmative power of the binomial quantity $a + b$. This demonstration therefore deserves to be generally known and studied by Mathematicians, as the true foundation of this celebrated and most useful Theorem. And upon that account I have re-printed the said three first chapters of the second Part of that excellent Treatise *De Arte Conjectandi*, in the foregoing part of this volume, in the author's original Latin text, with some explanatory notes on a few of the most difficult passages of it, and have afterwards added a very full Translation of the same three chapters, with some examples and additions of my own, which I thought might be useful to my readers, and which I have taken care to distinguish from the other parts which are translated from the Author's text. And I hope that, by thus exhibiting this part of that excellent work in an English dress, and removing the difficulties that occur in the original, in consequence of the Author's extreme conciseness, I shall induce the young Students of the Mathematicks in England, to make themselves acquainted with this masterly and scientific demonstration of this most important Theorem, which seems hitherto to have been adopted by too many Mathematicians, upon the mere ground of induction, and the experience of its truth in the several trials they have made of it, without endeavouring to find a demonstration of it.

Another

Another demonstration of it, in the same case of Integral Powers, will be given in the ensuing part of this Discourse.

9: But, though the demonstration of this proposition given by Mr. James Bernoulli in this excellent Treatise, *De Arte Conjectandi*, Part 2d, Chapter 3d, (and which may be seen above in this volume in the original Latin text of Mr. James Bernoulli in page 28, and in my Translation of it in pages 115 and 116), is the first, and, in my opinion, the best that has yet been given of it, yet I doubt not that the Mathematical Reader will be pleased to see another demonstration of it, that is somewhat shorter than Mr. Bernoulli's (inasmuch as it does not require the previous knowledge of the Doctrine of Permutations and Combinations, and the properties of the Figurate Numbers), and yet is equally accurate and conclusive. Such a demonstration I shall therefore now endeavour to lay before him in the remaining part of this Discourse.

10. Now in order to discover the general relation of the terms of the series that is equal to $a + b^m$ to each other, when m denotes any whole number whatsoever, it will be proper in the first place to examine their relation to each other when m is equal to some particular whole numbers, and those not large ones, that they may be more easily managed and their properties more readily seen into. And, if, when we have examined these particular serieses that are equal to certain particular values of $a + b^m$, when m is equal to certain small whole numbers, we can find any common properties that belong equally to all of them, and can also perceive that the same properties must likewise belong to all the serieses that shall be equal to any other values of $a + b^m$, as well as to those which we have considered; or, if we cannot immediately perceive this to be the case, but can find some method of demonstrating that it is so; we shall then arrive at the

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knowledge

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knowledge of the general relation of the terms of the series that is equal to $a+b^m$, to each other, which is the object of our pursuit. We will therefore raise the binomial quantity $a+b$ to its square, and cube, and fourth power, and fifth power, and sixth power, by multiplication; which may be done in the manner following.

$$\begin{array}{r} 1 \\ \hline a+b \end{array} = a+b^0.$$

$$\begin{array}{r} a+b \\ \hline a+b \end{array} = a+b^1.$$

$$\begin{array}{r} aa+ab \\ + ab+bb \\ \hline \end{array}$$

$$\begin{array}{r} aa+2ab+bb \\ \hline a+b \end{array} = a+b^2.$$

$$\begin{array}{r} a^3+2a^2b+ab^2 \\ + a^2b+2ab^2+b^3 \\ \hline \end{array}$$

$$\begin{array}{r} a^3+3a^2b+3ab^2+b^3 \\ \hline a+b \end{array} = a+b^3.$$

$$\begin{array}{r} a^4+3a^3b+3a^2b^2+ab^3 \\ + a^2b+3a^2b^2+3ab^3+b^4 \\ \hline \end{array}$$

$$\begin{array}{r} a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ \hline a+b \end{array} = a+b^4.$$

$$\begin{array}{r} a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\ + a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline \end{array}$$

$$\begin{array}{r} a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ \hline a+b \end{array} = a+b^5.$$

$$\begin{array}{r} a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5 \\ + a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6 \\ \hline \end{array}$$

$$\begin{array}{r} a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 \\ \hline \end{array} = a+b^6.$$

Observa-

Observations on the terms of the foregoing Serieses that are equal to $a + b$, $a + b$, $a + b$, $a + b$, $a + b$, and $a + b$, explaining the Composition of the Literal parts of the said terms.

11. If we examine the composition of the foregoing products, or serieses, which are obtained by continual multiplications by the binomial quantity $a + b$, the first observation that will occur to us will be, that the first term of the series $aa + 2ab + bb$, which is equal to the square of $a + b$, is aa or a^2 ; and that the first term of the series $a^3 + 3a^2b + 3ab^2 + b^3$, which is equal to the cube of $a + b$, is a^3 ; and that the first term of the series $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which is equal to the fourth power of $a + b$, is a^4 ; and that the first term of the series $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, which is equal to the fifth power of $a + b$, is a^5 ; and that the first term of the series $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, is a^6 ; or that the first term of the series that is equal to any one of the said five powers of the binomial quantity $a + b$, is the same power of the single quantity a .

And it is easy to see that, if we were to continue these multiplications by the binomial quantity $a + b$ ever so far, the same thing would take place with respect to the first terms of the following products, or serieses, which are equal to any higher powers of the binomial quantity $a + b$, let their number be ever so great; or that, if the letter m be any number, how great soever, the first term of the product, or series, that is equal to $a + b$, or the m th power of the binomial quantity $a + b$, will be a^m , or the same power of the single quantity a .

For, as the first term of every new product, or series, is produced by the multiplication of the first term of the next preceeding

preceeding product, or series, by a , or $1 \times a$, the co-efficient of the first term of the new series, which is the product of the said multiplication, must be the same with the co-efficient of the first term of the next preceeding series, which is the multiplicand of the said multiplication. And consequently, as the co-efficient of the said multiplicand, or first term of the preceeding series is originally 1, (namely, when $a+b$ is multiplied into $a+b$, in order to produce the series $aa+2ab+bb$, which is equal to its square), the co-efficients of the first terms of all the following products, or serieses, which are equal to $\overline{a+b}^1$, $\overline{a+b}^2$, $\overline{a+b}^3$, $\overline{a+b}^4$, $\overline{a+b}^5$, $\overline{a+b}^6$, $\overline{a+b}^7$, $\overline{a+b}^8$, $\overline{a+b}^9$, $\overline{a+b}^{10}$, &c, *ad infinitum*, must likewise all be equal to 1. Q. E. D.

12. The second observation that will occur to us, is, that the indexes of the several successive powers of a in the terms of every product, or series; that is equal to any power of the binomial quantity $a+b$, decrease continually by an unit, and that the indexes of the powers of b in the terms of the said products, or serieses, increase by an unit at the same time. Thus, in the series $a^2+2ab+b^2$, or (as it is sometimes expressed in Sir Isaac Newton's Notation of Indexes, because a^0 is $= 1$, and b^0 is likewise $= 1$), $a^2b^0+2a^1b^1+a^0b^2$, which is equal to the square of the binomial quantity $a+b$, the indexes of the powers of a are 2, 1, and 0, or 2, $2-1$, and $2-2$, and the indexes of the powers of b are 0, 1, and 2, or 0, $0+1$, and $0+2$; and in the series $a^3+3a^2b+3ab^2+b^3$, or $a^3b^0+3a^2b^1+3a^1b^2+a^0b^3$, which is equal to the cube of the binomial quantity $a+b$, the indexes of the powers of a are 3, 2, 1, and 0, or 3, $3-1$, $3-2$, and $3-3$, and the indexes of the powers of b are 0, 1, 2, and 3. And the same thing takes place in the following products, or serieses, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, and $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, and $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, which are equal to the fourth, fifth, and sixth powers of $a+b$. And it is easy to see that the same thing will likewise take place in the terms of the products, or serieses, that are equal to

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to any higher powers of $a+b$ whatsoever, if the said multiplications by $a+b$ were to be continued till the serieses equal to such higher powers were produced. Therefore the literal parts of the second term, and of all the following terms of each of the said products, or serieses, may always be generated, or derived, from the first term of it, by the continual multiplication of it by the fraction $\frac{b}{a}$.

13. But, by the first observation, the first term of the series which is equal to $\overline{a+b}^m$, or the m th power of the binomial quantity $a+b$, when m is any whole number whatsoever, is a^m .

It follows therefore, in the 3d place, that the literal parts of the terms of the series that is equal to $\overline{a+b}^m$, will be a^m , $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, $a^{m-5}b^5$, $a^{m-6}b^6$, $a^{m-7}b^7$, $a^{m-8}b^8$, $a^{m-9}b^9$, &c, till we come to the quantity $a^{m-m}b^m$, (or a^0b^m , or $1 \times b^m$), or b^m .

And thus we have discovered the composition of the literal parts of the terms of the series which is equal to $\overline{a+b}^m$, as fully as can be desired. And we have likewise discovered that the co-efficient of the first term, of which the literal part is a^m , is always 1, or that the first term of the said series is a^m itself, and not any multiple of a^m . Q. E. I.

14. In the 4th place it is evident that all the terms of every product, or series, arising from the multiplication of the binomial quantity $a+b$ into itself, must be connected together by the sign $+$, or added to each other. And consequently the literal parts of the series that is equal to $\overline{a+b}^m$, will

will be $a^m + a^{m-1}b + a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 +$
 $a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + a^{m-8}b^8 + a^{m-9}b^9 + \&c.$
 $+ b^m.$

*Of the numeral co-efficients of the second and other following terms
of the product, or series, that is equal to $a + b$ raised to the m th power.*

15. We come now to inquire into the numeral co-efficients of the second and other following terms of the product, or series, which is equal to $a + b$ raised to the m th power of the binomial quantity $a + b$.

Now the numeral co-efficient of the second term of this series will always be equal to m , or the index of the power to which the binomial quantity $a + b$ is to be raised. This may be demonstrated in the manner following.

In raising the several powers of the binomial quantity $a + b$ by the continual multiplication of that quantity into itself, in the manner above exemplified in art. 10, it is evident that the said second term of every new product, or series, that is equal to a new power of $a + b$, is always produced by adding the product of the multiplication of the first term of the series that is equal to the next lower power of $a + b$ (of which first term we have seen that 1 is always the co-efficient) by b to the product of the multiplication of the second term of the said foregoing series by a ; the effect of which addition is, to increase the co-efficient of the second term of the new series by an unit, or so as to make it exceed the co-efficient of the second term of the foregoing series by an unit. Thus, the second term, $2ab$, of the series

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$2ab + b^2$, which is equal to the square of $a + b$, is produced by the addition of the product ba , or ab , (which arises from the multiplication of a , the first term of the former series $a + b$, by b ,) to the product ab , which arises from the multiplication of the second term b of the former series $a + b$ by a ; the effect of which addition is, to make the co-efficient, 2, of the second term $2ab$ in the new series, exceed the co-efficient, 1, of the second term b of the former series, by an unit. And, in like manner, the second term, $3a^2b$, of the series $a^3 + 3a^2b + 3ab^2 + b^3$, which is equal to the cube of the binomial quantity $a + b$, is produced by the addition of the product a^2b (which arises from the multiplication of a^2 , the first term of the former series $a^2 + 2ab + b^2$, by b), to the product $2a^2b$, which arises from the multiplication of the second term $2ab$ of the said former series $a^2 + 2ab + b^2$ by a ; the effect of which addition is, to make the co-efficient, 3, of the second term $3a^2b$ of the new series, exceed the co-efficient, 2, of the second term $2ab$ of the former series, by an unit. And, in like manner, $4a^3b$, the second term of the next series, is $= a^3 \times b + 3a^2b \times a$, or $a^3b + 3a^3b = 1 + 3 \mid \times a^3b$; and $5a^4b$, the second term of the next series, is $= a^4 \times b + 4a^3b \times a = a^4b + 4a^4b = 1 + 4 \mid \times a^4b$; and $6a^5b$, the second term of the next series, is $= a^5 \times b + 5a^4b \times a = a^5b + 5a^5b = 1 + 5 \mid \times a^5b$. And this, it is easy to see, must be the case in any higher powers whatsoever of $a + b$, if we were to continue the multiplications by $a + b$ till the serieses that were equal to such higher powers of $a + b$ were produced. And consequently, since in the first power of the binomial quantity $a + b$, to wit, in the said quantity itself, the co-efficient of the second term b , to wit, 1, is equal to the index of the said first power, which is also 1, and in the second, and third, and fourth, and fifth, and sixth powers of the said binomial quantity, the co-efficient of the second term of the series that is equal to each of the said powers of $a + b$ is also equal to the index of the said powers; it follows that in all higher powers whatsoever of the said binomial quantity $a + b$, the co-efficient of the second term of the series which is equal to every such power

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will be equal to the index of the said power; or, in other words, the co-efficient of the second term of the series which is equal to $\overline{a+b}^m$ will always be equal to the index m .

Q. E. D.

16. From what has been shewn in the foregoing articles we may conclude with certainty, that the two first terms of the series that is equal to $\overline{a+b}^m$, when m is equal to any whole number whatsoever, will be $a^m + m \times a^{m-1}b$, and that the literal parts of the following terms of the said series will be $a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + a^{m-8}b^8 + a^{m-9}b^9 + \&c. + a^{m-m}b^m$, (or a^0b^m , or $1b^m$, or) $+ b^m$. It remains that we inquire what will be the numeral co-efficients of the said third and other following terms of the said series, or by what law, or rule, they may be generated, or derived, from the two first co-efficients, 1 and m . This is a matter of considerable difficulty; and I am not acquainted with any direct and scientific method of investigating this law of the generation of the said co-efficients, besides that of Mr. James Bernoulli above-mentioned, which is grounded on the Doctrine of Permutations and Combinations, and the properties of the Figurate numbers. But I can point out a manner of considering the subject and attempting to find this law of generation, which seems likely enough to have occurred to a Mathematician who was in pursuit of this inquiry, and which, if it had occurred to him, would have led him directly to form a just conjecture concerning this law by which these co-efficients are to be generated; after which he would have been induced to try the law, so discovered by conjecture, in some easy particular instances, and, having found it to be true in all of them, he would naturally conclude that it was true in all other cases whatsoever. This conjectural method of investigation, I conceive, may have been as follows.

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A Conjectural Investigation of the Law by which the co-efficients of the third, and fourth, and fifth, and other following terms of the series which is equal to $\overline{a+b}^m$, or the m th power of the binomial quantity $a+b$, may be generated, or derived, from 1 and m , the co-efficients of the two first terms of the said series.

17. Now, in order to discover the manner in which these co-efficients may be derived from the two first co-efficients 1 and m , I should think it would be natural to examine the co-efficients of the terms of the series that is equal to $\overline{a+b}^m$ in some of the lower powers of $a+b$ which we have actually raised by multiplication, as, for example, in the series which is equal to $\overline{a+b}^6$, and which we have found above in art. 10, to be $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. In this series the co-efficients of the terms are 1, 6, 15, 20, 15, 6, and 1; and our object is to discover, 1st, by what number, integral or fractional, the second of these co-efficients, to wit, 6, ought to be multiplied in order to produce the third co-efficient, to wit, 15; and, 2dly, by what number, integral or fractional, the third co-efficient, to wit, 15, ought to be multiplied in order to produce the fourth co-efficient 20; and, 3dly, by what number the fourth co-efficient, 20, ought to be multiplied in order to produce the fifth co-efficient 15; and, 4thly, by what number the fifth co-efficient, 15, ought to be multiplied in order to produce the sixth co-efficient 6; and, lastly, by what number the sixth co-efficient, 6, ought to be multiplied in order to produce the seventh and last co-efficient 1. Now these multiplying numbers are evidently $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$. For 6

$\times \frac{15}{6}$ is = 15, and $15 \times \frac{20}{15}$ is = 20, and $20 \times \frac{15}{20}$ is = 15,

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and

and $15 \times \frac{6}{15}$ is $= 6$, and $6 \times \frac{1}{6}$ is $= 1$. We must therefore now examine these five generating fractions $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, together with the preceeding generating fraction $\frac{6}{1}$, by the multiplication of which into the first co-efficient 1 the second co-efficient 6 is produced; and must endeavour to find out some remarkable, or regular, property in them, which we may reasonably suppose to belong also to the co-efficients of the terms of other powers of $a+b$, as well as to those of the terms of this, its sixth power. And, in order to examine these fractions with the greater ease, it seems natural, in the first place, to reduce them to their lowest denominations, by dividing both their numerators and their denominators by the factors which are common to them both. Now, if this be done, the said generating fractions $\frac{6}{1}$, $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, will be found to be equal to $(\frac{6}{1}$, $\frac{5 \times 3}{2 \times 3}$, $\frac{4 \times 5}{3 \times 5}$, $\frac{3 \times 5}{4 \times 5}$, $\frac{2 \times 3}{5 \times 3}$, and $\frac{1}{6}$, or) $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$; in which last fractions it is impossible not to observe that the numerators regularly decrease by an unit from 6, which is the index of the power to which the binomial quantity $a+b$ has been raised, to 1, and the denominators regularly increase at the same time by the same quantity of an unit from 1 to the said index 6. This regularity is very striking, and naturally raises a suspicion that the same thing may take place in the generating fractions of the co-efficients of the terms of the serieses that are equal to other powers of the binomial quantity $a+b$, and is an inducement to try it in the other serieses that have been produced above in art. 10, by multiplication, and which are equal to $\overline{a+b}^1$, $\overline{a+b}^2$, $\overline{a+b}^3$, and $\overline{a+b}^4$. We will therefore now proceed to try it in those instances.

18. Now

18. Now we have seen in art. 10, that $\overline{a+b}^5$ is = the series $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, in which the numeral co-efficients of the terms are 1, 5, 10, 10, 5, and 1. Therefore the generating fractions, by the multiplication of which the second of these co-efficients, to wit, 5, is derived from the first, or 1, and every following co-efficient from that which is next before it, will be $\frac{5}{1}$, $\frac{10}{5}$, $\frac{10}{10}$, $\frac{5}{10}$, and $\frac{1}{5}$; which are respectively equal to the fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$. And in these last fractions we cannot but observe that the numerators 5, 4, 3, 2, and 1, regularly decrease by an unit from 5, or the index of the power to which the binomial quantity $a+b$ is raised, to 1, and the denominators 1, 2, 3, 4, and 5, regularly increase at the same time, by the same quantity of an unit from 1 to the said index 5. It appears therefore that the same rule takes place amongst these generating fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$, as took place amongst the generating fractions $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, of the co-efficients of the terms of the former series which was equal to $\overline{a+b}^6$.

19. We will now try whether the same rule will take place in the series which is equal to the fourth power of $a+b$.

This series is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, in which the numeral co-efficients of the terms are 1, 4, 6, 4, and 1. Now the generating fractions by the multiplication of which the second of these co-efficients, to wit, 4, is generated from the first, or 1, and every following co-efficient is generated from that which is next before it, are evidently $\frac{4}{1}$, $\frac{6}{4}$, $\frac{4}{6}$, and

and $\frac{1}{4}$; which are respectively equal to $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{1}{4}$.

And in these last fractions the numerators 4, 3, 2, and 1, regularly decrease by an unit from 4, which is the index of the power to which the binomial quantity $a + b$ has been raised, to 1, and the denominators 1, 2, 3, and 4, regularly increase at the same time by an unit from 1 to the said index 4.

20. We come next to the series which is equal to the cube of $a + b$.

This series is $a^3 + 3a^2b + 3ab^2 + b^3$, in which the co-efficients of the terms are 1, 3, 3, and 1. Now the generating fractions, by the multiplication of which the second of these co-efficients is derived from the first, and the third from the second, and the fourth from the third, are evidently $\frac{3}{1}$, $\frac{3}{3}$, and $\frac{1}{3}$, which are respectively equal to $\frac{3}{1}$, $\frac{2}{2}$, and $\frac{1}{3}$. And in these last fractions the numerators 3, 2, and 1, decrease regularly by an unit from 3, which is the index of the power to which the binomial quantity $a + b$ has been raised, to 1, and the denominators 1, 2, and 3, increase regularly at the same time by an unit from 1 to the said index 3.

21. And the same thing takes place in the series which is equal to the square of $a + b$. For this series is $a^2 + 2ab + b^2$, in which the co-efficients of the terms are 1, 2, and 1. Now the generating fractions, by the multiplication of which the second co-efficient 2 is derived from the first co-efficient 1, and the third co-efficient 1 is derived from the second co-efficient 2, are evidently $\frac{2}{1}$, and $\frac{1}{2}$; which admit of no reduction to lower denominations. And in these fractions $\frac{2}{1}$ and $\frac{1}{2}$, the numerators 2 and 1 decrease by an unit, as
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in the former cases, from 2, which is the index of the power to which the binomial quantity $a + b$ has been raised, to 1, and the denominators 1 and 2 increase at the same time by an unit from 1 to the said index 2.

22. It appears therefore that this law of the generating fractions of the numeral co-efficients of the terms of the serieses that are equal to the powers of the binomial quantity $a + b$, takes place in the cases of the square, the cube, the fourth power, the fifth power, and the sixth power, of the said binomial quantity. This is a very strong ground for conjecturing that the same law will take place in the generating fractions of the numeral co-efficients of the terms of the serieses which are equal to the powers of the said binomial quantity in all other cases whatsoever; or that, if the index of the power to which the said binomial quantity is raised be any whole number whatsoever, denoted by the letter m , the generating fractions, by the continual multiplication of which the numeral co-efficients of the second and other following terms of the series which is equal to $(a + b)^m$, or the m th power of the said binomial quantity, may be derived from 1, or the co-efficient of the first term, a^m , or $1 \times a^m$, of the said series, will be $\frac{m}{1}$, or (as it is sometimes called) $\frac{m-0}{1}$, and $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is = 0, or till the said series is terminated, or exhausted.

23. And the ground for conjecturing that this is a general law that takes place among the generating fractions of the terms of these serieses in all cases, or when the index is equal to any whole number whatsoever, will become still stronger if we try it in a few more examples of serieses that are equal to higher powers of the binomial quantity $a + b$, than the sixth power. I shall therefore now proceed to try it

it in the serieses which are equal to $a + \sqrt[1]{b}$, $a + \sqrt[2]{b}$, and $a + \sqrt[3]{b}$.

24. Now $\overline{a + \sqrt[1]{b}}^7$ is ($= \overline{a + \sqrt[1]{b}}^6 \times \overline{a + \sqrt[1]{b}} = a^7 + 6a^6b + 15a^5b^2 + 20a^4b^3 + 15a^3b^4 + 6ab^5 + b^7 \times a + b$) $= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$; in which series the numeral co-efficients of the terms are 1, 7, 21, 35, 35, 21, 7, and 1. The generating fractions, by the multiplication of which the second of these co-efficients, to wit, 7, is derived from the first co-efficient 1, and the third and other following co-efficients are derived from those

which immediately preceed them, are evidently $\frac{7}{1}$, $\frac{21}{7}$, $\frac{35}{21}$,

$\frac{35}{35}$, $\frac{21}{35}$, $\frac{7}{21}$, and $\frac{1}{7}$; which are respectively equal to $\frac{7}{1}$,

$\frac{6}{2}$, $\frac{5}{3}$, $\frac{4}{4}$, $\frac{3}{5}$, $\frac{2}{6}$, and $\frac{1}{7}$; in which last fractions the nu-

merators 7, 6, 5, 4, 3, 2, and 1, regularly decrease by an unit from 7, (which is the index of the power to which the binomial quantity $a + b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, and 7, regularly increase at the same time by an unit from 1 to the said index 7; agreeably to what was observed in the five former examples.

25. And $\overline{a + \sqrt[1]{b}}^8$ is ($= \overline{a + \sqrt[1]{b}}^7 \times \overline{a + \sqrt[1]{b}} = a^8 + 7a^7b + 21a^6b^2 + 35a^5b^3 + 35a^4b^4 + 21a^3b^5 + 7a^2b^6 + b^7 \times a + b$) $= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$; in which series the numeral co-efficients of the terms are 1, 8, 28, 56, 70, 56, 28, 8, and 1. The

generating fractions of these co-efficients are evidently $\frac{8}{1}$, $\frac{28}{8}$,

$\frac{56}{28}$, $\frac{70}{56}$, $\frac{56}{70}$, $\frac{28}{56}$, $\frac{8}{28}$, and $\frac{1}{8}$; which are respectively equal to

$\frac{8}{1}$, $\frac{7}{2}$, $\frac{6}{3}$, $\frac{5}{4}$, $\frac{4}{5}$, $\frac{3}{6}$, $\frac{2}{7}$, and $\frac{1}{8}$; in which last fractions

the numerators 8, 7, 6, 5, 4, 3, 2, and 1, decrease regularly by an unit from 8, (which is the index of the power to

which the binomial quantity $a + b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, and 8, regularly increase at the same time by an unit from 1 to the said index 8; agreeably to what has been observed in the six former examples.

26. And, lastly, $\overline{a + b}^9$ is $(= \overline{a + b}^8 \times \overline{a + b} = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8 \times \overline{a + b}) = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$; in which series the numeral co-efficients of the terms are 1, 9, 36, 84, 126, 126, 84, 36, 9, and 1. The generating fractions of these co-efficients are evidently $\frac{9}{1}, \frac{36}{9}, \frac{84}{36}, \frac{126}{84}, \frac{126}{126}, \frac{84}{126}, \frac{36}{84}, \frac{9}{36}$, and $\frac{1}{9}$; which are respectively equal to $\frac{9}{1}, \frac{8}{2}, \frac{7}{3}, \frac{6}{4}, \frac{5}{5}, \frac{4}{6}, \frac{3}{7}, \frac{2}{8}$, and $\frac{1}{9}$; in which last fractions the numerators 9, 8, 7, 6, 5, 4, 3, 2, and 1, decrease regularly by an unit from 9, (which is the index of the power to which the binomial quantity $a + b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, 8, and 9, regularly increase at the same time by an unit from 1 to the said index 9; agreeably to what has been observed in all the former examples.

27. After observing this law of the co-efficients to take place in so many different examples, it would be impossible for our mathematical investigator not to conclude with a very high degree of confidence that it would take place in all other cases whatsoever; or that, when the index m is equal to any whole number whatsoever, the generating fractions of the numeral co-efficients of the terms of the series that is equal to $\overline{a + b}^m$, will be $\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \frac{m-4}{5}, \frac{m-5}{6}, \frac{m-6}{7}, \frac{m-7}{8}$, &c, till we come to the fraction $\frac{m-m}{m+1}$, which is

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= 0,

= 0, or till the said series of fractions is terminated, or exhausted. And then it would follow, from what has been shewn above concerning the literal parts of the terms of the said series, that the said series which is equal to $\overline{a+b}^m$, or the m th power of the binomial quantity $a+b$, would be $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \frac{m-5}{6} F a^{m-6} b^6 +$ &c, continued to b^m . Q. E. I.

28. This method of discovering (by a conjecture grounded on some trials in particular examples) that the generating fractions by which the numeral co-efficients of the third, and fourth, and other following terms of the series that is equal to $\overline{a+b}^m$ (or any integral power of the binomial quantity $a+b$), are derived from m (the index of the power to which the said binomial quantity is raised), or from the co-efficient of the second term of the said series (which is always equal to the said index) are $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, is suggested by Professor Saunderson, in the second volume of his Algebra, in the chapter on the Binomial Theorem; where the Reader will find a good explanation and illustration of the said celebrated Theorem, by a variety of examples, both in the case of Integral powers, and in the case of Roots and other Fractional powers, and even in the case of Negative powers, and of powers that are both fractional and negative; but no demonstration of it in any case, not even in that of Integral and Affirmative powers.

29. We have now shewn with demonstrative certainty that the literal parts of the terms of the series which is equal to $\overline{a+b}^m$, or the m th power of the binomial quantity $a+b$, when

when the letter m denotes any affirmative whole number whatsoever, are $a^m + a^{m-1}b + a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + \&c$, till we come to the term $a^{m-m}b^m$, (or $a^0 \times b^m$, or $1 \times b^m$), or b^m , and that the numeral co-efficient of the literal part of the first term of the said series is 1, and the numeral co-efficient of the literal part of the second term of it is m , and consequently that the two first whole terms of the said series are $1 \times a^m$, and $m \times a^{m-1}b$, or $a^m + m \times a^{m-1}b$, or $a^m + \frac{m}{1} \times a^{m-1}b$, or $a^m + \frac{m-0}{1} \times a^{m-1}b$. And we have also shewn that in the serieses which are equal to $\overline{a+b^2}$, $\overline{a+b^3}$, $\overline{a+b^4}$, $\overline{a+b^5}$, $\overline{a+b^6}$, $\overline{a+b^7}$, $\overline{a+b^8}$, and $\overline{a+b^9}$, or when the index m is equal to 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the said serieses are derived from m , or the numeral co-efficient of the second term, by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, and $\frac{m-8}{9}$; which affords a very strong ground for conjecturing that the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the serieses that are equal to any higher powers whatsoever of the binomial quantity $a+b$, will, in like manner, be derived from m , the co-efficient of the second term, by the continual multiplication of the same generating fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, $\frac{m-9}{10}$, $\frac{m-10}{11}$, $\frac{m-11}{12}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is $= 0$, or till the said series of generating fractions is terminated, or exhausted. Now this conjecture

jecture may be changed into absolute certainty, and the said law of the generation of the co-efficients may be shewn to take place in all the serieses that are equal to the quantity $\overline{a+b}^m$, or the powers of the binomial quantity $a+b$, when the index m of the said quantity $\overline{a+b}^m$, is equal to any whole number, how great soever, by shewing that, if it takes place when the index m is equal to any one particular number, (as we have seen that it does when the index m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9), it must also take place when the index m is greater by an unit than in the former case. For then it will follow that it must be true likewise when the index m is greater by any multitude of units than in the former case, or when it is equal to any other whole number, how great soever. This we shall now proceed to shew in the remaining part of this discourse.

Of the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the Series that is equal to $\overline{a+b}^m$, and the law of the generation of the said co-efficients from m , the co-efficient of the second term of the said series, and from each other.

30. In order to demonstrate the law of the generation of these co-efficients, it will be convenient to get rid of the powers of a and b , in the terms of the series that is equal to $\overline{a+b}^m$, and to fix our attention only on the generation of the numeral co-efficients of the third, fourth, fifth, sixth, and other following terms of the said series. This may be done by supposing a and b to be, each of them, equal to 1,

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and consequently $a+b$ to be equal to $1+1$, and $\overline{a+b}^m$ to be equal to $\overline{1+1}^m$. For, as all the powers of both a and b will, on this supposition, be equal to 1, the Binomial Theorem set forth above in art. 1, will then be reduced to this, to wit, that $\overline{1+1}^m$ will be equal to the series $1 + \frac{m}{1}$
 $+ \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$
 $\times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c$, con-
 tinued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c$,
 $\times \frac{m-\sqrt{n-1}}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times$
 $\frac{m-4}{5} \times \&c$, $\times \frac{m-m+1}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times$
 $\frac{m-3}{4} \times \frac{m-4}{5} \times \&c$, $\times \frac{1}{m}$, or to the term 1. For the last
 term of this series must always be 1; because the numerators of the several factors in it form a decreasing progression of numbers, decreasing by an unit, from m to 1; and the denominators of the same factors form an increasing progression of numbers, increasing by an unit, from 1 to m ; and consequently the product of the multiplication of all the denominators is equal to the product of the multiplication of all the numerators, and therefore the product of the multiplication of all the said factors, or fractions, $\frac{m}{1}$, $\frac{m-1}{2}$,
 $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, into each other, or the last term of the series, must always be equal to 1.

We are therefore now to demonstrate that $\overline{1+1}^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1}$
 $\times$

$\times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c, + 1.$ And this we propose to do by shewing,

by abstract and general reasonings, that, if this Theorem is true when the index m is of any particular value, as, for example, when it is equal to 9, it must likewise be true when the index m is increased by an unit, or that, if n be taken $= m+1$, the quantity $1+i$, or the n th power of the binomial quantity $1+i$, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c,$ continued to the term $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \&c, \times \frac{1}{n}$, or to the term 1.

31. To facilitate the demonstration of this proposition, it will be convenient to premise the following Lemma.

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If the terms of the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c, + 1$ (in which n represents any whole number whatsoever), be set down twice together in two parallel lines, or rows, one under the other, but with the terms in the lower row advanced one step further to the right-

right-hand than the terms in the upper row, so that the first term in the lower row shall stand under the second term of the upper row, and the second term in the lower row shall stand under the third term in the upper row, and the third, fourth, fifth, sixth, and other following terms in the lower row shall stand under the fourth, fifth, sixth, seventh, and other following terms in the upper row, respectively; and both rows are continued to the same number of terms, namely, to the whole number of terms in the said series, or to $m+1$ terms; and then the terms in the lower row (each of which, it is evident, will consist of one factor less than the corresponding term, or term standing immediately above it in the upper row) be reduced to the same denomination as the terms that stand immediately above them in the upper row, and, after being so reduced, are added to the said terms that stand immediately above them in the said upper row;—upon these suppositions the new series of terms arising from this addition of the said two rows of terms to each other, will

be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2}$
 $\times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$
 $\times \frac{m-3}{4} \times \frac{m+1}{5} + \&c + 1$; in which series the last term

is 1, as well as in the two serieses from the addition of which this series arises; and the numerators of the last factors in all the terms, except the last, are always equal to $m+1$, instead of being equal to $m-1$, $m-2$, $m-3$, $m-4$, &c, as in the two foregoing serieses; and the number of terms in the said new series is $m+2$, instead of $m+1$, which is the number of terms in each of the said foregoing serieses,

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32. This will appear by setting down the said series

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, + 1 \text{ twice over, in the manner}$$

that has been just described; which may be done as follows:

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c.$$

$$1 + \frac{m}{1} \qquad + \frac{m}{1} \times \frac{m-1}{2} \qquad + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \qquad + \&c.$$

In these two rows of terms it is evident, in the first place, that the terms in the upper row, after the two first terms 1. and $\frac{m}{1}$, consist of two, three, and four, and more, factors,

every new term having one more factor than the term next before it; and, 2dly, that the terms in the lower row that stand immediately under the third, fourth, fifth, and other following terms in the upper row, consist of one factor less than the corresponding terms, or terms immediately over them in the upper row; and, 3dly, that the terms in the lower row consist of the very same factors as the corresponding terms in the upper row, excepting that they want the last factors of the said terms in the upper row. And hence it follows, that, in order to reduce the terms in the lower row to the same denomination as the terms in the upper row, we must multiply them by factors that shall have the same denominators as the last, or additional factors in the upper row, and which must have their numerators equal to their denominators, so as to make each of them equal to 1, to the end that the magnitudes of the said lower terms may not be altered by the multiplication of them by the said new factors. Thus, for example, the second term of the

lower

lower row, to wit, $\frac{m}{1}$, must be multiplied into the factor $\frac{2}{2}$, in order to bring it to the same denomination as the third term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, without altering its magnitude; and the third term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, must be multiplied into the factor $\frac{3}{3}$, in order to bring it to the same denomination as the fourth term of the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, without altering its magnitude; and the fourth term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, must be multiplied into the factor $\frac{4}{4}$, in order to bring it to the same denomination as the fifth term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, without altering its magnitude; and, for the like reason, the fifth, and sixth, and seventh, and other following terms in the lower row must be multiplied into the several factors $\frac{5}{5}$, and $\frac{6}{6}$, and $\frac{7}{7}$, &c, respectively; after which multiplications the two rows of terms that are to be added to each other, will be as follows, to wit,

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$+ 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c.$$

33. And, if these two rows of terms (being now brought to the same denominations) are added together in the manner above described; that is, every term in the lower row to the term that is immediately above it, the sum thence resulting will be the series

$$1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c, \text{ in which}$$

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the numerator of the last factor in every term is always $m + 1$, instead of $m - 1$, $m - 2$, $m - 3$, $m - 4$, &c.

And "That this must be the case in all the following terms of the said new series as well as in the few terms of it that have been here set down," will be evident from this consideration, to wit, That the denominator of the last factor of every term in the upper of the two rows of terms that are added together is always greater by an unit than the number which is subtracted from m in the numerator of the same factor. For from thence it follows that the denominator of the new multiplying fraction in the corresponding term of the lower row (which is always equal to the denominator of the said last factor in the upper row,) must always be greater by an unit than the number which is subtracted from m in the numerator of the last factor of the said upper term. And, therefore, the numerator of the said new multiplying fraction in the lower row (which is always equal to its denominator,) must also always be greater by an unit than the number which is subtracted from m in the numerator of the last factor of the said upper term; the consequence of which, in adding the lower term to the upper term, is to convert the numerator of the last factor in the upper term from $m - 1$, or $m - 2$, or $m - 3$, or the excess of m above some other number, into $m + 1$. Q. E. D.

34. And the number of terms in the new series, arising from the addition of the two former in the manner that has been described, will be greater by one than the number of the terms in either of the two added series: because the lower row of terms, consisting of the same number of terms as the upper row, and being placed one term further to the right-hand, must extend one term beyond it; and consequently, as the number of terms in each of the two rows of terms is $m + 1$, the number of terms in the new series, arising from the addition of the two rows together, must be $m + 2$. Q. E. D.

35. And, lastly, the last term of the said new series must be the same as the last term of the old series, or of the lower
row

row of terms; because, as the lower row of terms extends one term beyond the upper row, the last term in the lower row will not have any term over it in the upper row to which it is to be added, and consequently will continue the same in the new series $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2}$

$\times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$, as in the old series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$. But we have seen above, in

art. 30, that the last term of the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$,

is 1. Therefore the last term in the new series $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$, will also be 1. Q. E. D.

36. Coroll. 1. Now let the order of the numerators m , $m-1$, $m-2$, $m-3$, $m-4$, $\&c$, and $m+1$, of the factors of the third and other following terms of the last series

$1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$, be changed, by making $m+1$

the numerator of the first factor of every term, instead of being the numerator of the last factor. The said series

will then be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c$.

Now this change in the order of the numerators of the several factors of the terms will create no change in the values,

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or

of magnitudes, of the several terms themselves; because the products arising from the multiplication of the same numbers are always the same, in whatever order the numbers are multiplied. Therefore the foregoing series, after this change in the order of the numerators of the several factors of its terms, will still be of the same magnitude as before, and consequently will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described; that is, the series $1 + \frac{m+1}{1} + \frac{m+1}{1} \times$

$\frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$
 $\&c, + 1$, will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described.

37. Coroll. 2. Now let n be $= m + 1$. Then will $n-1$ be $= m$, and $n-2$ will be $= m-1$, and $n-3$ will be $= m-2$, and $n-4$ will be $= m-3$; and, in like manner, $n-5$, $n-6$, $n-7$, $\&c$, will be equal to $m-4$, $m-5$, $m-6$, $\&c$, respectively. And consequently the series obtained in the foregoing Corollary, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1}$

$\times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4}$
 $+ \&c, + 1$, consisting of $m+2$ terms, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times$
 $\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c, + 1$, consisting of $n+1$ terms,

Therefore the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times$
 $\frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c, + 1$, consisting of $n+1$ terms, will be equal to the sum that arises by adding the two aforesaid rows of terms together in the manner above described.

The

The Demonstration of the principal Proposition.

38. These things being premised, the main proposition stated at the end of art. 30, to wit, that, if m denote any whole number whatsoever, the quantity $1+1^m$, or the m th power of the binomial quantity $1+1$, will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c$, continued to $m+1$ terms, or to the term 1, may be demonstrated in the manner following.

39. The product that arises by multiplying the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$, into $1+1$ is the sum that arises by setting down the said series twice following in two parallel rows, one under the other, with the terms in the lower row advanced one term further to the right-hand than the terms in the upper row, in the manner above described, and then adding the terms in the lower row to the corresponding terms in the upper row. And the $m+1$ th power of $1+1$ is the product of the multiplication of the m th power of $1+1$ into $1+1$. Therefore, if in any particular value of m the m th power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$, + 1, consisting of $m+1$ terms, the $m+1$ th power of $1+1$ will be equal to the sum that arises by setting down the said series twice following in two parallel rows in the manner above described, and adding the said two rows of terms

terms together. But, by the second Corollary of the foregoing Lemma, if n be $= m + 1$, the sum arising from the addition of the said two rows of terms is the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c, + 1$, consisting of $n+1$ terms. Therefore, if in any particular value of m the m th power of $1 + 1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, + 1$, consisting of $m+1$ terms, it will follow that the $m+1$ th, or n th, or next higher power, of $1 + 1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$ terms. But it has been shewn in art. 17, 18, 19, &c, - - - 26, that when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, the m th power of $1 + 1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$, consisting of $m+1$ terms. Therefore, if n be equal to $9+1$, or 10, the $9+1$ th power, or 10th power, or n th power, of $1 + 1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$, or $10+1$, or 11, terms. And in the same manner it may be proved that, since, when m is $= 10$, the m th power of $1 + 1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, + 1$, consisting of $m+1$, or $10+1$, or 11, terms, the $m+1$ th, or

or $10+1$ th, or 11 th, or (putting $n = m+1 = 10+1 = 11$) the n th, power of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$, or $11+1$, or 12 , terms.

And so we may proceed from number to number *ad infinitum*. And consequently, whatever be the whole number denoted by m , it will always be true that $\overline{1+1}^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c + 1$, consisting of $m+1$ terms. Q. E. D.

The foregoing Demonstration expressed in a more concise Manner.

40. The foregoing reasonings may be expressed in a more concise manner as follows. If n be $= m+1$, and it be true in any particular value of m that $\overline{1+1}^m$ is $=$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$, it will also be true that $\overline{1+1}^n$ will be $= 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c$.

For $\overline{1+1}^n$ is $= \overline{1+1}^{m+1} = \overline{1+1}^m \times \overline{1+1} =$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$

$$\begin{aligned}
& \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, \text{ multiplied into } 1 + 1 = \\
& 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c. \\
& + 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \&c. \\
& = 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c. \\
& + 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c. \\
& = 1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c. \\
& = 1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c. \\
& = 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c.
\end{aligned}$$

But it has been shewn in art. 17, 18, 19, &c, - - - 26, that, when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, $1 + 1^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c.$ Therefore, if n be $\pm 9+1$,

or 10, $1+1^n$, or $1+1^{10}$, will be $=$ the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c.$ And it may be shewn in like manner, that, if n be put for 11, 12, 13, 14, &c, *ad infinitum* successively, $1 + 1^n$ will, in all these suppositions, be always equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c;$ and therefore the proposition is universally true, whatever be the whole number denoted by the letter n .

Q. E. D.

41. This

41. This demonstration of the binomial theorem, in the case of integral powers, is nearly the same with that given by Mr. John Stewart, of Aberdeen, in the 6th Section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis by Equations of an infinite number of Terms*. See his edition of Newton's Treatise on the Quadrature of Curves, and of the said Tract intitled *Analysis, &c.* with his learned Comments on both, in one volume, quarto, published at London, in the year 1745, page 471, Art. 155.

Of the Powers of a Residual Quantity $a - b$, when their Indexes are whole Numbers.

42. We have hitherto been considering the integral powers of a binomial quantity $a + b$, or of the sum of two single quantities a and b ; and we have seen that, if the said binomial quantity $a + b$ be raised to any power of which a whole number denoted by m is the index, the quantity $a + b$ raised to the m th power of $a + b$, will be equal to the series $a^m + \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c. + b^m$, or (if we put $A = 1$, $B = \frac{m}{1} A$, $C = \frac{m-1}{2} B$, $D = \frac{m-2}{3} C$, $E = \frac{m-3}{4} D$, $F = \frac{m-4}{5} E$, and $G, H, I, K, L, \&c. = \frac{m-5}{6} F, \frac{m-6}{7} G, \frac{m-7}{8} H, \frac{m-8}{9} I, \frac{m-9}{10} K, \&c.$ respectively,) to the series $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \&c. + b^m$;

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in

in which all the terms after the first term a^m are marked with the sign +, or are added to the said first term. We will now proceed to consider the value of $\overline{a-b}^m$, or the m th power of the residual quantity $a-b$, or of the difference of the two quantities a and b , upon a supposition that a is the greater of the two.

43. Now, if a be supposed to be greater than b , and m be any whole number whatsoever, the quantity $\overline{a-b}^m$, or the m th power of the residual quantity, or difference, $a-b$, will be equal to the series $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c$, or (if we put A , as before, $= 1$, and $B = \frac{m}{1} A$, and $C = \frac{m-1}{2} B$, and $D = \frac{m-2}{3} C$, and $E = \frac{m-3}{4} D$, and $F = \frac{m-4}{5} E$, and $G, H, I, K, L, \&c, = \frac{m-5}{6} F, \frac{m-6}{7} G, \frac{m-7}{8} H, \frac{m-8}{9} I, \frac{m-9}{10} K, \&c$, respectively,) to the series $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 + \&c$, which consists of exactly the same terms as the series that is equal to $\overline{a+l}^m$, or the same power of the binomial quantity $a+b$, but with the sign — prefixed to the second, and fourth, and sixth, and every following even term in the series, which denotes that the said terms are not to be added to the first term a^m , and to the third, and fifth, and other following odd terms, (as they were in the former series, which was equal to $\overline{a+l}^m$), but to be subtracted from them.

44. That

44. That this must be so, will be evident from considering the manner in which the several powers of the residual quantity $a - b$ are generated from each other by the continual multiplication of $a - b$, of which we will now exhibit a specimen with respect to a few of its lowest powers. The second, third, fourth, and fifth powers of $a - b$ are derived from $a - b$ itself by the following multiplications.

$$\begin{array}{r} a-b \\ a-b \\ \hline aa-ab \\ -ab+bb \end{array}$$

$$\begin{array}{r} aa-2ab+bb = \overline{a-b}^2. \\ a-b \end{array}$$

$$\begin{array}{r} a^3-2a^2b+ab^2 \\ -a^2b+2ab^2-b^3 \end{array}$$

$$\begin{array}{r} a^3-3a^2b+3ab^2-b^3 = \overline{a-b}^3. \\ a-b \end{array}$$

$$\begin{array}{r} a^4-3a^3b+3a^2b^2-ab^3 \\ -a^3b+3a^2b^2-3ab^3+b^4 \end{array}$$

$$\begin{array}{r} a^4-4a^3b+6a^2b^2-4ab^3+b^4 = \overline{a-b}^4. \\ a-b \end{array}$$

$$\begin{array}{r} a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\ -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \end{array}$$

$$\begin{array}{r} a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 = \overline{a-b}^5. \end{array}$$

45. From these operations it is evident that, wherever the odd powers of b occur in the said powers of $a - b$, the terms are marked with the sign $-$, and that, wherever the even powers of b occur in the said powers of $a - b$, the terms are marked with the sign $+$. And the same thing, it is evident, must happen in all higher powers of $a - b$ whatsoever, as well in those that have been here set down, because

b is marked with the sign $-$ in the two original factors $a - b$ and $a - b$; whence it follows, from the nature of algebraick multiplication, that, whenever b is multiplied into itself an even number of times, the product will be marked with the sign $+$; and, whenever it is multiplied into itself an odd number of times, the product will be marked with the sign $-$. And it is further evident, from the foregoing multiplications, that the odd powers of b occur in the second, and fourth, and sixth, terms of the foregoing products, and that the even powers of b occur in the third and fifth terms of them. And it is easy to see that the odd powers of b will occur in like manner in the eighth, and tenth, and twelfth, and other following even terms of all higher powers of $a - b$ whatsoever, and that the even powers of b will occur in like manner in the seventh, and ninth, and eleventh, and other following odd terms of the said higher powers of $a - b$. And it is also evident, from the foregoing multiplications, that the terms themselves of which the several powers of $a - b$ will be composed, are exactly the same with the terms of which the same powers of $a + b$ are composed. And hence it follows that the series which is equal to $\overline{a - b}^m$ will be the same with the series which is equal to $\overline{a + b}^m$, when the sign $-$ has been prefixed to the second, and fourth, and sixth, and other following even terms of it, instead of the sign $+$, or that $\overline{a - b}^m$, or the m th power of the residual quantity

$$a - b, \text{ will be equal to the series } a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c, \text{ or } a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 + \&c.$$

Q. E. D.

A DIS-

A
DISCOURSE
OF
COMBINATIONS,
ALTERNATIONS,
AND
ALIQUOT PARTS.

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OF
COMBINATIONS,
ALTERNATIONS,
AND
ALIQUOT PARTS.

CHAP. I.

Of the variety of Elections, or Choice, in taking or leaving One or more, out of a certain Number of things proposed.

FOR the better understanding of what is proposed ; suppose we a certain number of counters or other things exposed ; as, for instance, 7 ; $abcdefg$: The question is, what variety, or how many cases there may be, of taking from thence one, or two of them ; as a, b, c, d , &c. Or, ab, ac, ad, bc, bd , &c. Or, three's, as abc, abd, acd, bdc , &c. Or, fours, fives, &c. Or all, or none ? And the like if any other number of things were so exposed.

In order to the solution whereof, I shall here insert a table, borrowed from my Arithmetick of Infinites, Prop. 132, 169, 183, 189, &c. (because there will be often occasion of having recourse to it.) And then proceed to propositions thereunto relating.

To

		To be left.											
		0	1	2	3	4	5	6	7	8	9	10	
Numbers.	Monadicks.	1	1	1	1	1	1	1	1	1	1	1	0
	Laterals.	1	2	3	4	5	6	7	8	9	10		1
	Triangulars.	1	3	6	10	15	21	28	36	45			2
	Pyramidals.	1	4	10	20	35	56	84	120				3
	Triang. Triang.	1	5	19	35	70	126	210					4
	Triang. Pyram.	1	6	21	56	126	280						5
	Pyram. Pyram.	1	7	28	84	210							6
	&c.	1	8	36	120								7
		1	9	45									8
		1	10										9
	1											10	

To be taken.

Now, as to the construction of this table, we are to observe, that, (the first ^{line} being all units,) the following numbers are, in every place, the aggregate of all those in the line next above it, so far. As for example; for the three first in the uppermost line, 1, 1, 1, we have in the second line (under the last of them) the number 3, which is the aggregate of them. And, in like manner, we have in the next place 4, which is the aggregate of 1, 1, 1, 1, (And so of the rest.) And, in the lines following, likewise: So for 1, 2, 3, (the three foremost of the second line,) we have in the third line (under the last of them) the number 6, equal to all of them: and so every where. This premised, the propositions follow.

1. It is manifest, that, if we would *take none*, that is, if we would *leave all*, there can be but one case thereof, whatever be the number of things exposed. (For this admits of no variety.) Which (in the table) is expressed in the first (transverse) line, where the numbers are all *Monadicks*, or units.

2. The same happens, if we would *take all*, (or *leave none*.) For here also there can be no variety of choice, whatever be the number of things exposed, *a, b, c, &c.* And this, in the table, we express in the first (erect) column, where also the numbers are all *Monadicks*.

*Leftmost columns for the leavings & horizontal
columns for the takings.*

3. If

3. If we would *take One*, it is manifest, that there are as many cases or varieties of choice, as is the number of things. For that *One* may be any one of them, as *a, b, c, d, e, f, g*; which is expressed in the secondnd line, where the numbers are in their natural order or consecution, 1, 2, 3, &c, which I call *Laterals*, *Horizontal*
Alternative

4. The same happens, if, taking all the rest, we *leave One*; that is, if we take *All but One*. For it is manifest, there is the same variety of *leaving One* as of *taking One*, as *abcdef, abcdeg, abcdg, abcefg, abdefg, acdefg, bcdefg*, which is signified in the secondnd column, where the numbers are also *Laterals*. *Effect of*
logical
column

5. If we would *take Two*; It is manifest, that we may first take *a*, combined with any other of the rest; as *ab, ac, ad, ae, af, ag*; the number of which combinations are therefore as many as the number of things wanting one. We may then take *b* (omitting its combination with *a*, as being already taken,) combined with every of those which follow it; as *bc, bd, be, bf, bg*; the number of which combinations are therefore as many as the number of things exposed, *wanting Two*. In like manner, *c*, (omitting its combinations with *a* and *b*, because *ca, cb*, are but the same with *ac, bc*, already taken,) may be further combined with every of those which follow it, (which are so many as is the number of things exposed *wanting Three*,) as *cd, ce, cf, cg*; and the fourth *d*, (omitting *da, db, dc*, as being the same with *ad, bd, cd*, already taken,) may be further combined with every of those which follow, (which are as many as the number of things *wanting Four*,) as *de, df, dg*. And in like manner for the fifth, sixth, &c; each of which affords new combinations fewer by one than that next before it, till at length we come to 1, as *ef, eg*, and *fg*. So that the number of all these combinations, is the aggregate of all the numbers in the same line so far; that is, in the present case, (the num-

<i>ab</i>	<i>ac</i>	<i>ad</i>	<i>ae</i>	<i>af</i>	<i>ag</i>	6
<i>bc</i>	<i>bd</i>	<i>be</i>	<i>bf</i>	<i>bg</i>		5
<i>cd</i>	<i>ce</i>	<i>cf</i>	<i>cg</i>			4
<i>de</i>	<i>df</i>	<i>dg</i>				3
<i>ef</i>	<i>eg</i>					2
<i>fg</i>						1
						21

ber of things exposed being 7,) the combinations are, $6 + 5 + 4 + 3 + 2 + 1 = 21$. To which answers (in the third transverse, or horizontal line of the foregoing Table of the Figurative numbers,) the *Triangular* number 21, just under the number 6, (which is less by one than the number of things exposed.) Such *Triangular* numbers, being the aggregate of all the *Laterals* so far. And universally, (whatever be the number of things exposed) the number of Two's, is a *Triangular number*, whose side is less by one than the number of things exposed.

6. The same happens, if we are to take *All but Two*; for there is the same variety of leaving Two, as of taking Two; that is, in both cases, so many as is the triangular number, whose side is less by one than the number of things exposed, which (in the table,) is signified in the third column, whose numbers are the same with those of the third line.

7. If we would take *Three*, it is manifest, that first, *ab*, (the first and second,) may be combined with every of those that follow; the number of which are as many as the things exposed *wanting Two*, (which therefore afford us so many different Triads, or Three's,) as *abc, abd, abe, abf, abg*. Then that *ac* (the first and third,) may be combined (omitting *acb*, as being the same with *abc* already taken,) with every of those that follow, (which therefore afford us so many new *Three's*, as is the number of things *wanting Three*,) as *acd, ace, acf, acg*. And, in like manner, *a* coupled with those that follow, (as *ad, ae, af*,) may each of them be further combined with their respective subsequents, affording each of them new Triads, fewer by one than that next before it, till at length we come to 1, as *ade, adf, adg*, and *ae, aef, aeg*, and *afg*. (But *ag* affords none, because *g* being the last, there is none remaining with which it might be combined.) The aggregate of all which, is a *Triangular number* (as being an aggregate of *Laterals*,) whose side is less by two, than the number of things exposed; that is, in the present

<i>abc</i>	<i>abd</i>	<i>abe</i>	<i>abf</i>	<i>abg</i>	5
<i>acd</i>	<i>ace</i>	<i>acf</i>	<i>acg</i>		4
	<i>ade</i>	<i>adf</i>	<i>adg</i>		3
		<i>aef</i>	<i>aeg</i>		2
			<i>afg</i>		1
					15

present case, $5 + 4 + 3 + 2 + 1 = 15$; which is a triangular number of the side 5; which is less by two, than 7, the number of things exposed, in all which, *a* is one of the Ingredients.

In like manner (omitting all the Triads wherein *a* is an Ingredient, as being already taken,) *bc* (the second and third) may be further combined with each of those that follow *d, e, f, g*, affording us as many new Triads as did *ac*, (which was before so compounded,) that is, so many as is the number of things wanting *Three*. And then again *bd, be, bf*, afford as many as *ad, ae, af*, did before. Which afford us a new Triangular number, whose side is less by one than that we had before; that is, $4 + 3 + 2 + 1 = 10$, whose side is 4; in all which Triads *b* is the leader.

$$\begin{array}{r|l} bcd & 4 \\ bce & 3 \\ bcf & 2 \\ bcd & 1 \\ \hline & 10 \end{array}$$

In the same manner may be shewed, that (omitting the combinations of *a* and *b*,) those Triads wherein *c* is the leader, will give another Triangular number, whose side is yet less by one, and so onward continually till we come at 1: as $3 + 2 + 1 = 6$, a triangular number whose side is 3; and $2 + 1 = 3$, a Triangular number, whose side is 2; and 1, a Triangular number whose side is also 1.

$$\begin{array}{r|l} cde & 3 \\ cdf & 2 \\ cde & 1 \\ \hline & 6 \end{array}$$

$$\begin{array}{r|l} def & 2 \\ def & 1 \\ \hline & 3 \end{array}$$

$$\begin{array}{r|l} efg & 1 \\ \hline & 1 \end{array}$$

And then the aggregate of these Triangulars is 35, a *Pyramidal* number, which (in the fourth line,) stands next under 15, the greatest of them, whose side is less by two, than the number of things exposed; that is, a *Pyramidal number whose side is less by two than the number of things exposed*. And so many are the several Triads which may be had in such number of things exposed; that is, in the present case, $15 +$

$$\begin{array}{r|l} 5, 4, 3, 2, 1 & 15 \\ 4, 3, 2, 1 & 10 \\ 3, 2, 1 & 6 \\ 2, 1 & 3 \\ 1 & 1 \\ \hline & 35 \end{array}$$

2 N 2

10 T,

$10 + 6 + 3 + 1 = 35$, which is represented in the fourth line, which is of pyramidal numbers.

8. The same happens, if instead of taking Three, we take *All but Three*. For the same variety of cases happens, if now we take what were before left, and leave what were then taken. And as that is represented in the fourth line, *vertical* so this is in the fourth column.

9. If we would take *Four*; then, with *a*, may be made so many Fours (or Quaternions,) as may be formed Triads of those that follow, (as *b, c, d, e, f, g*;) that is, (by art. 7,) a pyramidal number whose side is less by Two than the number of these; that is, less by Three than the number of things exposed; that is, in the present case, 20; which is a Pyramidal number of the side 4, which is less by Three, than 7.

In like manner, (omitting *a*;) there may with *b*, be so many Quaternions formed, as may be Triads of those that follow it, (as *c, d, e, f, g*;) that is, a Pyramidal number whose side is less by 1, than that foregoing; that is, 10; whose side 3 is less by 4 than 7.

And (omitting *a, b*;) there may with *c* be formed so many Quaternions, as may of those that follow it (*d, e, f, g*;) be formed Triads; that is, a Pyramidal number whose side is yet less by 1: that is, 4, whose side is 2. And so onward, till we come at 1.

And then the aggregate of all these Pyramidals, that is, the number in the fifth line, next under the greatest of them, is (what they call) a *Trianguli-triangular* number, whose side is less by three than the number of things exposed. That is, in the present case, (where the number of things is 7,) $20 + 10 + 4 + 1 = 35$, (a trianguli-triangular number, of the side $4 = 7 - 3$;) is the number of different Quaternions which may be had when the things exposed are 7.

(If any like not the name of *trianguli-triangular*, and so of the rest that follow; I am content to change them. For I am not fond of them, but use them because I find them.)

Which

Which number is the same which before we had for Three's; which hence comes to pass, because, when the number of things is 7, the number 4, is the same with *All wanting 3*; where the variety is the same as if 3 were taken; as is shewed in the preceeding article.

10. The same happens, (for the reasons already shewed,) if we were to take *All wanting Four*. And as that is to be found in the fifth line, so this, in the fifth column, whose numbers are the same with those of the fifth line.

11. In the same manner will be shewed, that, if we would take *Five* (or *All but Five*,) the varieties are then so many as is the aggregate of the numbers in the fifth line, ending with that whose side is *less by Four* than the number 15 of things exposed. That is, the number in the sixth 5 line (which is *Trianguli-pyramidal*) next under the 1 greatest of those, whose side is *less by Four* than the 21 number of things exposed. That is, in the present case, $15 + 5 + 1 = 21$, a *Trianguli-pyramidal* number, whose side is $31 = 7 - 4$. And so, if *Six* are to be taken, (or *All but Six*,) the varieties are so many, as is the aggregate of numbers in the sixth line (or the number answering thereunto in the seventh,) ending with that whose 6 side is *less by Five* than the number of things exposed. 1 And so for *Seven*, *Eight*, &c, (or all but seven, eight, 7 &c,) we are to take the numbers of the following lines, ending with that whose side is less by one, than that for the line next above. As, in the present, (where 7 is the number of things exposed,) the number of sixes is 7; the number of sevens is 1.

12. All these varieties of choice, for any number of things exposed, are found in the Table foregoing, in a rank of numbers obliquely descending; in which that number which is the number of things exposed, is to be found in the second line, and again in the second column, both which are of Laterals. As, in the present case (where 7 is the number of things exposed,) in the oblique descent passing by

by 7 in the second line, and again in the second column; we have the numbers 1, 7, 21, 35, 35, 21, 7, 1; which represent the variety of cases for taking, 0, 1, 2, 3, 4, 5, 6, 7. And the like for any other number of things exposed.

13. And these numbers (as appears upon view,) are the same with those which are called *uncies*, prefixed to the proportionals that constitute the respective powers of a binomial root; or, (which is the same) the respective powers of $1 + 1$ considered as a binomial root. That is, the root, square, cube, fourth, fifth power, &c, of $1 + 1$, according as the number of things exposed are 1, 2, 3, 4, 5, 6, &c.

14. The table thus begun, is easily continued as far as there is occasion: for the number of each place, is the aggregate of two numbers, whereof one is next above it, and the other next before it, as $15 = 5 + 10$, $20 = 10 + 10$, $35 = 20 + 15$. And so every where.

15. Having therefore any number of things exposed, let that number be sought in the second line, (which is of *Laterals*), and again in the second column; and then, in the sloping rank of numbers passing through these two, we have the number of cases for taking 0, 1, 2, 3, 4, &c, in such order as the index on the side directs; and likewise for taking *All but* 0, 1, 2, 3, 4, &c, in such order as the index on the top directs.

16. And if we would have the sum of all these varieties (for any such number of things proposed) all together, it is had by adding the numbers of such sloping rank; as in the present case, $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128$.

17. Which number is always that power of the number 2, (that is, of $1 + 1$) which is of so many dimensions as is the number of things exposed, (or that power whose exponent

is

is such number,) that is, the product of so many two's continually multiplied, (as, in the present case, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$;) or, 1 so many times doubled as is the number of things exposed. That is, for 0, it is 1. (For here, to take all, or to leave all, is but one and the same case.) For 1, it is (the side) $1 + 1$. For 2, (the square) $1 + 2 + 1 = 4$. For 3, (the cube) $1 + 3 + 3 + 1 = 8$. For 4, (the biquadrate) $1 + 4 + 6 + 4 + 1 = 16$. And so of the rest.

18. And thus far we have considered the variety of cases concerning taking or leaving, *None, One, Two, Three, &c.*, of any number of things exposed, without regarding the order of them, so that *abc, acb, bac, bca, &c.*, are reputed for one and the same case. But if the different alternations, or changes of order, in the same things, be accounted as different cases; this we are to consider in the next Chapter. And if therein, some two or more are indifferently reputed as one and the same, or indifferently to be taken each for other; what abatement of the former number will hereupon arise, is considered in the same Chapter.

19. If, by *Combination*, we understand the taking of two or more, (but not of one, or none;) then, out of the number of cases before found, we must abate so many as is the number of things exposed, and one more. For, of those, so many as is the number of things exposed, answers to the cases of 1. And one more, answers to the case of taking *None*. But all the rest are combinations in that sense. For though *Combination* (as coming from *Bini*;) in its proper signification extend only to the taking of *couples*, (or Two's;) yet in common acceptation the word is now used of greater numbers. And, in English also, we scruple not to say, that Three, or Four, (or more than so,) are *coupled* together, that is, connected.

20. If, out of the former number of cases, we please to exclude that of *taking None*, or 0, (because, *to take none, is not to take*;) then is the number of cases fewer by one, than

than is above expressed. And so we have the cases of taking one or more. And so many are the number of *Divisors* of a number compounded of so many different Prime numbers continually multiplied, as are the cases of taking one or more of so many things exposed.

21. And if further we abate one more (which answers to the case of taking all;) then have we the number of *Aliquot Parts* of a number so composed of different Primes or Incomposite numbers. The number of *Aliquot Parts* being fewer by one, than is the number of divisors.

I shall subjoin to this Chapter (as properly appertaining to this place,) an Explication of the *Rule of Combination*, which I find in *Buckley's Arithmetick*, at the end of *Sesson's Logick*, (in the Cambridge edition;) which (because obscure,) Mr. George Fairfax (a Teacher of the Mathematics then in Oxford,) desired me to explain; to whom (Sept. 12, 1674.) I gave the explication under written; Consonant to the doctrine of this Treatise, (which had been long before written, and was the subject of divers public Lectures in Oxford, in the years 1671, 1672.)

REGULA COMBINATIONIS.

*Quot fuerint Numeri, quos Combinare velimus;
Tot sint et series, quibus est proportio dupla;
Quarum principium ducatur semper ab Uno.
Omnes hæc series conjunge per Additionem:
Producto, numerum quot Combinatio constat,
Aufer. Quod superest, numerum citas; unde patebit,
Quot faciant numeros distinctos, undique siquis
Propositos numeros velit in se Multiplicare.
Si nihil à summâ prædictâ surripiatur;
Restabunt partes Aliquotæ, quæ numerabunt
Illum, qui numeros est inter Maximus omnes,
Ex ductu in sese numerorum provenientem.*

I have taken the liberty, to alter the pointing (so as to make the sense the clearer,) and to restore (in the second verse)

verse) *sint*, for *sunt*; and (in the third verse) *principium*, for *principio*; which had been misprinted. And (in the fifth verse) I have restored *numerus*, for *numeros*; for it is but one Number that is to be subducted; namely, the Number of those Numbers which are to be Combined. My Explication was this:

“ Let as many Numbers as you please, be proposed to be
“ Combined: Suppose *Five*, which we will call *a b c d e*.

1	<i>a</i>	<i>ab</i>	<i>abc</i>	<i>abcd</i>	<i>abcde</i>	
2	<i>b</i>	<i>ac</i>	<i>abd</i>	<i>abce</i>		
4	<i>c</i>	<i>ad</i>	<i>abe</i>	<i>abde</i>		1
8	<i>d</i>	<i>ae</i>	<i>acd</i>	<i>acde</i>		
16	<i>e</i>	<i>bc</i>	<i>ace</i>	<i>bcd</i>		
—	—	<i>bd</i>	<i>ade</i>	—	5	
31	5	<i>be</i>	<i>bcd</i>	5	10	10
—5		<i>cd</i>	<i>bce</i>		10	10
—		<i>ce</i>	<i>bde</i>		5	5
26		<i>de</i>	<i>cde</i>		1	1
		—	—		—	—
		10	10		31	26

“ Put, in so many Lines, Numbers, in duple proportion,
“ beginning with 1.

“ The Sum (31) is the Number of Sumptions, or Elections;
“ wherein, one or more of them, may several ways be
“ taken.

“ Hence subduct (5) the Number of the Numbers pro-
“ posed; because each of them may once be taken singly.

“ And the Remainder (26) shews how many ways they
“ may be taken in Combination; (namely, Two or more at
“ once.)

“ And, consequently; how many Products may be had
“ by the Multiplication of any two or more of them so
“ taken.

“ But the same Sum (31) without such Subduction, shews
“ how many Aliquot Parts there are in the greatest of those
“ Products, (that is, in the Number made by the continual
“ Multiplication of all the Numbers proposed,) *abcde*.
“ For every one of those Sumptions, are Aliquot Parts of

2 0

“ *abcde*,

“ *abcde*, except the last, (which is the whole,) and instead thereof, *i* is also an Aliquot Part; which makes the number of Aliquot Parts, the same with the Number of Sumptions.

“ Only here is to be understood, (which the Rule should have intimated;) that, all the Numbers proposed, are to be Prime Numbers, and each distinct from the other. For if any of them be Compound Numbers, or any Two of them be the same, the Rule for Aliquot Parts will not hold.”

CH A P. II.

Of Alternations, or the different Change of Order, in any Number of Things proposed.

SUPPOSE we a certain Number of things exposed, different each from other, as *a, b, c, d, e, &c.* The question is, how many ways the order of these may be varied? as, for instance, how many changes may be Rung upon a certain Number of Bells; or, how many ways (by way of Anagram) a certain Number of (different) Letters may be differently ordered?

- | | |
|--|--|
| $\begin{array}{l} a \} 1 \\ ab \} 2 \\ ba \} 2 \\ \hline 1 \times 2 = 2 \end{array}$ | <p>1. If the thing exposed be but <i>One</i>, as <i>a</i>, it is certain, that the order can be but one. That is 1.</p> <p>2. If <i>Two</i> be exposed, as <i>a, b</i>, it is also manifest, that they may be taken in a double order, as <i>ab, ba</i>, and no more. That is $1 \times 2 = 2$.</p> |
|--|--|

3. If

3. If *Three* be exposed; as *a, b, c*: Then, beginning with *a*, the other two *b, c*, may (by art. 2,) be disposed according to Two different orders, as *bc, cb*; whence arise Two Changes (or varieties of order) beginning with *a*, as *abc, acb*: And, in like manner it may be shewed, that there be as many beginning with *b*; because the other two, *a, c*, may be so varied, as *bac, bca*. And again as many beginning with *c*, as *cab, cba*. And therefore, in all, Three times Two. That is, $1 \times 2, \times 3 = 6$.

$\left. \begin{array}{l} abc \\ acb \\ bca \\ bac \\ cab \\ cba \end{array} \right\} 2$
 $2 \times 3 = 6$

4. If *Four* be exposed, as *a, b, c, d*; Then, beginning with *a*, the other Three may (by art. preceeding) be disposed six several ways. And (by the same reason) as many beginning with *b*, and as many beginning with *c*, and as many beginning with *d*. And therefore, in all, Four times six, or 24. That is, the Number answering to the case ~~now~~ foregoing, so many times taken as is the Number of things here exposed. That is, $1 \times 2 \times 3, \times 4 = 6 \times 4 = 24$.

$\left. \begin{array}{l} abcd \\ abdc \\ acbd \\ acdb \\ aadb \\ adcb \end{array} \right\} 6$
 $\left. \begin{array}{l} bacd \\ badc \\ bcad \\ bcda \\ bdac \\ bdca \end{array} \right\} 6$

5. And in like manner it may be shewed, that this Number 24 Multiplied by 5, that is $120 = 24 \times 5 = 1 \times 2 \times 3 \times 4 \times 5$, is the number of alternations (or changes of order) of *Five* things exposed. (Or, the Number of Changes on Five Bells.) For each of these five being put in the first place, the other four will (by art. preceeding) admit of 24 varieties, that is, in all, five times 24. And, in like manner, this Number 120 Multiplied by 6, shews the Number of Alternations of 6 things exposed; and so onward, by continual Multiplication by the consequent Numbers 7, 8, 9, &c.

$\left. \begin{array}{l} cabd \\ cadb \\ cbad \\ cbda \\ cdab \\ cdba \end{array} \right\} 6$
 $\left. \begin{array}{l} dabc \\ dacb \\ dbac \\ dbca \\ dcab \\ dcba \end{array} \right\} 6$
 $4 \times 6 = 24$

6. That is, how many so ever of Numbers, in their natural Consecution, beginning from 1, being continually Multiplied,

2 O 2

tiplied, give us the Number of Alternations (or Change of order) of which so many things are capable as is the last of the Numbers so Multiplied, As for instance, the Number of Changes in Ringing Five Bells, is $1 \times 2 \times 3 \times 4 \times 5 = 120$. In Six Bells, $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 120 \times 6 = 720$. In Seven Bells, $720 \times 7 = 5040$. In Eight Bells, $5040 \times 8 = 40320$. And so onward, as far as we please.

Thus *Vossius* tells us, (*Cap. 7, De Scientiis Mathematicis*,) That if an Host promise to entertain seven Guests so long as they sit every day in a different order, this extends to 14 years. He means, *almost* so many years, namely, 5040 days, which of 14 years wants 73 or 74 days, according as the Leap-years may chance to fall.

7. This Number of Alternations, according as the Number of things exposed doth increase, will proceed to a vast Multitude beyond what at first one would expect. As for Example, the 24 Letters will admit of so many Varieties or Alternations in Changing their order, as that if so many Bells were to be Rung according to all those Changes, it could not have been dispatched (as the Learned *John Gerard Vossius*, in the place last cited, doth observe,) from the beginning of the World to this day. I add; no, nor if for every Minute of an hour which hath passed, there had passed *Ten Thousand Thousand Years*; as will appear by the following Computation.

1	1 X
2	2 X
6	3 X
24	4 X
120	5 X
720	6 X
5,040	7 X
40,320	8 X
362,880	9 X
3,628,800	10 X
39,916,800	11 X
479,001,600	12 X
6,227,020,800	13 X
87,178,291,200	14 X
1,307,674,368,000	15 X
20,922,789,888,000	16 X
355,687,428,096,000	17 X
6,402,373,705,728,000	18 X
121,645,100,408,832,000	19 X
2,432,902,008,176,640,000	20 X
51,090,942,171,709,440,000	21 X
1,124,000,727,777,607,680,000	22 X
25,852,016,738,884,976,640,000	23 X
620,448,401,733,239,439,360,000	24 X

In 1 year.
 365 $\frac{1}{4}$ days.
 X 24
 1460
 730
 6.
 8766 hours.
 X 60
 525,960 Minutes
 In 6000 years.
 3,155,760,000 Minutes
 X 5
 15,778,800,000 Changes.
 525,960 Min. in 1 year.
 946728000000
 1420092
 788940
 315576
 788940
 8,299,017,648,000,000
 10,000,000
 82,990,176,480,000,000,000,000

For, supposing in one year,
 365 $\frac{1}{4}$ days; and, from the be-
 ginning of the World, to have
 passed 6000 years; (both of
 which suppositions are at the
 largest,) and therefore the Num-
 ber of minutes in all that time,
 3,155,760,000. Suppose we
 then, in every Minute of an hour,
 5 Changes to be dispatched, that
 is, (because of 24 Bells) 120
 strokes successively one after an-
 other, (which allowance is also
 at the largest;) and therefore,
 in 6000 years, 15,778,800,000
 Changes, which Number if we
 Multiply by 525,960, (the Num-
 ber

ber of Minutes in one year,) we have 8,299,017,648,000,000 for the Number of Changes to be dispatched in so many years as there have been Minutes, which Multiplied by 10,000,000, (Ten Thousand Thousand, or Ten Millions,) will be but 82,990,176,480,000,000,000,000, which is less than 620,448,401,733,239,439,360,000, the Number of Changes whereof 24 Bells are capable.

Nay, if we should proceed no further than to 14 Bells, and allow 10 Changes (that is, 140 strokes) to every Minute, the Number of Minutes requisite to Ring them all would be 8,717,829,120, (a tenth part of the Number of Changes,) which is more than double (almost treble) the Number of Minutes in 6000 Years; and would require more than 16 Thousand Years (yea, more than 16,575 Years) to Ring them all.

8. Hence it may appear, how many ways the Letters of a Name or Word, (supposing them to be all several,) may be differently disposed by way of Anagram; (out of which those that are of use may be selected, neglecting the rest,) by art. 6. For Example, the Word *ROMA*, (consisting of four different Letters) may admit of Changes $24 = 1 \times 2 \times 3 \times 4$.

<i>Roma</i>	orma	mroa	arom
roam	oram	mrao	armo
rmoa	omra	mora	aorm
rmao	omar	moar	aomr
raom	oarm	maro	amro
ramo	oamr	maor	amor

Of which (in Latin) these seven are only useful; *Roma*, *ramo*, *oram*, *mora*, *maro*, *armo*, *amor*. The other forms are useless, as affording no Latin Word of known signification.

9. But in case some one or more of the Letters do occur more than once, the Number of Alternations so found as before, must be divided by such Number or Numbers as such repetitions do require: Namely, if the same Letter do twice occur, we are to divide by 2; if three times, by 6; if four times,

times, by 24; and so onward, according to the varieties that such a Number is capable of. For, if the Letters *a* and *b* be reputed for the same; then, whereas (the rest remaining as before) *ab* and *ba* would every where afford two varieties, they are in this case to pass for one, and therefore the Number of cases will be only half so many as otherwise they would be. In like manner (the rest remaining as before) *abc* would every where (according as they may change places one with another) afford six varieties; but in case the three Letters *a, b, c* be considered as being all the same, or as being *a, a, a*, these Six cases must then pass but for one. And in like manner, if *abcd* be severals, they afford (the rest remaining as before) 24 varieties; but, if the same, these 24 must pass but for one: And the like in other cases. And, if more Letters be so repeated, there must be for each of them such division.

For Example, the Word *MESSES* having 6 Letters, if they were all different, the Alternations would be $720 = 1 \times 2 \times 3 \times 4 \times 5 \times 6$. But because the Letter *e* comes twice, that Number is to be divided by 2. (For if instead of *ee*, we put *en*, then *messns* and *mnssee* would be two forms, both which are now Co-incident in *messes*: And so every where.) Again, because the Letter *s* comes three times, we are (for the like reason) to divide by 6. (For if those three were three different, they would in every position of the rest, afford 6 cases, all which are now Co-incident in *sss*.) And therefore, (because both happen,) 720 being divided by 2, and again by 6, the different varieties will be $\frac{720}{2 \times 6} = 60$.

meesss	emesss	esmsse	smeess	seesms	ssmsee
mesess	emsess	esemss	sme ses	seessm	sse mes
<i>messes</i>	emsses	esesms	smesse	sesmes	ssemse
messe se	emasse	esessm	smsees	sesmse	sseems
mseess	eesmss	essmes	smseee	sesems	sseesm
mseses	eesmss	essmse	smsee	sesesm	sse sme
msesse	eesmss	essmes	semess	sessme	ssesem
mssees	eesmss	essesm	sem ses	sessem	sssmee
msseee	eesmss	esssme	semsee	ssmees	ssseme
msseee	eesmss	esssem	seemss	ssmeee	ssseem

Of

Of all which varieties, there is none beside *messes* itself, that affords an useful Anagram.

In like manner we may shew, that the Letters *abbccddddd* will admit of $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3628800}{2 \times 6 \times 24 = 288} = 12600$

several varieties : And *abbccdd*, of, $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040}{2 \times 2 \times 2 = 8}$

$= 630$: And *aaabbcccc*, of $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320}{6 \times 2 \times 6 = 72} =$

560. And the like in other cases however varied.

10. The converse of this, is of like use, when what was considered but as one and the same several times repeated, comes afterward to be distinguished. For then the Number before found, is to be so often Multiplied as the Number of things so distinguished shall require.

As, in the Word *messes* before mentioned, where *sss* are considered but as one Letter thrice repeated, and *ee* as the same twice repeated, the Number of different positions is 60 ; but if *sss* be distinguished as three severals, and *ee* as two severals, the Number of all will be $60 \times 6 \times 2 = 720$.

Thus *Voffius*, Cap. 7, *De Scientiis Mathematicis*, tells us that this verse,

Rex, lex, sol, lux, dux, fons, mons, spes, pax, petra, Christus.

which (consisting of 11 Words) may be turned (absolutely) 39,916,800 ways ; and so as to preserve the Rules of an Hexameter verse, be turned 3,628,800 ways, he should rather have said 3,265,920. That is, the 9 Monosyllables (which may promiscuously take each other's place) 362,880 times ; and *Christus* is capable of 9 (not 10) different positions ; that is, in the first, second, third, fourth, fifth, sixth, seventh, eighth, (but not in the ninth, and tenth,) and in the last place ; (and *petra* confined, by the nature of the verse, to the place next before the last spondee.) That is, $362,880 \times 9 = 3,265,920$ ways :

He

He says also that the verse

Tot tibi sunt dotes, virgo, quot sidera celo;

may be turned absolutely 46320 ways; and, so as to preserve the verse 1022 ways; which is very true, (and I have been told, of somebody, who, in praise of the Virgin Mary, had made a Book of that verse turned so many ways, which was wont to be reputed the Number of the Fixed Stars, according to the ancient Catalogue of them.) But it is true also, that it may be turned many more ways than so, and yet preserve the verse true: Namely, 2628, retaining the quantity of the last Syllables in *tibi* and *virgo* as before; and 468, Changing their quantity in *virgo tibi*. That is, in all 3096 ways. As will appear by the Scheme adjoyned, and the brief Explication, (or Demonstration) of it: which is thus to be understood.

Tot, sunt, quot, which may promiscuously supply each other's place, are (in verse 1, 2, 3, &c,) set down only in this order, and so pass but for one case; but are capable of six varieties; which case I call $a = 6$. And the like for *dotes, virgo, celo*; which case I call $b = 6$. And again, *tot tibi* may change place with *sidera*; which case I call $c = 2$. And, because all these happen in verse 1, the varieties thereby represented, are $abc = 72 = 6 \times 6 \times 2$. And so of the rest, as the Scheme directs.

1.	<i>Tot tibi sunt dotes virgo quot sidera celo.</i>	$abc = 72$
2.	<i>quot virgo</i>	$abcd = 144$
3.	<i>quot dotes</i>	$ace = 1152$
4.	<i>dotes sunt virgo quot</i>	$abcf = 144$
5.	<i>sunt dotes quot virgo tibi</i>	$agh = 180$
6.	<i>quot dotes tibi virgo</i>	$abi = 324$
7.	<i>virgo tibi</i>	$abi = 324$
8.	<i>dotes tibi sunt virgo quot</i>	$ab = 36$
9.	<i>quot virgo</i>	$ak = 108$
10.	<i>sunt virgo quot tibi</i>	$ablm = 144$

virgo tibi 2628

2 P

11. *Virgo*

11.	<i>Virgo tibi tot sunt dotes quot sidera celo:</i>	<i>an</i>	=	36
12.	<i>quot dotes</i>	<i>an</i>	=	36
13.	<i>dotes sunt quot</i>	<i>ao</i>	=	36
14.	<i>Tot sunt virgo tibi dotes</i>	<i>an</i>	=	36
15.	<i>quot dotes</i>	<i>an</i>	=	36
16.	<i>dotes virgo tibi quot</i>	<i>an</i>	=	36
17.	<i>Tot dotes sunt</i>	<i>ap</i>	=	12
18.	<i>sunt dotes sidera celo virgo tibi quot</i>	<i>aq</i>	=	144
19.	<i>dotes sunt</i>	<i>apr</i>	=	24
20.	<i>celo sunt sidera</i>	<i>ap</i>	=	12
21.	<i>sidera tot dotes sunt celo</i>	<i>apr</i>	=	24
22.	<i>celo sunt</i>	<i>ap</i>	=	12
23.	<i>dotes tot</i>	<i>apr</i>	=	24

virgo tibi 468

tibi virgo 2628

3096

Tot, sunt, quot, a = 6. *dotes, virgo, celo, b* = 6.

tot tibi, sidera, c = 2. *tot tibi, virgo, d* = 2.

Tot tibi, sunt quot, dotes, virgo, celo; e = 120 — 24 =
 120 $\times \frac{1}{3}$ = 96.

(Because *tot tibi* cannot supply the place of *celo*, as of the rest.) *Tot tibi, dotes, f* = 2. *Tot sunt, dotes, virgo, celo, g* = 24. *Quot tibi, sidera, k* = $1\frac{1}{4}$.

(Because when *tot sunt*, or its Equivalent *sunt quot*, comes next before *tibi*, which is a fourth part of the cases contained in *g*, then will *quot tibi*, change with *sidera*; which adds $\frac{1}{4}$ of what was before.) *Tot sunt*, (and *sunt quot*), *dotes, virgo, celo, i* = 9.

(Because *dotes, virgo, celo*, contained in *b*, may each of them change with *tot sunt*, which Multiplies by 4, or adds a Triple to what was before, as at *g*, and $\frac{1}{3}$ of that Triple, or $\frac{1}{4}$ of that Quadruple, as at *b*; that is, it adds a Quadruple or Multiplies by 5: And again, each of them with *sunt quot*, which, for the same reason, adds another Quadruple: Therefore both together, add an Octuple, or Multiply by 9.)

Dotes,

Dotes, sunt quot, virgo, celo, k = 24 — 6 = 24 × $\frac{1}{2}$ = 18.

(Because, if *sunt quot* supply the place of *dotes*, it will be Co-incident with some of the cases of ver. 3.)

<i>Quot tibi, fidera, e = 2.</i>	<i>virgo, quot tibi, m = 2.</i>
<i>tot sunt, dotes, celo, n = 6.</i>	<i>dotes, sunt quot, celo, o = 6.</i>
<i>dotes, celo, p = 2.</i>	<i>tot sunt, dotes, fidera, celo, q = 24.</i>
<i>fidera, celo, r = 2.</i>	

I will not be positive that there may not be some other Changes: (and then, those may be added to these:) Or, that most of these be twice repeated, (and if so, those are to be abated out of the number:) But I do not, at present, discern either the one and other*.

CHAP. III.

Of the Divisors, and Aliquot Parts, of a Number proposed.

1. BY *Number*, I here understand only Integer Numbers, as 1, 2, 3, 4, 5, &c. Not Fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. Or Mixed, as $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, &c. Much less Surds, as $\sqrt{2}$, $\sqrt{5}$, $\sqrt{6}$, &c.

2. By the *Divisor* of a Number, I here understand, such Integer as doth measure such Number; that is, being once or oftener taken doth equal it. So, of the Number 6, the Divisors are, 1, 2, 3, 6: Because 6, once taken; and 3, twice taken; and 2, thrice; and 1, six times taken; do equal 6.

$$1)6(6; 2)6(3; 3)6(2; 6)6(1. \quad 6 = 1 \times 6 = 2 \times 3 = 3 \times 2 = 6 \times 1.$$

* The number of all the possible variations of the words in this Hexameter Verse, without destroying the measure of it, has been investigated with greater accuracy by Mr. *James Bernouilli*, in the second part of his excellent Treatise, *De Arte Conjectandi*, and is there found to be 3312. See above, pages 8, 9, and 10.

2 P 2

3. By

3. By *Aliquot Part* of a Number, I understand such a Divisor as is less than it. As of 6, the Aliquot Parts are 1, 2, 3; but not 6. For, though 6 be also a Divisor of itself; yet not an Aliquot Part; because the Word *Part* implies somewhat less than the whole.

4. The Number of Aliquot Parts, therefore, is always less by one than the Number of Divisors. Because all the Divisors except one, are Aliquot Parts; all the Aliquot Parts are Divisors, and there is likewise one more Divisor of the Number, to wit, the whole Number itself.

5. So that, the Number of Divisors being given, the Number of Aliquot Parts is given also. And contrarywise; if this, then that. As, of the Number 6, the Divisors being 4, the Aliquot Parts are 3, (that is, $4 - 1$.) And, these being 3, the Divisors are $4 = 3 + 1$.

6. It is manifest, that the Number 1, hath no Aliquot Part, and but one Divisor, that is 1. Because there is no Number less than itself that may be a part of it: But it measures itself; and therefore is its own Divisor.

7. Any other Prime Number hath one Aliquot Part, and Two Divisors. For a *Prime Number*, we call, such as is measured (beside itself) by no other Number but an Unit. As 2, 3, 5, 7, 11, &c. Each of which are measured by 1, and by itself; but not by any other Number. And hath therefore 2 Divisors, and 1 Aliquot Part; but no more.

8. Every *Power* of a *Prime Number* (other than of 1, which here is understood to be excluded,) hath so many Aliquot Parts as are the dimensions of such Power; and one Divisor more than so. As (supposing a, b, c , &c, to be so many Prime Numbers;) a hath two divisors (1 and a ;) a^2 or aa hath three, (1, a , aa ;) a^3 , or aaa , hath four, (1, a , aa , aaa ;) and so of the rest. That is, the Number of Divisors is one more than the Number of Dimensions. Because 1, and all the Degrees of such Power (not higher than itself) are Divisors; but not any other Number, if a be a Prime. That is, one more than the Number of Dimensions: Of which the greatest Divisor (being the whole) is not an Aliquot Part; and

and therefore the Aliquot Parts are just so many as are the Dimensions. Thus of 8 (the Cube of 2) the Divisors are four, (1, 2, 4, 8;) the Aliquot Parts are three, (1, 2, 4;) Of 81 (the Biquadrate of 3) the Divisors are five, (1, 3, 9, 27, 81;) the Aliquot Parts are four, (1, 3, 9, 27,) just so many as are the Dimensions. That is, (of such Biquadrate) the Divisors are 1, a , aa , aaa , $aaaa$; the Aliquot Parts 1, a , aa , aaa ; and so every where: For, though the highest Dimension came not into the Number of Aliquot Parts, yet 1 being supernumerary, makes the Aliquot Parts just as many as the Dimensions.

9. If a Prime Number, or any Power thereof, be Multiplied by any other Prime Number, or any Power hereof; the Product hath so many Divisors, as is the Number of Divisors in That, Multiplied by the Number of Divisors in This; and, therefore, the Aliquot Parts fewer by one than so.

For Example: Let a , b , be two different Prime Numbers, (suppose 2, 3;) and certain Powers thereof, as a^3 , b^3 , (that is 8, 9,) the Product a^3b^3 , (that is, $72 = 8 \times 9$.) Now for as much as the Divisors of the former 1, a , aa , aaa ; (that is, 1, 2, 4, 8,) divide a^3 (that is 8;) not only these, or (which is the same) every of these Multiplied by 1; but also every of them Multiplied by b , and by bb , (that is by 3, and by 9,) will divide a^3b^3 . That is, every of the Divisors of a^3 , Multiplied into every of the Divisors of b^3 ; will divide a^3b^3 .

1	1	}	4.	The Divisors of a^3 Multiplied by 1.
a	2			
aa	4			
aaa	8			
b	3	}	4.	The same Multiplied by b .
ab	6			
aab	12			
$aaab$	24			
bb	9	}	4.	The same Multiplied by bb .
abb	18			
$aabb$	36			
$aaabb$	72			
12 = 4 × 3.				

The

The Number therefore of all ; is the Number of 1, a , aa , aaa , (that is 4,) so many times taken as is the Number of 1, b , bb , (that is, 3 times;) That is, $4 \times 3 = 12$: The Number of Divisors therefore is 12; and of Aliquot Parts, 11.

10. If a Product made by the Multiplication of different Prime Numbers, or of their Powers by one another, be further Multiplied by another Prime Number different from every of those: The Number of Divisors in this new Product, will be so many as is the Number of Divisors in that first Product Multiplied by the Number of Divisors in the new Multiplier.

For Example: The Number of Divisors in the Product but now mentioned a^3b^2 , is 12; (as is already shewed:) if therefore this be Multiplied by any other Prime Number, as c (suppose 5,) different from a , b ; (whose Divisors therefore are two, 1 and c ;) the Divisors of the Product a^3b^2c (that is, of $72 \times 5 = 360$) will be $12 \times 2 = 24$. Namely, all those (before found) which divided a^3b^2 , will also divide a^3b^2c ; or (which is the same) all those Multiplied by 1 (which is one of the Divisors of c ;) and the same also Multiplied by c , (which are as many more;) and therefore both together are twice as many; that is, $12 \times 2 = 24$. Namely, 1, a , aa , aaa ; b , ab , aab , $aaab$; bb , abb , $aabb$, $aaabb$; c , ac , aac , $aaac$; bc , abc , $aabc$, $aaabc$; bbc , $abbc$, $cabbc$, $aaabbc$. That is, 1, 2, 4, 8; 3, 6, 12, 24; 9, 18, 36, 72; 5, 10, 20, 40; 15, 30, 60, 120; 45, 90, 180, 360.

And if, for the new Multiplier $c = 5$, where taken $cc = 25$, or $ccc = 125$; (the Number of whose Divisors are 3 or 4;) the Number of Divisors of the Product $a^3b^2c^2$, or $a^3b^2c^3$, would (accordingly) be $12 \times 3 = 36$, or $12 \times 4 = 48$. (And, in like manner, for any other Power of c .) For now not only the Divisors of a^3b^2 Multiplied, by 1, and by c ; but the same also Multiplied by cc , (which is a third time so many,) will be Divisors of $a^3b^2c^2$; and the same Multiplied by ccc , (which is as many a fourth time,) will be Divisors of $a^3b^2c^3$.

$\begin{array}{l} 1 \\ a \\ aa \\ aaa \end{array} \left. \begin{array}{l} 1 \\ 2 \\ 4 \\ 8 \end{array} \right\} 4$	$\begin{array}{l} c \\ ac \\ aac \\ aaac \end{array} \left. \begin{array}{l} 5 \\ 10 \\ 20 \\ 40 \end{array} \right\} 4$	$\begin{array}{l} cc \\ acc \\ aacc \\ aacc \end{array} \left. \begin{array}{l} 25 \\ 50 \\ 100 \\ 200 \end{array} \right\} 4$	$\begin{array}{l} ccc \\ acce \\ aacce \\ aaacce \end{array} \left. \begin{array}{l} 125 \\ 250 \\ 500 \\ 1000 \end{array} \right\} 4$
$\begin{array}{l} b \\ ab \\ aab \\ aabb \end{array} \left. \begin{array}{l} 3 \\ 6 \\ 12 \\ 24 \end{array} \right\} 4$	$\begin{array}{l} bc \\ abc \\ aabc \\ aabc \end{array} \left. \begin{array}{l} 15 \\ 30 \\ 60 \\ 120 \end{array} \right\} 4$	$\begin{array}{l} bcc \\ abcc \\ aabcc \\ aabcc \end{array} \left. \begin{array}{l} 75 \\ 150 \\ 300 \\ 600 \end{array} \right\} 4$	$\begin{array}{l} bccc \\ abccc \\ aabccc \\ aabccc \end{array} \left. \begin{array}{l} 375 \\ 750 \\ 1500 \\ 3000 \end{array} \right\} 4$
$\begin{array}{l} bb \\ abb \\ aabb \\ aaabb \end{array} \left. \begin{array}{l} 9 \\ 18 \\ 36 \\ 72 \end{array} \right\} 4$	$\begin{array}{l} bbc \\ abbc \\ aabc \\ aabc \end{array} \left. \begin{array}{l} 45 \\ 90 \\ 180 \\ 360 \end{array} \right\} 4$	$\begin{array}{l} bbcc \\ abbc \\ aabcc \\ aabcc \end{array} \left. \begin{array}{l} 225 \\ 450 \\ 900 \\ 1800 \end{array} \right\} 4$	$\begin{array}{l} bbccc \\ abcccc \\ aabcccc \\ aabcccc \end{array} \left. \begin{array}{l} 1125 \\ 2250 \\ 4500 \\ 9000 \end{array} \right\} 4$
$4 \times 3 = 12$	12	12	12
$12 \times 4 = 48$			

The same will in like manner be shewed, if this new Product a^3b^2c , (whose Divisors are 24,) be further Multiplied by d , or dd , &c. Namely, the Divisors of a^3b^2cd will be $24 \times 2 = 48$; and, of $a^3b^2cd^2$, $24 \times 3 = 72$. And so forward.

Or (which comes to the same pass) if a^3b^2 (whose Divisors are $12 = 4 \times 3$;) be Multiplied by cd , (whose Divisors are $4 = 2 \times 2$;) or by ccd ; (whose Divisors are $2 \times 3 = 6$;) for then will the Divisors of a^3b^2cd be $12 \times 4 = 48$; and of $a^3b^2cd^2$, $12 \times 6 = 72$; as before.

And in like-manner, the same will hold, how many soever Prime Numbers, and what ever Powers of such Primes, be so continually Multiplied; provided always (which is heedfully to be attended,) that such Primes a , b , c , d , &c, be all different each from other.

II. If any Number however Compounded, be further multiplied by any of those Primes of which it was before Compounded, or by any Power of such Prime; the Number of Divisors thence arising, will be such as would have been by advancing that Prime so many Degrees higher, as is the Degree of such Multiplier.

As, for instance, if c , d , were the same Prime; then instead of cd , whose Divisors, if different, would have been

4 =

$4 = 2 \times 2$, ($1, c, d, cd$;) we are to take cc , whose Divisors are but 3, ($1, c, cc$;) because c, d , which would otherwise have been two different Divisors, are now but one and the same. And accordingly, the Divisors of a^3b^2cd , that is, (because $c = d$;) of $a^3b^2c^2$, will now be (not $12 \times 4 = 48$, as before,) but $12 \times 3 = 36$. So if a^3b^2c be Multiplied by d^2 , and $d = b$. For then $a^3b^2cd^2$ is the same with a^3b^4c ; and the number of Divisors (not $4 \times 3 \times 2 \times 3 = 72$, but) $4 \times 5 \times 2 = 40$. And the like in other cases, as is of itself manifest.

12. And, universally: *If a Number be made, by continual Multiplication of how many soever Prime Numbers, (different each from other,) or of any Powers of such Primes: The Number of Divisors of such Compound Number, is Compounded (by continual Multiplication) of the exponents of the Degrees of such Primes or their Powers so Compounded, increased (each of them) by 1. And such Number of Divisors, wanting 1, is the Number of Aliquot Parts.* (Which Theorem contains the main substance of the Doctrine of Aliquot Parts.

As, for the Number a^3b^2cd ; the exponents of the Degrees or Dimensions of the Primes a, b, c, d , are 3, 2, 1, 1; and these increased by 1, are 4, 3, 2, 2. These, continually Multiplied, give us the Number of Divisors $4 \times 3 \times 2 \times 2 = 48$; and, of Aliquot Parts $48 - 1 = 47$. (And, in like manner, for any other Number however Compounded.) As is evident by what is before Demonstrated.

Hence we may gather the solution of the following Problems.

13. Any Number being proposed; to find how many Divisors it hath; and, how many Aliquot Parts.

Divide the Number proposed (and the Quotients arising from such Division) continually, by Prime Numbers (or the Powers of such) according as it is capable, till we come to 1. And we shall thereby find, of how many different Prime Numbers, and what Powers of them, the Number proposed is Compounded: which being done, we have the Number of Divisors

Divisors, and of Aliquot Parts, by the proposition foregoing.

As for Example; Let the Number so proposed be 5940; we shall find, upon Tryal, that it may be divided by 2, twice; by 3, three times; by 5, once; (by 7, not at all;) and by 11, once.

11)5)3)3)3)2)2) 5940(2970(1485(495(165(55(11(1

And may therefore be thus designed, a^2b^3cd ; where a, b, c , and d , denote the Numbers 2, 3, 5, and 11, respectively, and the exponents of a, b, c, d , are 2, 3, 1, 1; and these increased by 1, are 3, 4, 2, 2; which continually multiplied, are $3 \times 4 \times 2 \times 2 = 48$. So many therefore (by the proposition foregoing) are the Number of Divisors; and 47 the Number of Aliquot Parts.

14. Any Number being proposed; to find, what are the Divisors, and the Aliquot Parts thereof.

First find (as in the preceeding article) of what Prime Numbers, and what Powers of them, the Number proposed is Compounded. Then, taking any one of those Prime Numbers to whatever Degree it be advanced, set down in order all the Divisors of such Degree. Then Multiply every of these by every Divisor of such Degree as some other of those Primes is advanced to. And every of the Divisors hitherto found, by every Divisor of the Degree, to which a third Prime is advanced. And all these, by those of a fourth; and so onward, if yet there be more Primes. (In such manner as is to be seen above in art. 10.) And the Number arising from all those Multiplications, is the Number of the Divisors of the Number proposed: And all these Divisors, except itself, are the Aliquot Parts of it.

Thus for the Number $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 8 \times 9 \times 5$; suppose a^3b^2c . All the Divisors of $a^3 = 8$, are 1, a , aa , aaa ; that is, 1, 2, 4, 8. Let these be multiplied by all the Divisors of $b^2 = 9$; which are 1, b , bb ; that is, 1, 3, 9. And all the results of these, by the Divisors of c ; which are 1, c ; that is, 1, 5. So have we all the Divisors of 360.

1	<i>a</i>	<i>aa</i>	<i>aaa</i>	1	2	4	8
<i>b</i>	<i>ab</i>	<i>aab</i>	<i>aaab</i>	3	6	12	24
<i>bb</i>	<i>abb</i>	<i>aabb</i>	<i>aaabb</i>	9	18	36	72
<i>c</i>	<i>ac</i>	<i>aac</i>	<i>aaac</i>	5	10	20	40
<i>bc</i>	<i>abc</i>	<i>aabc</i>	<i>aaabc</i>	15	30	60	120
<i>bc</i>	<i>abb</i>	<i>aabb</i>	<i>aaabb</i>	45	90	180	360

And in like manner we may proceed, whatever Number be proposed, and howsoever Compounded.

But the same may also be done in divers other methods, (for we are not confined to proceed always in the same order,) which in the result will be the same with this. Provided always, in whatever order we proceed, that we be sure to take all the Prime Numbers, that are Ingredients of such Compound, with all the Degrees of them, and all the possible Combinations that may be made of them, not exceeding (in any) the Number of Dimensions which they have in the Compound. And, that we may be sure not to miss any, it will be convenient to proceed, if not in this, at least in some other regular order, that we may know when we have all. And some other forms of process we may after have occasion to mention.

15. To find a Number, which shall have just so many Divisors, or so many Aliquot Parts, as is proposed: And, in how many forms the same may be had; and, the least in each form; or the least of all, that may have so many.

The Number of Aliquot Parts proposed, increased by 1, is the Number of Divisors. This Number, we are to consider, how many ways it may be expressed in Integers; whether by one alone, or by the Multiplication of two or more: (As is to be after shewed in art. 17, 18.) And, as many ways as this may be done, so many forms there are of Numbers which have just so many Divisors: Namely, for every of the Integers by which such Number is to be expressed, so many different Prime Numbers are to be assigned; and such Degrees or Powers thereof, whose exponents are less by one than the respective Integers which they represent; and those

those Powers or Degrees, (continually multiplied, if there be more of them,) will have such Number of Divisors as is required.

As for Example: If a Number be required which shall have 99 Aliquot Parts, or, (which is the same) 100 Divisors. This Number 100, may be expressed by Integers (single, or multiplied into one another,) nine several ways: $100 = 50 \times 2 = 25 \times 4 = 25 \times 2 \times 2 = 20 \times 5 = 10 \times 10 = 10 \times 5 \times 2 = 5 \times 5 \times 4 = 5 \times 5 \times 2 \times 2$: And so many several forms there are of Numbers which shall have 100 Divisors, or 99 Aliquot Parts. Namely, if (for every of the forms wherein the Number

100 may be so designed) we take so many different Primes, as there are Integers in such designation; and each of them advanced respectively to such Degree whose exponent is less by one than the Integer it represents. As a^{99} , $a^{49}b$, $a^{24}b^3$, $a^{24}bc$, $a^{19}b^4$, $a^{9}b^9$, $a^{9}b^4c$, $a^4b^4c^3$, a^4b^4cd ; whatever be those Prime Numbers a , b , c , d , different each from other. (As appears from art. 12.) But not any other forms:

100	a^{99}
50×2	$a^{49}b$
25×4	$a^{24}b^3$
$25 \times 2 \times 2$	$a^{24}bc$
20×5	$a^{19}b^4$
10×10	a^9b^9
$10 \times 5 \times 2$	a^9b^4c
$5 \times 5 \times 4$	$a^4b^4c^3$
$5 \times 5 \times 2 \times 2$	a^4b^4cd

As may be thence shewed, in case any other form be assigned. As, for instance, if any form be assigned wherein (whatsoever be the other Ingredients) there is the bare Square of a Prime Number, (such as in none of these appears) as e^2 . For whatever be the Number which the rest of the ingredients design, that Number (because of e^2) is to be Tripled (by art. 9.) But 100 is not the Triple of any Integer (as not being divisible by 3 :) And therefore cannot be so designed. And in like manner may be shewed, (with such variation as the case shall require,) concerning any other form, different from those assigned.

Now for finding the least Number in each form, that shall have so many Divisors; no more is to be done, but for a , b , c , d , &c, or so many of them as occur in each form respectively, to take so many of the smallest Primes, 2, 3, 5, 7, &c. And, of these, still to assign the lesser for that which is to have the greater Number of Dimensions.

2 Q 2

(As

(As is of itself manifest.) So for the form a^2b^4c , it is manifest, that if for a, b, c , we take 2, 3, 5, the number must needs be less, than if we take 2, 3, 7; or 3, 7, 11, or any other numbers: And, (supposing those three to be taken,) it must needs be less if we assign $a = 2, b = 3, c = 5$, than if we assign them any otherwise. Because, in the composition, a is oftener to be repeated than b , and b than c .

Now when it appears, which is the least in each form; it is easily determined upon view, which is the least of all. As, in the present case, putting $a = 2, b = 3, c = 5, d = 7$; it is easy to judge that a^4b^4cd , that is, $16 \times 81 \times 5 \times 7 = 45360$, is the smallest number that can have 100 Divisors. For it is, to $a^4b^4c^2$; as $d = 7$, to $cc = 9$: And it is, to a^2c^4c ; as $d = 7$, to $a^5 = 32$: And, to a^2b^9 ; as $cd = 35$, to $a^3b^5 = 7776$. And so of the rest.

And, for the most part, those are the smaller numbers wherein more Primes be ingredients; than where fewer Primes, but in higher Degrees; as $ab = 2 \times 3 = 6$, is less than $a^3 = 8$; though each of them have four Divisors. But it is not always so; for $a^3b = 8 \times 3 = 24$, is less than $abc = 2 \times 3 \times 5 = 30$; (though the number of Divisors be eight in each.) For here one Degree of a greater Prime $c = 5$, doth over-balance two Degrees of a lesser $aa = 4$.

16. It appears moreover, That, wherever the number of Divisors is odd, such Number is a Square: And, contrary-wise, of every Square Number, the number of Divisors is odd. And, of every Non-quadrate Number, the number of Divisors is even: And, wherever the number of Divisors is even, such Number is a Non-quadrate Number.

For every Divisor divides the Number proposed by some other Divisor, (whereof when one is the Divisor, the other is the Quotient;) except only the Square Root, (where the Divisor and Quotient are the same.) All other Divisors therefore go by couples, and make an even Number: To which when the Square Root is to be added (which is the case

case of all Square Numbers, and of these only;) this being solitary, makes the number of Divisors odd.

1	36	1	<i>aabb</i>	1	360	1	<i>aaabbe</i>
2	18	<i>a</i>	<i>abb</i>	2	180	<i>a</i>	<i>aabbe</i>
3	12	<i>b</i>	<i>aab</i>	3	120	<i>b</i>	<i>aaabe</i>
4	9	<i>aa</i>	<i>bb</i>	4	90	<i>aa</i>	<i>abbe</i>
6			<i>ab x'</i>	5	72	<i>c</i>	<i>aaabb</i> ^{<i>423(ab)²</i>}
				6	60	<i>ab</i>	<i>aabc</i>
1	72	1	<i>aaabb</i>	8	45	<i>aaa</i>	<i>bbe</i>
2	36	<i>a</i>	<i>aabb</i>	9	40	<i>bb</i>	<i>aaac</i>
3	24	<i>b</i>	<i>aaab</i>	10	36	<i>ac</i>	<i>aabb</i>
4	18	<i>aa</i>	<i>abb</i>	12	30	<i>aab</i>	<i>abc</i>
6	12	<i>ab</i>	<i>aab</i>	15	24	<i>bc</i>	<i>aaab</i>
8	9	<i>aaa</i>	<i>bb</i>	18	20	<i>abb</i>	<i>aac</i>

17. A Number being proposed; to find, how many different ways it may be designed by Integers; whether singly or by the continual Multiplication of more than one.

First find out (by art. 14,) what are all the Divisors of such proposed Number. Then, considering them all singly (beginning at the greatest and so proceeding to the lesser; that, by keeping such order, we may be the more sure not to miss any;) inquire, what Number doth with every of these compose the Number proposed; and if this chance to be a Compound, let this in like manner be resolved into its Components, (and so onward as long as the Component is itself a Compound;) whereby, having thus run through them all, we shall meet with all the ways whereby the Number proposed may so be designed by Integers.

As for Example: Let such Number proposed, be 360; whose Divisors (found by art. 14,) are 360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 24, 20, 18, 15, 12, 10, 9, 8, 6, 5, 4, 3, 2, 1, where we shall find the first designation to be 360, (or 360×1 .) Then 180×2 , 120×3 , 90×4 , and (because $4 = 2 \times 2$,) $90 \times 2 \times 2$. Then 72×5 , 60×6 ; and (because $6 = 3 \times 2$) $40 \times 3 \times 3$, 36×10

36×10
 $36 \times 5 \times 2$
 30×12
 $30 \times 6 \times 2$
 $30 \times 4 \times 3$
 $30 \times 3 \times 2$
 24×15
 $24 \times 5 \times 3$
 20×18
 $20 \times 9 \times 2$
 $20 \times 6 \times 3$
 $20 \times 3 \times 3 \times 2$
 $18 \times 10 \times 2$
 $18 \times 5 \times 4$
 $18 \times 5 \times 2 \times 2$
 $15 \times 12 \times 2$
 $15 \times 8 \times 3$
 $15 \times 6 \times 4$
 $15 \times 6 \times 2 \times 2$
 $15 \times 4 \times 3 \times 2$
 $15 \times 3 \times 2 \times 2 \times 2$
 $12 \times 10 \times 3$
 $12 \times 6 \times 5$
 $12 \times 5 \times 3 \times 2$
 $10 \times 9 \times 4$
 $10 \times 9 \times 2 \times 2$
 $10 \times 6 \times 6$
 $10 \times 6 \times 3 \times 2$
 $10 \times 4 \times 3 \times 3$
 $10 \times 3 \times 2 \times 2 \times 2$
 $9 \times 8 \times 5$
 $9 \times 5 \times 4 \times 2$
 $9 \times 5 \times 2 \times 2 \times 2$
 $8 \times 5 \times 3 \times 3$
 $8 \times 5 \times 2 \times 2$
 $6 \times 5 \times 4 \times 3$
 $6 \times 5 \times 3 \times 2 \times 2$
 $5 \times 4 \times 3 \times 3 \times 2$
 $5 \times 3 \times 3 \times 2 \times 2 \times 2$

$60 \times 3 \times 2$. Then 45×8 ; and (because $8 = 4 \times 2 = 2 \times 2 \times 2$.) $45 \times 4 \times 2$, $45 \times 2 \times 2 \times 2$. Then 40×9 ; and (because $9 = 3 \times 3$.) $40 \times 3 \times 3$. Then 36×10 ; and (because $10 = 5 \times 2$.) $36 \times 5 \times 2$. Then 30×12 ; and (because $12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2$.) $30 \times 6 \times 2$, $30 \times 4 \times 3$, $30 \times 3 \times 2 \times 2$. Then 24×15 , and (because $15 = 5 \times 3$.) $24 \times 5 \times 3$. Then 20×18 , and (because $18 = 9 \times 2 = 6 \times 3 = 3 \times 3 \times 2$.) $20 \times 9 \times 2$, $20 \times 6 \times 3$, $20 \times 3 \times 3 \times 2$. Then, (omitting 18×20 , as being the same with 20×18 ; and resolving $20 = 10 \times 2 = 5 \times 4 = 5 \times 2 \times 2$;) $18 \times 10 \times 2$, $18 \times 5 \times 4$, $18 \times 5 \times 2 \times 2$. Then (omitting 15×24 , as being the same with 24×15 ; and to every where when a greater follows a less, as being had before; and resolving $24 = 12 \times 2 = 8 \times 3 = 6 \times 4 = 6 \times 2 \times 2 = 4 \times 3 \times 2 = 3 \times 2 \times 2 \times 2$;) $15 \times 12 \times 2$, $15 \times 8 \times 3$, $15 \times 6 \times 4$, $15 \times 6 \times 2 \times 2$, $15 \times 4 \times 3 \times 2$, $15 \times 3 \times 2 \times 2 \times 2$. In like manner (omitting such Combinations of 12 as have been already,) $12 \times (30 = 15 \times 2 =) 10 \times 3$, $12 \times 6 \times 5$, $12 \times 5 \times 3 \times 2$. In like manner, $10 \times (36 = 18 \times 2 = 12 \times 3 =) 9 \times 4$, $10 \times 9 \times 2 \times 2$, $10 \times 6 \times 6$, $10 \times 6 \times 3 \times 2$, $10 \times 4 \times 3 \times 3$, $10 \times 3 \times 3 \times 2 \times 2$. Then $9 \times (40 = 20 \times 2 = 10 \times 4 =) 8 \times 5$, $9 \times 5 \times 4 \times 2$, $9 \times 5 \times 2 \times 2 \times 2$. Then $8 \times (45 =) 5 \times 3 \times 3$. Then $6 \times (60 =) 6 \times 5 \times 2$, $6 \times 5 \times 4 \times 3$, $6 \times 5 \times 3 \times 2 \times 2$. Lastly, $5 \times (72 =) 4 \times 3 \times 3 \times 2$, $5 \times 3 \times 3 \times 2 \times 2 \times 2$. (The Divisors 4, 3, 2, 1, afford no new cases; because every of them is less than 5, and cannot without it, or some greater Number, make up 360.) Which forms (in Number 52) are all the forms in which 360, may thus be expressed by

by Integers. And how, to every of these forms, we may fit so many forms of Numbers which shall have 360 Divisors, is before shewed in art. 15. As, for $5 \times 3 \times 3 \times 2 \times 2 \times 2$, $a^4 b^3 c^2 def$: And so of the rest.

But, why I have here omitted (for instance) 5×72 , $5 \times 36 \times 2$, $5 \times 24 \times 3$, $5 \times 18 \times 4$, $5 \times 18 \times 2 \times 2$, $5 \times 12 \times 6$, $5 \times 12 \times 3 \times 2$, $5 \times 9 \times 8$, $5 \times 9 \times 4 \times 2$, $5 \times 9 \times 2 \times 2 \times 2$, $5 \times 8 \times 3 \times 3$, $5 \times 6 \times 6 \times 2$, $5 \times 6 \times 4 \times 3$, $5 \times 6 \times 3 \times 2 \times 2$, and others of like kind; the cause is evident: Because, the Numbers 72, 36, 24, 18, 12, 9, 8, 6, being greater than 5, all the Combinations which have these ingredients were had before. For 5×72 , is but the same with 72×5 ; and so of the rest. And it is so ordered all along, that whenever a greater Number comes to follow a lesser, we may know that that case was (or should have been) had before.

But it is no way necessary that we should always observe this order; for the same will hold, in whatever method we proceed: provided we be sure to take them all, in whatever order.

18. The same also may be thus had, if the Number itself (of Divisors required) or the form thereof, be so expressed in Species, as it may thence appear in what form it is compounded of the ingredient Primes: As if we put $a^3 b^2 c$, for the Number $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$; or for any other Number which is compounded of the Third Degree of one Prime, Multiplied by the Second Degree of another Prime, and this by a Third Prime.

For, however we are not by this directed how to proceed (as before) from the greater to the lesser in a continual order, (because the Second or Third Degree of a lesser Prime, may possibly be greater than the first of some greater Prime;) yet we may thus, though in another order, meet with them all.

And it will be then convenient (beginning with 1,) to take the Species or Symbols, first singly, one by one, (as a , b , c ,) in such order as they follow in the Alphabet. And then by Two's, (as aa , ab , ac , bb , &c,) and here, first those

those that begin with a ; and here again aa before ab , and this before ac , 1 , &c, and then those that begin with b ; and here (omitting ba , as being the same with ab which was had before,) beginning with bb , or (in case there be not a second b) with bc , and so onward: And then by Threes, and Fours, and so onward as there is occasion; observing all along, as the case will permit, the Alphabetical order, (that we may be the more sure not to miss any.) Placing always, over against each, the correspondent Divisor; which doth, with it, constitute the Number proposed. As, against aa , putting abb , which, with it, compleats $aaabb$.

1	a^3b^2c		1	$aaabb$
a	a^2b^2c		a	$aabbc$
b	a^3bc		b	$aaabc$
c	a^3b^2		c	$aaabb$
a^2	ab^2c		aa	$abbc$
ab	a^2bc	Or thus rather,	ab	$aabc$
ac	a^2b^2		ac	$aabb$
b^2	a^3c		bb	$aaac$
bc	a^3b		bc	$aaab$
a^3	b^2c		aaa	bbc
a^2b	abc		aab	abc
a^2c	ab^2		aac	abb

And this we are to pursue so far, till, in that opposite rank, we meet with the same (in the case of a Square Number proposed,) or, (if not a Square Number,) that which was next to follow, in the first rank. (As here, against $aaac$, we have abb ; which was next to have followed if the first rank had proceeded.) For, when we be come so far, those which were to have followed in the continuation of the first rank, do follow (in the same order, but going backward,) in the latter rank, till we come to the greatest of all.

And having thus disposed all the Divisors in due order; we may then, (beginning with the greatest, and so proceeding backward to the least,) compound each with its opposite, which

which stands against it. (As *cl* *baaa*, *cbbaa* $\times$ *a*, *cbaaa* $\times$ *b*, &c.) And when that second Component is itself a Compound, we are to resolve it into its Components; (as *cbaxaa*, *cbaxaxa*, &c.) and so continually till it be resolved into Primes.

When we have thus dispatched all the Divisors of the latter rank (for till then, there is no danger,) we are to take heed, that some of the Compositions already taken, be not taken a second time in another order; and when they do so occur a second time, we are to pass them by. And accordingly, when I come at *caa*, I do not Compound this with the whole of *bba* which stands against it; (because this hath been already considered, and there joined in all the Compositions that it is capable of;) but with all these Components of *bba*, which had not before been fully considered. And when I come at *cb*, I omit, not only the whole of *baaa*, (which stands against it) but all the Components of it which have three Members, (because not only those of Four, but even of Three Components, have been fully dispatched, before we come at *cb* which hath but

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<i>cbbaaa</i>	360
<i>cbbaaxa</i>	180x2
<i>cbbaaxb</i>	120x3
<i>bbaaaaxc</i>	72x5
<i>cbbaaxaa</i>	90x4
<i>xaxa</i>	90x2x2
<i>cbbaaxba</i>	60x6
<i>xbxa</i>	60x3x2
<i>bbaaaxca</i>	36x10
<i>xcxa</i>	36x5x2
<i>caaaxbb</i>	40x9
<i>xbxb</i>	40x3x3
<i>baaaxbc</i>	24x15
<i>xbxc</i>	24x5x3
<i>cbbxaaa</i>	45x8
<i>xaaxa</i>	45x4x2
<i>xaxaxa</i>	45x2x2x2
<i>cbbaaxbaa</i>	30x12
<i>xbaxa</i>	30x6x2
<i>xaaxb</i>	30x4x3
<i>xbxaxa</i>	30x3x2x2
<i>bbbaaxcaa</i>	18x20
<i>xcaxa</i>	18x10x2
<i>xaaxc</i>	18x4x5
<i>xcxaxa</i>	18x5x2x2
<i>caaaxbbxa</i>	20x9x2
<i>xbaxb</i>	20x6x3
<i>xbxbxa</i>	20x3x3x2
<i>baaaxcbxa</i>	12x15x2
<i>xcaxb</i>	12x10x3
<i>xbaxc</i>	12x6x5
<i>xcxbxa</i>	12x5x3x2
<i>aaaxcbxb</i>	8x15x3
<i>xbbx</i>	8x9x5
<i>xcxbxb</i>	8x5x3x3
<i>cbxbaxaa</i>	15x6x4
<i>xaxa</i>	15x6x2x2
<i>xaaxbxa</i>	15x4x3x2
<i>xbxaxaxa</i>	15x3x2x2x2
	<i>bbxaxaxa</i>

<i>blxcaxaa</i>	$9 \times 10 \times 4$
<i>xaxa</i>	$9 \times 10 \times 2 \times 2$
<i>xaaxcxa</i>	$9 \times 4 \times 5 \times 2$
<i>xcxaxaxa</i>	$9 \times 5 \times 2 \times 2 \times 2$
<i>caxbaxba</i>	$10 \times 6 \times 6$
<i>xbxa</i>	$10 \times 6 \times 3 \times 2$
<i>xaaxbxb</i>	$10 \times 4 \times 3 \times 3$
<i>xbxbxaxa</i>	$10 \times 3 \times 3 \times 2 \times 2$
<i>baxbaxcxa</i>	$6 \times 6 \times 5 \times 2$
<i>xaaxcxb</i>	$6 \times 4 \times 5 \times 3$
<i>xcxbxaxa</i>	$6 \times 5 \times 3 \times 2 \times 2$
<i>aaxcxbxbxa</i>	$4 \times 5 \times 3 \times 3 \times 2$
<i>cxbxbxaxaxa</i>	$5 \times 3 \times 3 \times 2 \times 2 \times 2$

two Components.) And when I come at *ca*, I omit *caxbbxaa*, &c, because *bb* had been before considered. And in like manner, at *ba*, I omit all the Compositions wherein *cb*, *bb*, *ca*, were ingredients; because these had been before considered. And in like manner, at *aa*, and *c*, I omit all those of two Members which might be Compounded with them; because already had. As is to be seen in the order adjoined.

And over against the forms thus expressed in Species; I have set the Numbers answering to them; which are the same with those at art. 17, but not in the same order. Because here I was guided by the forms of Composition, in directing the order; but, there, by the bigness of the Numbers.

Having thus laid the Foundation of this Doctrine of Divisors and Aliquot Parts; I shall give some Examples of Operations concerning them.

Examples of the foregoing Operations.

19. Of the Number 110,880: How many are the Divisors, and Aliquot Parts? And which be they?

The Number 110,880 divided, as is directed at art. 13, is resolved into these Primes; 2, 2, 2, 2, 2, 3, 3, 5, 7, 11. And is therefore in this form a^5b^3cde .

11) 5) 7) 3) 3) 2) 2) 2) 2) 110880 (55440 (27720 (13860
(6930 (3465 (1155 (385 (77 (11 (1

Or, I might at first cut off the Cypher; and, for it, set down two Divisors 2, 5: And then, because it is obvious to view, that 11,088 is divisible by 11; I might next set down 11 for another Divisor. (Because by this means we come

come the sooner to small Numbers.) And then divide the Quotient 1008 by 2, and 3, as oft as I can; which done, we shall have 7 for the last Divisor. Or, I might have divided 11,088 by 9; (and for it set down two Divisors 3, 3 :) For it is obvious also to view that it may be so divided; because the Figures put together without regard had to the places, (as is usual in the proofs of Multiplication and Division,) may be so divided; or, casting away 9 as oft as may be, nothing remains; or, I may so do, for the same reason, with 1008; or, take any the like advantage for expedition, as the view shall direct. For it matters not, in what order we find the Component Primes, so we have them all.

The Number therefore appearing in this form a^2b^2cde ; it is manifest (by art. 12,) that the Number of Divisors is $6 \times 3 \times 2 \times 2 \times 2 = 144$; and, of Aliquot Parts, $144 - 1 = 143$. And those, (according to the method of art. 18,) are found to be these that follow.

i	i	110880	aaaaabbcd
a	2	55440	aaaabbcd
b	3	36960	aaaaabcde
c	5	22176	aaaaabbde
d	7	15840	aaaaabbce
e	11	10080	aaaaabbcd
aa	4	27720	aaabbcd
ab	6	18480	aaaabcde
ac	10	11088	aaaabbde
ad	14	7920	aaaabbce
ae	22	5040	aaaabbcd
bb	9	12320	aaaaacde
bc	15	7392	aaaaabde
bd	21	5280	aaaaabce
be	33	3360	aaaaabcd
cd	35	3168	aaaabbfe
ce	55	2016	aaaaabbdd
de	77	1440	aaaaabbcc
aaa	8	13860	aabbcd
aab	12	9240	aaabcde
aac	20	5544	aaabbde
		2 R 2	aad

<i>aad</i>	28	3960	<i>aaabbbce</i>
<i>aae</i>	44	2520	<i>aaabbbcd</i>
<i>abb</i>	18	6160	<i>aaaacde</i>
<i>abc</i>	30	3696	<i>aaaabde</i>
<i>abd</i>	42	2640	<i>aaaabce</i>
<i>abe</i>	66	1680	<i>aaaabcd</i>
<i>acd</i>	70	1584	<i>aaaabbe</i>
<i>ace</i>	110	1008	<i>aaaabbd</i>
<i>ade</i>	154	720	<i>aaaabbc</i>
<i>bbc</i>	45	2464	<i>aaaaade</i>
<i>bbd</i>	63	1760	<i>aaaaace</i>
<i>bbe</i>	99	1120	<i>aaaaacd</i>
<i>bcd</i>	105	1056	<i>aaaabbe</i>
<i>bce</i>	165	672	<i>aaaaabd</i>
<i>bde</i>	231	480	<i>aaaaabc</i>
<i>cde</i>	385	288	<i>aaaaabb</i>
<i>aaaa</i>	16	6930	<i>abbcede</i>
<i>aaab</i>	24	4620	<i>aabcede</i>
<i>aaac</i>	40	2772	<i>aabbde</i>
<i>aaad</i>	56	1980	<i>aabbce</i>
<i>aaae</i>	88	1260	<i>aabbcd</i>
<i>aabb</i>	36	3080	<i>aaacde</i>
<i>aabc</i>	60	1848	<i>aaabde</i>
<i>aabd</i>	84	1320	<i>aaabce</i>
<i>aabe</i>	132	840	<i>aaabcd</i>
<i>aacd</i>	140	792	<i>aaabbe</i>
<i>aace</i>	220	504	<i>aaabbd</i>
<i>aade</i>	308	360	<i>aaabbc</i>
<i>abbc</i>	90	1232	<i>aaaaade</i>
<i>abbd</i>	126	880	<i>aaaaace</i>
<i>abbe</i>	198	560	<i>aaaacd</i>
<i>abcd</i>	210	528	<i>aaaabe</i>
<i>abce</i>	330	336	<i>aaaabd</i>
<i>abde</i>	462	240	<i>aaaabc</i>
<i>acde</i>	770	144	<i>aaaabb</i>
<i>bbcd</i>	315	352	<i>aaaaae</i>
<i>bbce</i>	495	224	<i>aaaaad</i>
<i>bbde</i>	693	160	<i>aaaaac</i>
<i>bcde</i>	1155	96	<i>aaaaab</i>

aaaaa

<i>aaaaa</i>	32	3465	<i>bbcde</i>
<i>aaaab</i>	48	2310	<i>abcde</i>
<i>aaaac</i>	80	1386	<i>abbde</i>
<i>aaaad</i>	112	990	<i>abbce</i>
<i>aaaae</i>	176	630	<i>abbcd</i>
<i>aaabb</i>	72	1540	<i>aacde</i>
<i>aaabc</i>	120	924	<i>aabde</i>
<i>aaabd</i>	168	660	<i>aabce</i>
<i>aaabe</i>	264	420	<i>aabcd</i>
<i>aaacd</i>	280	396	<i>aabbe</i>
<i>aaace</i>	440	252	<i>aabbd</i>
<i>aaade</i>	616	180	<i>aabbc</i>

The same, ordered according to the greatness of the Numbers, will stand thus :

<i>1</i>	1	110880	<i>aaaaabbcd</i>
<i>a</i>	2	55440	<i>aaaabbcd</i>
<i>b</i>	3	36960	<i>aaaaabcd</i>
<i>aa</i>	4	27720	<i>aaabbcd</i>
<i>c</i>	5	22176	<i>aaaaabbde</i>
<i>ab</i>	6	18480	<i>aaaabde</i>
<i>d</i>	7	15840	<i>aaaaabbce</i>
<i>aaa</i>	8	13860	<i>aabbcd</i>
<i>bb</i>	9	12320	<i>aaaaacde</i>
<i>ac</i>	10	11088	<i>aaaabbde</i>
<i>e</i>	11	10080	<i>aaaaabbcd</i>
<i>aab</i>	12	9240	<i>aaabde</i>
<i>ad</i>	14	7920	<i>aaaabbce</i>
<i>bc</i>	15	7392	<i>aaaaabde</i>
<i>aaa</i>	16	6930	<i>abbcd</i>
<i>abb</i>	18	6160	<i>aaaacde</i>
<i>aac</i>	20	5544	<i>aaabde</i>
<i>bd</i>	21	5280	<i>aaaaabce</i>
<i>ae</i>	22	5040	<i>aaaabbcd</i>
<i>aab</i>	24	4620	<i>abcde</i>
<i>aad</i>	28	3960	<i>aaabbce</i>
<i>abc</i>	30	3696	<i>aaaabde</i>
<i>aaaa</i>	32	3465	<i>bbcd</i>

be

be	33	3360	aaaaabed
cd	35	3168	aaaaabbe
abb	36	3080	aaacde
aac	40	2772	aabbde
abd	42	2640	aaaabce
aac	44	2520	aaabbcd
bbe	45	2464	aaaaade
aaaab	48	2310	abcde
ce	55	2016	aaaaabbd
aaad	56	1980	aabbce
aabc	60	1848	aaabde
bbd	63	1760	aaaace
abc	66	1680	aaaabcd
acd	70	1584	aaaabbe
aaabb	72	1540	aacde
de	77	1440	aaaaabbc
aaac	80	1386	abbde
aabd	84	1320	aaabce
aac	88	1260	aabbcd
abbc	90	1232	aaaade
aaaaab	96	1155	bcde
bbe	99	1120	aaaaatd
bcd	105	1056	aaaaabe
ace	110	1008	aaaabbd
aaaad	112	990	abbce
aaabc	120	924	abbde
abbd	126	880	aaaace
aabe	132	840	aaabcd
aacd	140	792	aaabbe
aaabb	144	770	acde
ade	154	720	aaaabbc
aaaaac	160	693	bbde
bce	165	672	aaaaabd
aaabd	168	660	aabce
aaaae	176	630	abbcd
aabbe	180	616	aaade
abbe	198	560	aaaacd
abcd	210	528	aaaabe
aace	220	504	aaabbid

aaaaad

<i>aaaaad</i>	224	495	<i>bbce</i>
<i>bde</i>	231	480	<i>aaaaabc</i>
<i>aaaabc</i>	240	462	<i>abde</i>
<i>aabbd</i>	252	440	<i>aaase</i>
<i>aaabc</i>	264	420	<i>aabcd</i>
<i>aaacd</i>	280	396	<i>aabbce</i>
<i>aaaaabb</i>	288	385	<i>cde</i>
<i>aade</i>	308	360	<i>aaabbc</i>
<i>bbcd</i>	315	352	<i>aaaaae</i>
<i>abce</i>	330	336	<i>aaaabd</i>

20. Of Numbers (for instance) which have 12 Divisors: To exhibit all the forms; and, all the Numbers in each form; not exceeding the Number 2048; (which is the lowest Number of the highest form;) according to art. 15. 18.

All the ways according to which 12 may be expressed by Integers (as in art. 17, 18,) are $12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2$: Which affords us these forms, a^{11} , a^5b , a^3b^2 , a^2bc . And in each of these, the Numbers are as follow; being in all 211.

a^{11} .

2048.

a^5b .

32	×	3	=	96
32	×	5	=	160
	×	7	=	224
	×	11	=	352
	×	13	=	416
	×	17	=	544
	×	19	=	608
	×	23	=	736
	×	29	=	928
	×	31	=	992
	×	37	=	1184
	×	41	=	1312
	×	43	=	1376
	×	47	=	1504

$$\times 53 = 1696$$

$$\times 59 = 1888$$

a^{15} . $243 \times 61 = 1952$

$$\times 2 = 486$$

$$\times 5 = 1215$$

$$\times 7 = 1701$$

a^3b^2 .

$$8 \times 9 = 72$$

$$\times 25 = 200$$

$$\times 49 = 392$$

$$\times 121 = 968$$

a^2b . $27 \times 196 = 1352$

$$\times 4 = 108$$

$$\times 25 = 675$$

$$\times 49 = 1323$$

$$125 \times 4 = 500$$

$$\times 9 = 1125$$

$${}^{343}_{a^2bc} \times 4 = 1372$$

$$4 \times 3 \times 5 = 60$$

$$\times 7 = 84$$

$$\times 11 = 132$$

$$\times 13 = 156$$

$$\times 17 = 204$$

$$\times 19 = 228$$

$$\times 23 = 276$$

$$\times 29 = 348$$

$$\times 31 = 372$$

$$\times 37 = 444$$

$$\times 41 = 492$$

$$\times 43 = 516$$

$$\times 47 = 564$$

$$\times 53 = 636$$

$$\times 59 = 708$$

$$\times 61 = 732$$

$$\times 67 = 804$$

$$\times 71 = 852$$

$$\times 73 = 876$$

$$\times 79 = 948$$

$$\times 83 = 996$$

$$\times 89 = 1068$$

$$\times 97 = 1164$$

$$\times 101 = 1212$$

$$\times 103 = 1236$$

$$\times 107 = 1284$$

$$\times 109 = 1308$$

$$\times 113 = 1356$$

$$\times 127 = 1524$$

$$\times 131 = 1572$$

$$\times 137 = 1644$$

$$\times 139 = 1668$$

$$\times 149 = 1788$$

$$\times 151 = 1812$$

$$\times 157 = 1884$$

$$\times 163 = 1956$$

$$\times 167 = 2004$$

$$4 \times 5 \times 7 = 140$$

$$\times 11 = 220$$

$$\times 13 = 260$$

$$\times 17 = 340$$

$$\times 19 = 380$$

$$\times 23 = 460$$

$$\times 29 = 580$$

$$\times 31 = 620$$

$$\times 37 = 740$$

$$\times 41 = 820$$

$$\times 43 = 860$$

$$\times 47 = 940$$

$$\times 53 = 1060$$

$$\times 59 = 1180$$

$$\times 61 = 1220$$

$$\times 67 = 1340$$

$$\times 71 = 1420$$

$$\times 73 = 1460$$

$$\times 79 = 1580$$

$$\times 83 = 1660$$

$$\times 89 = 1780$$

$$\times 97 = 1940$$

$$\times 101 = 2020$$

$$4 \times 7 \times 11 = 308$$

$$\times 13 = 364$$

$$\times 17 = 476$$

$$\times 19 = 532$$

$$\times 23 = 644$$

$$\times 29 = 812$$

$$\times 31 = 868$$

$$\times 37 = 1036$$

$$\times 41 = 1148$$

$$\times 43 = 1204$$

$$\times 47 = 1316$$

$$\times 53 = 1484$$

$$\times 59 = 1652$$

$$\times 61 = 1708$$

$$\times 67 = 1876$$

$$\times 71 = 1988$$

$$4 \times 7$$

4 × 7	×	73	=	2044
4 × 11	×	13	=	572
	×	17	=	748
	×	19	=	836
	×	23	=	1012
	×	29	=	1276
	×	31	=	1364
	×	37	=	1628
	×	41	=	1804
	×	43	=	1892
4 × 13	×	17	=	884
	×	19	=	988
	×	23	=	1196
	×	29	=	1508
	×	31	=	1612
	×	37	=	1924
4 × 17	×	19	=	1292
	×	23	=	1564
	×	29	=	1972
4 × 19	×	23	=	1748
9 × 2	×	5	=	90
	×	7	=	126
	×	11	=	198
	×	13	=	234
	×	17	=	306
	×	19	=	342
	×	23	=	414
	×	29	=	522
	×	31	=	558
	×	37	=	666
	×	41	=	738
	×	43	=	774
	×	47	=	846
	×	53	=	954
	×	59	=	1062
	×	61	=	1098
	×	67	=	1206
	×	71	=	1278
	×	73	=	1314

9 × 2	×	79	=	1422
	×	83	=	1498
	×	89	=	1602
	×	97	=	1746
	×	101	=	1818
	×	103	=	1854
	×	107	=	1926
	×	109	=	1962
	×	113	=	2034
9 × 5	×	7	=	315
	×	11	=	495
	×	13	=	585
	×	17	=	765
	×	19	=	855
	×	23	=	1035
	×	29	=	1305
	×	31	=	1395
	×	37	=	1665
	×	41	=	1845
	×	43	=	1935
9 × 7	×	11	=	693
	×	13	=	819
	×	17	=	1071
	×	19	=	1197
	×	23	=	1449
	×	29	=	1827
	×	31	=	1953
9 × 11	×	13	=	1287
	×	17	=	1683
	×	19	=	1881
9 × 13	×	17	=	1989
25 × 2	×	3	=	150
	×	7	=	350
	×	11	=	550
	×	13	=	650
	×	17	=	850
	×	19	=	950
	×	23	=	1150
	×	29	=	1450

2 S

25 × 2

$25 \times 2 \times 31 = 1550$	$49 \times 2 \times 17 = 1666$
$\times 37 = 1850$	$\times 19 = 1862$
$25 \times 3 \times 7 = 525$	$49 \times 3 \times 5 = 735$
$\times 11 = 825$	$\times 11 = 1617$
$\times 13 = 975$	$\times 13 = 1911$
$\times 17 = 1275$	$121 \times 2 \times 3 = 726$
$\times 19 = 1425$	$\times 5 = 1210$
$\times 23 = 1725$	$\times 7 = 1694$
$25 \times 7 \times 11 = 1925$	$121 \times 3 \times 5 = 1815$
$49 \times 2 \times 3 = 294$	$169 \times 2 \times 3 = 1014$
$\times 5 = 490$	$\times 5 = 1690$
$\times 11 = 1078$	$289 \times 2 \times 3 = 1734$
$\times 13 = 1274$	

These digested according to their natural order, stand thus:

60	306	516	735	940
72	308	522	736	948
84	315	525	738	950
90	340	532	740	954
96	342	544	748	968
108	348	550	765	975
126	350	558	774	988
132	352	564	804	992
140	364	572	812	996
150	372	580	819	1012
156	380	585	820	1014
160	392	608	825	1035
198	414	620	836	1036
200	416	636	846	1060
204	444	644	850	1062
220	460	650	852	1068
224	476	666	855	1071
228	486	675	860	1078
234	490	693	868	1098
260	492	708	876	1125
276	495	726	884	1148
294	500	732	928	1150

1164	1312	1524	1725	1924 .
1180	1314	1550	1734 .	1925
1184	1316	1564 .	1746	1926
1196	1323 .	1572	1748	1935
1197 .	1340	1580	1780	1940
1204	1352	1602	1788	1952 .
1206	1356	1612	1804 .	1953
1210	1364	1617 .	1812	1956
1212	1372 .	1628	1815	1962
1215 .	1376	1644	1818	1972
1220	1395	1652	1827	1988 .
1236	1420	1660	1845 .	1989
1274	1422	1665 .	1850	2004
1275	1425 .	1666	1854	2020
1276 .	1449	1668	1862	2034
1278	1450	1683	1876 .	2044 .
1284	1460	1690	1881 .	2048
1287	1484	1694 .	1884	
1292	1494 .	1696	1888	
1305 .	1504	1701	1892	
1308	1508	1708	1911	

Of the Numbers that are most convenient for the Purpose of dividing Large Quantities into Lesser equal Parts.

21. Those Numbers which (for the bigness of them) have the greatest Number of Divisors, and Aliquot Parts; have been wont to be made choice of, as most convenient for use; especially when there may be frequent occasion of dividing things so designed.

Hence it is, that the *English* Penny is divided into Four Farthings, (and almost all things in Four Quarters of a different Name,) because there is often occasion to divide into halves, and then again into halves. Hence also the *Roman* Pound,

2 S 2

1	1	1
2	2	a
4	3	a^2
6	4	ab
12	6	a^2b
24	8	a^3b
36	9	a^2bc
48	10	a^3b
60	12	a^2bc
120	16	a^3bc
180	18	a^2b^2c
240	20	a^3bc
360	24	a^2b^2c
720	30	a^3b^2c
840	32	a^2bcd
1260	36	a^3b^2cd
1680	40	a^4bcd
2520	48	a^3b^2cd
5040	60	a^4b^2cd
7560	64	a^3b^3cd
10080	72	a^4b^2cd
15120	80	a^4b^3cd
20160	84	a^5b^2cd
25200	90	$a^4b^2c^2d$
27720	96	a^3b^3cde
45360	100	a^4b^3cd
50400	108	$a^5b^2c^2d$
55440	120	a^4b^3cde
83160	128	a^3b^3cde
110880	144	a^4b^3cde
166320	160	a^4b^3cde
221760	168	$a^5b^2c^2de$
277200	180	$a^4b^3c^2de$
332640	192	a^5b^3cde
498960	200	a^4b^4cde
554400	216	$a^5b^3c^2de$
665280	224	a^6b^3cde

Pound, (and that which we now call the Pound *Troy Weight*,) is divided into 12 Ounces; and the *English* Shilling, into 12 pence; the Foot, into 12 Inches; the Zodiack, into 12 Signs; the Year, into 12 Months; because, beside the Division into Quarters, it is further divisible by 3. And for a like reason *Ptolemy* (and others after him) makes use of the Sexagenary division, of Integers into first Minutes, or small, or *minute*, parts of the first order; and of these, into Second Minutes, or Seconds, or small, or *minute*, parts of the Second order; and so onward: because 60 is divisible by 2 and 3, and likewise by 5. And the *Chinese* (or *Cathaians*) Number their Years (and other things) by Revolutions of 60. After this; 360 is looked on as most considerable, because it may be further divided by 2 and 3 once more: Which therefore is made the Number of Degrees in a Circle; admitting of 24 Divisors. And if this be not enough, each of these is divided into 60 Minutes; (that is, by 4, 3, 5, once, more;) and these into Seconds, and so forth. And the *English* Pound Sterling, is divided into 20 Shillings; which Number is divisible by 4 and 5, (as 12, the Number of Pence in a Shilling, is divisible by 4 and 3;) which was accounted

counted more convenient than to make another Collection of Shillings by 12 ; because this would not afford a division by 5. So that now 960 the Number of Farthings in a Pound Sterling, is for the first step (from Farthings to Pence) divisible by 4 ; for the second step (from Pence to Shillings) by 4 and 3 ; for the third step (from Shillings to Pounds) by 4 and 5. And (without taking notice of the division of Pence into Farthings) the Number of Pence in a Pound Sterling, 240, is capable of 20 Divisors ; and, of more than so many, no Number is capable which is not greater than it.

Of the foregoing Table of Numbers in page 316.

In pursuance of which notion, I have here Collected a Table of all those Numbers, which (of all not greater than themselves) have the greatest Number of Divisors ; (together with the Number of Divisors in each of them, and the Form of their Composition ;) as far as 665,280, which hath 224 Divisors. All which (except 1,) are made by the Composition of 2, 3, 5, 7, 11, (which I call *a, b, c, d, e,*) and the Powers of these, without admitting any other Prime. (But, if we would proceed to a greater Number of Divisors, we must further take in $f = 13$.) And, of these, some are of that nature, that none can have a greater Number of Divisors, which is not at least the double of them. Such are 1, 2, 6, 12, 60, 360, 2520 : But not any after these for a great way.

Of

Of the Use of a Table of Prime Numbers.

22. For resolution of some of the Questions above mentioned, (as in art. 13, 14, 17, 18, &c.) it is very convenient to have at hand a Table of Prime Numbers: That we may know, by what Numbers to make trial of the Divisions therein directed. And, because, in great Numbers, it would be tedious to make trial of all the Prime Numbers in order, it is convenient also to know, by what Prime such greater Numbers may be divided.

In order to which, it is evident, in the first place, that all even Numbers may be divided by 2; and, if the Quotient of such division be even also, it may be again divided by 2, and so continually as long as the Quotient is an even Number.

It is also evident, in the second place, that all Numbers ending in 5, are divisible by 5; and, if they end in 0, then by 2 and 5. And so continually, as long as the Quotient of such division ends in 0, or 5.

It is known also, in the 3d place, that, if the Figures of any Number being added promiscuously (without regarding the places wherein they stand) are divisible by 9, (or, casting away 9 as oft as may be, nothing remains,) such Number is also divisible by 9. As in 29097; when (the Nines being left out, and) $2 + 7 = 9$ being cast away, nothing remains; whence we may conclude, 'tis divisible by $9 = 3 \times 3$. And I add further, as a fourth observation, (though I do not find that others have taken notice of it,) that the same holds also as to the Number 3: That is, from the Figures so promiscuously added, if 3 being cast away as oft as may be, nothing remain, such Number is divisible by 3; Otherwise, it is not. As in 530,967; where, all the threes, nines and sixes being left out (as manifestly divisible by 3,) the rest $5 + 7 = 12$, is so also,
(or,

(or, which is the same, $1 + 2 = 3$;) so that all the threes, being cast away, nothing remains; whence we may conclude, that the whole Number is divisible (though not by 9) at last by 3.

The ground of this and the former Observation is one and the same: Because, the places increasing in decuple proportion, if from 10, or any Number of tens, we cast away all the nines or all the threes, there remains 1, or so many ones. So that, in case of such casting away of nines and threes, 1 and 10, have the same remainders; and so 2 and 20; 3 and 30, &c. And consequently 1, 10, 100, 1000, &c, 2, 20, 200, 2000, &c. So that the same Figure, as to this, is of the same influence in whatever place it stand.

Of Dr. Pell's Table of Prime Numbers.

23. Beside this, we have at the end of *Dr. Pell's Algebra*, (Translated and Published by *Thomas Branker*, in the Year 1668, with *Dr. Pell's* directions,) a Compendious Table of all odd Numbers (not ending in 5) as far as 100,000; shewing not only, which of them are Prime Numbers; but also by what smallest Prime Number every other of them may be divided.

So that, whatever Number be proposed, having divided it first by 2 and 5, (and if you will by 3 also,) as oft as may be, if it be capable of such division: If the result of such division do not exceed 100,000, we have direction in that Table, by what Prime it may be next divided; and then, by what Prime to divide the Quotient of such Division; and so continually, 'till we come to a Prime Number.

The reason why, in that Table, he omits all even Numbers, and all Numbers ending in 5, is obvious: to wit,
Because

Because it appears to view (without the help of a Table) that such are accordingly divisible by 2, or 5.

He might, for a like reason, have omitted also all that are divisible by 3, (because this would presently appear upon such promiscuous adding of the Figures as was but now mentioned ;) but that he could not well omit these, without disordering the Form of the Table.

Now, because, in such Tables, it is of great moment that they be carefully Computed, and exactly Printed, (because mistakes therein are not easily observed and Corrected by the Readers Eye,) I have taken care to examine that whole Table very exactly, (in the same method and with the same pains as if I were to compute it anew ;) and find that, though it had been Computed and Printed with great care, yet some few mistakes (and but a few) have escaped the Corrector's Eye. Most of which are noted in the Table of *Errata*, Printed with it. Beside which I have observed these that follow : Which (to save another Reader the like labour) I have thought fit (for his ease and satisfaction) here to note. And, these being also amended as is here directed (beside those noted in the Printed *Errata*,) the Table will then be very accurate ; and (I think,) without any Error.

Pag.	Numb.	For	Set	Pag.	Numb.	For	Set
3	5579	P	7	28	55609	3	P
5	9287	19	37	31	60701	01	101
8	14873	73	107		60799	63	163
11	20983	3	P	33	64499	13	P
16	30167	71	97		65479	3	P
	31001	29	29	34	67993	1	P
17	33409	47	P	38	75653	151	P
19	37583	13	7	41	80561	17	13
21	40049	19	29	43	85909	137	P
	40599	P	3	44	86993	79	P
	40759	3	P	47	93719	7	P
	41581	41	43	48	94769	41	97
24	46199	73	P	49	96109	3	13
27	53941	13	17		97487	3	13
28	54449	71	P				

Pag. 7, in the margin (after 43) for 37 set 47.

By

By the help of this Table, if we had the Number proposed 539,454,600, it is easy to resolve it into the Primes of which it is Composed. For first, (because of two Cyphers at the end) it is manifest that it may be divided twice by 2, and twice by 5. And then (because these Cyphers being cut off, the Remainder is yet an even Number) it may be a third time divided by 2; and the result will be 2,697,273. And, if this Number were not beyond the reach of the Table, I should seek it there; to see by what Prime it may be next divided. But, because it is too big for it; I find, upon consideration, that, the Figures being promiscuously added, and 9 cast away as oft as may be, nothing remains; and therefore that it may be divided by 9: Which being done, the next Quotient 299,697, may (for a like reason) be again divided (not by 9, but) by 3. And the Quotient 99899, is now come within the reach of this Table. And (without assaying the Prime Numbers 7, 11, 13, &c, in their order, till I come to a Prime Number by which it may be divided,) I find, by the Table, that it may be divided by 283, but not by any smaller Prime; and the Quotient of such division will be 353, another Prime. And therefore the Number proposed 539,454,600 is $= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 283 \times 353$.

But if, instead of 99,899, I had come to a Number greater than this Table, and yet not divisible by 2, 5, or 3; I must then (for want of such Table large enough) have been fain to make trial of the consequent Prime Numbers 7, 11, 13, &c, 'till by help of such I had brought it within the Compass of the Table; And, if no such can be found, before I come at a Prime as great as the Square Root of such Number; I may then conclude such Number to be a Prime.

CHAP. IV.

Monsieur FERMAT's Problems concerning Divisors and Aliquot Parts.

IT is here proper to consider of such Questions (concerning Aliquot Parts) as those on account of which Monsieur *Fermat* and Monsieur *Frenicle* did value themselves; as is to be seen in my *Commercium Epistolicum*, Epist. 1, 11, 12, 22, 25, 26, 31, 33. And in a Treatise purposely Published on this occasion by Monsieur *Frenicle*, intituled, *Solutio duorum Problematum, circa numeros Cubos & Quadratos, quæ tanquam insolubilia universis Europæ Mathematicis à Clarissimo Viro D. Fermat sunt proposita, &c.*, à D. B. F. D. B. inventa, &c. (that is, à Domino Bernardo Frenicle de Bessy.) Parisiis apud Jacobum Langlois, &c. 1657, in which he glories much that he was able to solve them. And amongst Monsieur *Fermat's* posthumous Works, (Published since his death) the Publisher is pleased to insert his formal Challenge of me to solve them (with some others Letters to and from Monsieur *Fermat*, concerning the same) in these Words :

Problemata proposita à D. Fermat.

Proponatur (si placet) Wallisio, & reliquis Angliæ Mathematicis, sequens Quæstio Numerica.

Invenire Cubum, qui, additus omnibus suis partibus aliquotis, conficiat Quadratum. Exempli gratiâ, Numerus 343 est Cubus à latere 7. Omnes ipsius partes aliquotæ sunt 1, 7, 49, quæ, adjunctæ ipsi 343, conficiunt numerum 400, qui est Quadratus à latere 20. Queritur alius Cubus numerus ejusdem naturæ.

Queritur etiam numerus Quadratus, qui, additus suis partibus aliquotis, conficiat numerum Cubum.

Has

Has solutiones expectamus: Quas, si Anglia aut Gallia Belgica & Celtica non dederint, Dabit Gallia Narbonensis; easque, in pignus nascentis amicitiae, Domino Digby offeret & dicabit.

But was not so kind (though he there insert also divers Letters to and from Monsieur *Fermat*, concerning the same) as to insert those of mine, wherein I solved these (and others of) his Problems: Nor those of Monsieur *Fermat*, wherein he acknowledgeth that I had so done. Which are to be seen in my *Commercium Epistolicum*, in Epist. 23, 28, 29, 47, and elsewhere.

To those two Problems, I added a third of a like nature:

Invenire duos numeros Quadratos, qui, partibus suis aliquotis additi, eandem efficiant summam. Exempli gratia, $16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1$. Inveniantur istiusmodi alii duo.

The whole Mystery of solving these (and such like) Questions, I there discover in Epist. 23, which depends on what is here delivered in art. 8, 9, 10, 11, 12, of the Chapter here next preceeding.

For, 1. A Number added to all its Aliquot Parts, is all one as the Aggregate of its Divisors. 2. The Divisors of any Power of a Prime Number, (as of a) is a Geometrical Progression from 1 to such Power; as, for instance, of a^5 , the Divisors are 1, a , aa , a^3 , a^4 , a^5 . 3. And therefore the sum of such Geometrical Progression is the Aggregate of those Divisors. 4. This Aggregate is conveniently expressed by the Primes which Compose it. 5. The Divisors of any Power, or Degree, of one such Prime, severally Multiplied into all those of any Power, or Degree, of any other Prime, give all the Divisors of the Compound of those Powers. 6. And therefore the Aggregate of those first into the Aggregate of those second, give the Aggregate of the Divisors of such Compound, (For, by the common practice of Multiplication, all the Members of one Number, or Aggregate, Multiplied severally into all the Members of another, are equivalent to the whole of the one, multiplied into the whole of the other.) 7. And therefore the Primes Composing this

last Aggregate, are the same with those of both the Aggregates which Compose it. 8. And the same is in like manner to be argued, in case any Power, or Degree, of a third, fourth, or further, Prime, be continually Multiplied with those foregoing: provided always, that they be all several Primes, and not any of the former Primes repeated; for, in such case we are to follow the direction of art. 11, Chap. preceeding.

As, for instance; supposing $a = 2$, and therefore $a^3 = 32$: All the Divisors hereof (or the Aggregate of such Divisors) are $1 + a + aa + a^3 + a^4 + a^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63 = 3 \times 3 \times 7$. And supposing $b = 3$, and therefore $b^4 = 81$: The Aggregate of the Divisors hereof are $1 + b + bb + b^3 + b^4 = 1 + 3 + 9 + 27 + 81 = 121 = 11 \times 11$: And therefore, of a^3b^4 , the Aggregate of Divisors is $63 \times 121 = 3 \times 3 \times 7, \times 11 \times 11$. And supposing further $c = 5$, and therefore $c^3 = 125$: The Aggregate of the Divisors hereof are $1 + c + cc + c^3 = 1 + 5 + 25 + 125 = 156 = 2 \times 2 \times 3 \times 13$: And therefore, of $a^3b^4c^3$, the Aggregate of Divisors is $63 \times 121 \times 156 = 3 \times 3 \times 7, \times 11 \times 11, \times 2 \times 2 \times 3 \times 13$, or $2 \times 2 \times 3 \times 3 \times 3 \times 7 \times 11 \times 11 \times 13$. And so onwards, in case of further Compositions.

Now, this being universal; it will be easy to make application thereof, to the particular cases proposed; or to any other of like nature.

As for Example.

I. The first Question, is, *To find a Cube Number, which added to all its Aliquot Parts will make a Square*; (that is, the Aggregate of whose Divisors shall be a Square Number.)

Here it is manifest, that such Cube Number must be either the Cube of some Prime, (or at least the second, third, fourth, or further Cube of such Prime; that is, some Power thereof whose exponent is divisible by 3;) or else Compounded by the continual Multiplication of such Cubes (first, second, third, and so forth,) of two or more such Prime Numbers. (For all such, will be Cube Numbers, and no other but such.)

Now

Now, if we can find any such Cube (first, second, third, &c.) of any one Prime Number, whereof the Aggregate of Divisors being expressed in Primes, those Primes will be all Pairs, (that is, each of them occurring an even Number of times;) such Aggregate ('tis manifest) will be a Square Number; and therefore such Cube, will be such as is required.

And such Cube is $343 = 7 \times 7 \times 7$; whose Divisors are $1 \times 7 \times 49 \times 343 = 400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$; which is the Square of $2 \times 2 \times 5 = 20$. *All these should have the sign of addition*

When the Cubes (first, second, third, or others,) of several Primes, have not their Aggregate of Divisors expressible by Pairs of Primes; yet may the Compound of Two, Three, or more of such Cubes continually Multiplied (which will also be a Cube Number,) have its Aggregate of Divisors (which is the Compound of the several Aggregates continually Multiplied) so expressed: Namely, if the Cubes so to be Compounded be so chosen as that, what Primes in expressing some of the Aggregates be single, may be Paired by like single Primes in some other of them.

Thus, for the Cube of 47, the Aggregate of Divisors (expressed in Primes) is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$; where (beside Pairs) we have 2, 3, 5, 13, 17; singly: And, for the Cube of 5, the Aggregate is $2 \times 2 \times 3 \times 13$, where (beside Pairs) we have 3, 13, solitary; which (joined to those before) serve to Pair 3, 13, but leave 2, 5, 17, yet solitary: And, for the Cube of 13, the Aggregate is $2 \times 2 \times 5 \times 7 \times 17$, which afford fellows to 5, 17, but leaves us 2, 7, yet solitary: And, for the Cube of 41, the Aggregate is $2 \times 2 \times 3 \times 7 \times 29 \times 29$; where (beside Pairs) we have 3, 7, solitary; which afford a fellow to 7, but leave 2, 3, solitary. So that for the Cube of $47 \times 5 \times 13 \times 41$, we have (beside Pairs) 2, 3, solitary. Which may thus be Paired.

For the Cube of 11, the Aggregate of Divisors is, $2 \times 2 \times 2 \times 3 \times 61$, where (beside Pairs) we have 2, 3, 61, solitary; which afford fellows to 2, 3, but leave 61, solitary: And, for the Cube of 27 (or the third Cube of 3, the Aggregate is $2 \times 2 \times 11 \times 11 \times 61$; which (beside Pairs) affords a fellow to 61. So that, for the Cube of $47 \times 5 \times 13 \times 41$

$\times 41 \times 11 \times 27$, (or $27 \times 5 \times 11 \times 13 \times 41 \times 47$) the Aggregate of Divisors, is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$, $\times 2 \times 2 \times 3 \times 13$, $\times 2 \times 2 \times 5 \times 7 \times 17$, $\times 2 \times 2 \times 3 \times 7 \times 29 \times 29$, $\times 2 \times 2 \times 2 \times 3 \times 61$, $\times 2 \times 2 \times 11 \times 11 \times 61$: Or (putting the Primes in order) $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 11 \times 11 \times 13 \times 13 \times 17 \times 17 \times 29 \times 29 \times 61 \times 61$; where we have 2, sixteen times; 3, four times; and 5, 7, 11, 13, 17, 29, 61, twice; which therefore (being all continually Multiplied) must needs afford a Square Number. Which was the thing required to be found in Monsieur Fermat's first Question.

In like manner; if with the Cube of $47 \times 5 \times 13 \times 41$ (as before) we Compound the Cubes of 2, and of 3, where we have the Aggregates 3×5 , and $2 \times 2 \times 2 \times 5$, which (beside Pairs) afford us 2, 3, solitary; which afford fellows to 2, 3, that were solitary before. And therefore for the Compound Cube of $47 \times 5 \times 13 \times 41 \times 2 \times 3$ (or $2 \times 3 \times 5 \times 13 \times 41 \times 47$) we shall have (in the Compound Aggregate of Divisors) these Primes Components, 2, fourteen times; 3 and 5, four times; 7, 13, 17, and 29, twice: Which being all continually Multiplied will also make a Square Number. Which was the thing required to be found in Monsieur Fermat's first Question.

These two Compound Cubes, if they be further Compounded with the Cube of 7 (which is no ingredient in either of them) will afford two more; whose Aggregate of Divisors will (beside the Primes in each of them respectively) have these farther Primes Components, 2, four times; and 5, twice: Which, being Compounded with the fore-mentioned Squares, will still afford Square Numbers.

So have we five Cubes, whose Aggregate of Divisors are Squares.

Roots of the Cubes.

7.

$$27 \times 5 \times 11 \times 13 \times 41 \times 47.$$

$$2 \times 3 \times 5 \times 13 \times 41 \times 47.$$

$$27 \times 5 \times 7 \times 11 \times 13 \times 41 \times 47.$$

$$2 \times 3 \times 5 \times 7 \times 13 \times 41 \times 47.$$

Roots

Roots of the Squares.

$$2 \times 2 \times 5.$$

$$2 \times (\text{Eight-times}) \times 3 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 29 \times 61.$$

$$2 \times (\text{Seven-times}) \times 3 \times 3 \times 5 \times 5 \times 7 \times 13 \times 17 \times 29.$$

$$2 \times (\text{Ten-times}) \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 \times 13 \times 17 \times 29 \times 61.$$

$$2 \times (\text{Nine-times}) \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 13 \times 17 \times 29.$$

In all which I make use of no Cube of a Prime which is not less than 100. And, in like manner, may other such Cubes be found; as is there shewed in Epist, 23, and 24. Such as these :

Roots of the Cubes.

$$2 \times 3 \times 5 \times 13 \times 17 \times 31 \times 41 \times 191.$$

$$2 \times 3 \times 5 \times 7 \times 13 \times 17 \times 31 \times 41 \times 191.$$

$$3 \times 3 \times 3 \times 5 \times 11 \times 13 \times 17 \times 31 \times 41 \times 191.$$

$$3 \times 3 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 31 \times 41 \times 191.$$

$$17 \times 31 \times 47 \times 191.$$

$$7 \times 17 \times 31 \times 47 \times 191.$$

Roots of the Squares.

$$2 \times (\text{Twelve-times}) \times 3 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13 \times 17 \times 29 \times 29 \times 37.$$

$$2 \times (\text{Fourteen-times}) \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 13 \times 17 \times 29 \times 29 \times 37.$$

$$2 \times (\text{Thirteen-times}) \times 3 \times 3 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 29 \times 29 \times 37 \times 61.$$

$$2 \times (\text{Fifteen-times}) \times 3 \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 \times 13 \times 17 \times 29 \times 29 \times 37 \times 61.$$

$$2 \times (\text{Ten-times}) \times 3 \times 3 \times 5 \times 13 \times 17 \times 29 \times 37.$$

$$2 \times (\text{Twelve-times}) \times 3 \times 3 \times 5 \times 5 \times 13 \times 17 \times 29 \times 37.$$

In all which I make use of no Cube of a Prime Number which is not less than 200.

But, in order to make these Inquiries for such Cubes; it is expedient to have at hand a Table of the Cubes of Prime Numbers (and of the second, third, or further Cubes, of the lesser of them,) or of the Roots of such Cubes; with the

the Aggregate of Divisors (in each of those Cubes) expressed in Primes.

And, to save the Reader the labour of computing such a-new, I here subjoin what I have at hand.

Roots of the Cubes.	Aggregate of their Divisors.
1	1
2	3×5
4	$127 = 55 = 5 \times 11$.
8	$3 \times 11 \times 31$
16	$8191 = 4364 = 7 \times 17 \times 257$.
32	$3 \times 5 \times 17 \times 257$
3	$2 \times 2 \times 2 \times 5$
9	1093
27	$2 \times 2 \times 11 \times 11 \times 61$
81	797161
243	$2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 17 \times 41 \times 193$
5	$2 \times 2 \times 3 \times 13$
25	19531
125	$2 \times 3 \times 11 \times 71 \times 521$
7	$2 \times 2 \times 2 \times 2 \times 5 \times 5$
11	$2 \times 2 \times 2 \times 3 \times 61$
13	$2 \times 2 \times 5 \times 7 \times 17$
17	$2 \times 2 \times 3 \times 3 \times 5 \times 29$
19	$2 \times 2 \times 2 \times 5 \times 181$
23	$2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 53$
29	$2 \times 2 \times 3 \times 5 \times 421$
31	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 13 \times 37$
37	$2 \times 2 \times 5 \times 2603$
41	$2 \times 2 \times 3 \times 7 \times 29 \times 29$
43	$2 \times 2 \times 2 \times 5 \times 5 \times 17 \times 37$
47	$2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$
53	$2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 281$
59	$2 \times 2 \times 2 \times 3 \times 5 \times 1741$
61	$2 \times 2 \times 31 \times 1861$
67	$2 \times 2 \times 2 \times 5 \times 17 \times 449$
71	$2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 2521$
73	$2 \times 2 \times 5 \times 13 \times 37 \times 41$

Roots

Roots of the Cubes.	Aggregates of their Divisors.
79	$2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 3121$
83	$2 \times 2 \times 2 \times 3 \times 5 \times 7 \times 13 \times 53$
89	$2 \times 2 \times 3 \times 3 \times 5 \times 17 \times 233$
97	$2 \times 2 \times 5 \times 7 \times 7 \times 941$
101	$2 \times 2 \times 3 \times 17 \times 5101$
103	$2 \times 2 \times 2 \times 2 \times 5 \times 13 \times 1061$
107	$2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 229$
109	$2 \times 2 \times 5 \times 11 \times 13 \times 457$
113	$2 \times 2 \times 3 \times 5 \times 19 \times 1277$
127	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 1613$
131	$2 \times 2 \times 13 \times 19 \times 2293$
137	$2 \times 2 \times 3 \times 5 \times 23 \times 1877$
139	$2 \times 2 \times 2 \times 5 \times 67627$
149	$2 \times 2 \times 3 \times 5 \times 5 \times 11 \times 101$
151	$2 \times 2 \times 2 \times 2 \times 13 \times 19 \times 877$
157	$2 \times 2 \times 5 \times 5 \times 17 \times 29 \times 79$
163	$2 \times 2 \times 41 \times 2657$
167	$2 \times 2 \times 2 \times 3 \times 7 \times 2789$
173	$2 \times 2 \times 3 \times 5 \times 29 \times 41 \times 73$
179	$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 37 \times 433$
181	$2 \times 2 \times 7 \times 13 \times 16381$
191	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 17 \times 29 \times 37$
193	$2 \times 2 \times 5 \times 5 \times 5 \times 14453$
197	$2 \times 2 \times 3 \times 3 \times 5 \times 42691$
199	$2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 15801$

If, in the Question proposed, it had been required that the Aggregate of Divisors (of the Cube sought) should be (not a Square Number, but) the *Double*, *Treble*, or otherwise *Multiple*, of a Square Number: The process would be just the same, (and the same Table will serve,) save that, then, the Aggregate is to be divisible by 2, 3, or such other Number as is the exponent of the proposed Multiple, and the rest of the Primes composing it to be all Pairs.

Thus, if the *Decuple* of a Square be required; the Cube of 3 will answer it; where the Aggregate is $2 \times 2 \times 2 \times 5$; that is, beside $2 \times 5 = 10$, the other Components are Pairs.

If the *Quadruple* of a Square (which must therefore itself
2 U
be

be a Square;) the Cube 7 Answers it; whose Aggregate is $2 \times 2 \times 2 \times 2 \times 5 \times 5$: Out of which, if we exempt $2 \times 2 = 4$, the rest are Pairs. And so will any other Cube whose Aggregate of Divisors is an even Square, and therefore divisible by 4.

If the *Sextuple* be required: The Cube of 27×11 answers it; where the Aggregate is $2 \times 2 \times 11 \times 11 \times 61, \times 2 \times 2 \times 2 \times 3 \times 61$. Whence if we exempt $2 \times 3 = 6$, the rest are Pairs: And so will also (for the same reason) the Cube of 2×3 ; where the Aggregate is $3 \times 5, \times 2 \times 2 \times 2 \times 5$. And the like in other cases.

But if such Multiple should be required, as that no Aggregate can be found (or not within certain limits) which, being divided by the Exponent of that Multiple, will leave the rest of the Prime Components Pairs; such case (at least within such limits) is an impossible case.

As, if we demand a Square's Multiple by 23, 43, or 47; and confine ourselves to the Cubes of the Table foregoing; it is manifest that (without assuming the Cube of some other Prime, or some further Cube of some of these,) it cannot be done. For here, amongst all the Prime Components of the Aggregates, the Numbers 43, and 47, come not at all; and though 23 come once (at the Cube of 137) yet it is there joyned with 1877, which (coming no more) cannot be Paired by any such Composition of the proposed Aggregates. (Remembring always, what was before noted, that the Aggregates for two or more Cubick Powers of the same Prime, are not here to be Compounded.) So that (within the limits of the Table) the case is not possible. And the like may be shewed of many others: I say, not possible *within the limits of this Table*. But, to say it is *not at all possible*, through the whole extent of all possible Numbers; is (I think) too bold an assertion for any to make out.

Of the Second Question proposed by Monsieur FERMAT.

See above, page 322.

II. The Second Question is, (*To find a Square Number, which*

which added to all its Aliquot Parts will make a Cube; that is, the Aggregate of whose Divisors shall be a Cubick Number.)

And here the process is much the same as before; save that here we shall need a Table of Square Numbers, (as there of Cubes,) with their Aggregate of Divisors expressed in Primes: And here we are to find out, or so to Compound, the Aggregates, as that the Primes expressing them may be (not Couples or Duplicates, as there, but) Triplicates: That is, that each Prime may occur three, six, nine, or other Number of times divisible by three.

But, though the process be much the same, yet the success will not be altogether so ready as there; because Triplicates of the Components will not be so easily adjusted as Duplicates. (And, for the same reasons, if Biquadrates, or Surfolids, or some higher Powers, were required; the process would still be much the same, but the trouble of finding such would still be increased.)

Such Table of Squares (because I have it at hand) I shall here subjoin; to save the Reader (who shall think fit to give himself the trouble of inquiring into such Questions) the labour of Computing the same again.

Roots of the Squares. | Aggregate of their Divisors.

1	1
2	7
4	~~31 = 21 = 7 × 3~~
8	~~127 = 23 × 5~~
16	~~511 = 7 × 73~~ <i>23263</i>
32	~~2047 = 23 × 89~~ <i>231037 = 157 × 7</i>
64	~~8191 = 23 × 11 × 31~~ <i>73 × 19 × 5</i>
128	~~32767 = 7 × 31 × 151~~
256	131071
3	13
9	121 = 11 × 11
27	1093
81	9841 = 13 × 757
243	88573 = 23 × 3851

2 U 2

Roots

Roots of
the Squares.

Aggregate of their Divisors.

5	31
25	781 = 11 × 71
125	19531
625	488281 = 19 × 31 × 829
7	57 = 3 × 19
49	2801
343	137257 = 29 × 4733
2401	6725601 = 3 × 3 × 19 × 37 × 1063
11	133 = 7 × 19
121	16105 = 5 × 3221
13	183 = 3 × 61
169	30941
17	307
289	88741
19	381 = 3 × 127
361	137561 = 151 × 911
23	553 = 7 × 79
29	871 = 13 × 67
31	993 = 3 × 331
37	1407 = 3 × 7 × 67
41	1723
43	1893 = 3 × 631
47	2257 = 37 × 61
53	2863 = 7 × 409
59	3541
61	3783 = 3 × 13 × 97
67	4557 = 3 × 7 × 7 × 31
71	5113
73	5403 = 3 × 1801
79	6321 = 3 × 7 × 7 × 43
83	6973 = 19 × 367
89	8011
97	9507 = 3 × 3169
101	10303
103	10713 = 3 × 3571
107	11557 = 7 × 13 × 127
109	11991 = 3 × 7 × 571

Roots

Roots of the Squares	Aggregate of their Divisors.
113	12883 = 13 × 991
127	16257 = 3 × 5419
131	17293
137	18907 = 7 × 37 × 73
139	19461 = 3 × 13 × 499
149	22351 = 7 × 3193
151	22953 = 3 × 7 × 1093
157	24807 = 3 × 8269
163	26733 = 3 × 7 × 19 × 67
167	28057
173	30103
179	32221 = 7 × 4603
181	32943 = 3 × 79 × 139
191	36673 = 7 × 13 × 13 × 31
193	37443 = 3 × 7 × 1783
197	39007 = 19 × 2053
199	39801 = 3 × 13267
211	44733 = 3 × 13 × 31 × 37
223	49953 = 3 × 16651
227	51757 = 73 × 709
229	52671 = 3 × 97 × 181
233	54523 = 7 × 7789
239	57361 = 19 × 3019
241	58323 = 3 × 19441
251	63253 = 43 × 1471
257	66307 = 61 × 1087
263	69433 = 7 × 7 × 13 × 109
269	72631 = 13 × 37 × 151
271	73713 = 3 × 24571
277	77007 = 3 × 7 × 3667
281	79243 = 109 × 727
283	80373 = 3 × 73 × 367
293	86143
307	94557 = 3 × 43 × 733
311	97033 = 19 × 5107
313	98283 = 3 × 181 × 181
317	100807 = 7 × 14401

Roots

Roots of
the Squares.

Aggregate of their Divisors.

331	109893 = 3 × 7 × 5233
337	113907 = 3 × 43 × 883
347	120757 = 7 × 13 × 1327
349	122151 = 3 × 19 × 2143
353	124963 = 19 × 6577
359	129241 = 7 × 37 × 499
367	135037 = 7 × 101 × 191
373	139503 = 3 × 7 × 7 × 13 × 73
379	144021 = 3 × 61 × 787
383	147073
389	151711 = 7 × 21673
397	158007 = 3 × 31 × 1699
401	161203 = 7 × 23029
409	167691 = 3 × 55897
419	175981 = 13 × 13537
421	177663 = 3 × 59221
431	186193 = 7 × 67 × 397
433	187923 = 3 × 37 × 1693
439	193161 = 3 × 31 × 31 × 67
443	196693
449	202051 = 97 × 2083
457	209307 = 3 × 7 × 9967
461	213083 = 13 × 37 × 443
463	214833 = 3 × 19 × 3769
467	218557 = 19 × 11503
479	229921 = 43 × 5347
487	237657 = 3 × 7 × 11317
491	241573 = 37 × 6529
499	249501 = 3 × 7 × 109 × 109

Now it is manifest, upon view, that (if we confine ourselves to the limits of this Table) many of these Numbers are not of use to the present purpose. Because many of the Primes (amongst the Aggregates) come but once; as 5. 29. 71. 89. 101. 139. 191. 307. 331. 397. 409. 443. 571. 631. 709. 727. 733. 757. 787. 829. 883. 911. 991. 1063. 1087. 1327. 1471. 1693. 1699. 1723. 1783. 1801. 2053. 2083. 2143.

2143. 2801. 3019. 3169. 3193. 3221. 3541. 3571. 3667.
 3769. 3851. 4603. 4733. 5107. 5113. 5233. 5347. 5419.
 6529. 6577. 7789. 8011. 8191. 8269. 9967. 10303. 11317.
 11503. 13267. 13537. 14401. 16651. 17293. 19441. 19531.
 21673. 23029. 24571. 28057. 30103. 30941. 55897. 59221.
 86143. 88741. 131071. 147073. 196693. Others but twice
 (not thrice) as 23. 79. 367. 499. 1093. And therefore
 cannot by any Composition (within these limits) make a
 Cube. And, consequently, all the Squares to which any of
 them belong, are to be laid aside as not of use. And those
 are, the Squares of 32, 64, 256, 27, 81, 243, 25, 125,
 625, 49, 343, 2401, 121, 169, 17, 289, 361, 23, 31,
 41, 43, 53, 59, 71, 73, 83, 89, 97, 101, 103, 109, 113,
 127, 131, 139, 149, 151, 157, 167, 173, 179, 181, 193,
 197, 199, 223, 227, 233, 239, 241, 251, 257, 271, 277,
 281, 283, 293, 307, 311, 317, 331, 337, 347, 349, 353,
 359, 367, 379, 383, 389, 397, 401, 409, 419, 421, 431,
 433, 443, 449, 457, 461, 463, 467, 479, 487, 491.
 (And the Square of 1, is, in this case, insignificant; be-
 cause a Multiplication by 1 makes no alteration.) And,
 these being laid aside, we must also lay aside the Squares of
 128, 9, 13, 47, 61, 79, 229, 269. Because, in those that
 remain, 43 occurs but once; and 11, 61, 97, 151, but
 twice. And, those being laid aside, we must also lay aside
 the Squares of 137, 211, 313, because, in those now re-
 maining, 37, 181, occur but twice. And (137 being laid
 aside) the Squares of 16, 373, must also be laid aside;
 because now 73 comes but twice.

So that we have now but these few left for consideration, to
 wit, the Squares of 2, 4, 8, 3, 5, 7, 11, 19, 29, 37, 67, 107,
 163, 191, 263, 439, 499. Which, with their Aggregates,
 stand thus :

2 7	5 31	29 13 × 67	163 3 × 7 × 19 × 67
4 31	7 3 × 19	37 3 × 7 × 67	191 7 × 13 × 13 × 31
8 127	11 7 × 19	67 3 × 7 × 7 × 31	263 7 × 7 × 13 × 109
3 13	19 3 × 127	107 7 × 13 × 127	439 3 × 31 × 31 × 67
			499 3 × 7 × 109 × 109

In

In which there is no Prime (amongst the Aggregates) which doth not occur at least three times. That is, 3 seven times; 7 eleven times; 13 and 31 six times; 67 four times; 19, 109, 127, three times.

Of these I will first consider 127; which, because it comes but thrice, we must take all or none of them. If all, then this (at 107) brings in 13; which must therefore be trebled. And it must be done one of these three ways, either by taking in the Squares of 3 and 29; or of 3 and 263; or of 191 alone.

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31

If the first way, this (at 29) brings in 67. Which (that it may be trebled) brings in two of these 3 Squares 37, 163, 439. Of which, if 163 be one, this (because of 19) brings in the Squares 7 and 11. And if, for the other, we take the Square of 37; this brings in 3 and 7 a fourth time, and therefore either each of them must come in twice more (that we may have them six times) or else 37 must here be laid aside. Now if, for 3 twice, we take (for one of them) the Square of 439, this brings in a fourth 67; which must not be (unless we could have it six times, which we cannot.) Therefore, if at all, this 3 twice, must be supplied by the Squares of 67 and 499 (for there is no other supply;) which brings in 109 twice; and this (that it may be tripled) requires the Square of 263. But, with this, comes in 13 a fourth time; and therefore (that we may have it six times) we must take in the Square of 191. But, by this time, we have 7 ten times; which must not be, unless we could (which we cannot) have it twelve times. Therefore the Square of 37 must here be laid aside. If then (retaining that of 163) we take (instead of 37) the Square of 439; this brings in 3 a fourth time; which therefore we must have twice more. But not from the Square of 37 (because already laid by, and because it would bring in a fourth 67;) therefore,

therefore, if at all, from the Squares of 67 and 499 (as before,) which requires that of 263; and, this, that of 191, as before. But now we have 31 a fourth time, which requires it twice more; which is not to be had, save at the Squares of 4 and 5; whereof that of 4 is not to be admitted, as being included in that of 8 already taken. So that the Square of 163 cannot be taken either with that of 37 or of 439, and must therefore be laid aside; (and, with it, the Squares of 7 and 11.) And consequently (retaining that of 3 and of 29,) we must (for trebling of 67) take the Squares of 37 and 439. And here we have

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
439	3, 31, 31, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31
4	31
5	31

31 twice, and must therefore have it a third time: But not from the Square of 4; (because included in that of 8;) Therefore either from that of 5, or of 191. If from that of 5; we shall want a third 7 (having yet but two;) which we cannot have from the Square of 2 (because included in 8;) nor from 163 (because already

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
5	31

rejected;) nor from that of 11 (because already excluded with that of 163;) nor from that of 191, because this would bring in a fourth 31, (which may not be, because we cannot have it six times without the Square of 4, which is included in that of 8;) nor from that of 69 (for the same reason;) nor from that of 499, because this cannot stand without that of 263; nor from both these together; because then we shall have it five times, but cannot have it a sixth; (all the rest wherein 7 is found, being already excluded.) Therefore (omitting that of 5) we must (if at all) have a third 31 from the Square of 191. But this brings in a fourth and fifth 13; which (for a sixth) will require the Square of 263; and this (because of 109) the

2 X

Square

8 | 127
 19 | 3, 127
 107 | 7, 13, 127
 3 | 13
 29 | 13, 67
 37 | 3, 7, 67
 439 | 3, 31, 31, 67
 191 | 7, 13, 13, 31
 263 | 7, 7, 13, 109
 499 | 3, 7, 109, 109

consequently, this first way (by the Squares of 3 and 29) doth not succeed.

8 | 127
 19 | 3, 127
 107 | 7, 13, 127
 3 | 13
 263 | 7, 7, 13, 109
 499 | 3, 7, 109, 109
 7 | 3, 19
 163 | 3, 7, 19, 67
 11 | 7, 19
 37 | 3, 7, 67
 439 | 3, 31, 31, 67
 67 | 3, 7, 7, 31

Square of 499. And this (beside Triplicates) brings in a fourth 3; (which therefore will afford, not a Cube, but the Triple of a Cube, if that had been required;) we want therefore 3 twice more (to make it up six times;) but can have neither of them from the Squares of 7 or 163 (as being already excluded,) nor from that of 67, (as bringing in a fourth 3,) and therefore not at all. And,

The second way of supplying 13 twice, (which at the Square of 107 were wanting;) is, from the Squares of 3 and 263: Which (because of 109) requires that of 499. And, because (amongst the Aggregates) we have 3 twice; we must have it a third time. If, for this, we take in the Square of 7, or of 163; either of these (because of 19) brings in the other, and that of 11. And now, because of 67 once, we must have it twice more. But not from the Square

of 29 (being already excluded as not to be taken with that of 3;) and therefore from the Squares of 37, and 439. And, by this time we have 3 six times (and more than so, we may not have it, unless we could have it nine times;) and 7 we have 7 times, and therefore must have it twice more: But, not from the Square of 2 (as being included in that of 8;) nor from that of 191, (because this would bring in 13 a fourth and a fifth time, which would require a sixth, from the Square of 29 already rejected;) therefore, if at all, from the Square of 67. But neither can this be, (because it brings in a seventh 3; which may not be, there being no more to make it up nine times:) And, consequently, the third 3 (wanting at the Square of 499) is not to be supplied

supplied from the Squares of 7, or of 163. If then (omitting these two) we should take (for a third 3) the Square of 37 or of 439, either of these (because of 67) would bring in the other, and also require that of 29, or of 163, already rejected. If then (omitting

these of 37 and 439) we take (for a third 3) the Square of 67; this brings in 31, which is therefore to be Tripled. But not from the Square of 4 (as included in that of 8;) nor from the Square of 191 (because that would bring in a fourth and fifth 13, which would require a sixth from the Square of 29 already rejected;) nor from the Square of 439 (because of 67 there, which would bring in that of 29, or 37, or 163, already rejected) nor from the Square of 5, because (though that would afford a second 31,) a third would yet be wanting, and not to be had. And, consequently, (there being no other place from whence to fetch a third 3) this second way will not succeed.

The third way for supplying 13 twice, (which at the Square of 107 were wanting) is (omitting the Squares of 3, 29, 263,) from the Square of 191. And, because here we have 31 once, this must be Tripled. But not from the Square of 4: (as included in 8;) And therefore, if at all, either from that of 439 (where it is twice,) or from the Squares of 5 and 67. If from that of 439; then 67 (here

found) must be Tripled; but not from the Square of 29 (as already excluded,) therefore from those of 37, and 163; and this last (because of 19) calls in those of 7 and 11. But, by this time, we have 3 five times, and therefore

2 X 2

should

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
37	3, 7, 67
439	3, 31, 31, 67

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
67	13, 7, 7, 31

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31

should have it a sixth time; but not from the Square of 499 (for that would recall that of 263 already rejected;) therefore, if at all, from that of 67; but we shall then have 7 seven times; which is not to be admitted, since we cannot have it nine times. Therefore (omitting that of 439, and therefore those of 37 and 163) take we those of 5 and 67.

8|127
19|3, 127
107|7, 13, 127
191|7, 13, 13, 31
5|31
67|3, 7, 7, 31

And, by this time; we have 7 four times; and therefore, if at all, we must have it twice more. But not from the Square of 2 (as included in 8;) nor from that of 37 or 163 (as already rejected, with that of 439;) nor from that of 11 (which, because of 19, would bring us back to that of 163 already rejected;) nor from 499 (which, because of 109, would bring us back to that of 263 already laid aside;) and therefore not at all. So that this third way fails also: And, consequently, the Squares of 8, 19, 107, (where we meet with 127,) must all be laid aside.

We have then but these left to be further considered.

2|7 5'31 29|13, 67 163|3,7,19,67 439|3,31,31,67
4|31 7'3,19 37|3,7,67 391|7,13,13,31 499|3,7,109,109
3|13 11|7,19 67|3,7,7,31 263|7,7,13,109

And here we will begin with the Prime 109; which, because it comes but once at the Square of 263, and twice at that of 499; these must either both be taken, or both omitted.

263|7, 7, 13, 109
499|3, 7, 109, 109
3|13
29|13, 67
37|3, 7, 67
163|3, 7, 19, 67
7|3, 19
11|7, 19
67|3, 7, 7, 31
439|3, 31, 31, 67

And because, in these, we have 13 once; this must be taken twice more. And therefore either from the Squares of 3 and 29, or from that of 191 above; (since we have it now but five times in all.)

If the first way; then, because of 67 once, we must take it twice more; from two Squares of these three, 37, 163, 439. First, let those be the Squares

Squares of 37 and 163; therefore (because of 19) we must take also those of 7 and 11. And, by this time, we have 3 four times, (and this affords us, not a Cube, but the Triple of a Cube, if that were required;) we must therefore take it twice more; which is only to be had at the Squares of 67 and 439, (for now we have it but six times in all,) but this brings in a fourth 67 which cannot be admitted. Secondly, let it be the Squares of 37 and of 439: which brings in 31 twice, and we must therefore have it a third time. Which if we take from the Square of 67; this brings in a fourth 3; which will require two more, from the Squares of 7 and 163; which will bring in a fourth 67. If from the Square of 191; this brings in a fourth and fifth 13, which cannot be admitted, because we have not a sixth. If from the Square either of 4, or of 5; either of these (beside Triplicates) would leave us 7 four times (which would afford, not a Cube, but the Septuple of a Cube, if that had been required;) but this requires 7 twice more. Neither of which can be had from the Squares of 67, or 191, (as being already rejected;) nor from that of 163 (as bringing in a fourth 67;) and therefore, if at all, from the Squares of 2 and 11. But this would bring in 19; and therefore (to Triple it) will call in the Squares of 7 and 163; (which last is already rejected, and would bring in a fourth 67;) therefore not at all. Thirdly, (omitting that of 37) let this 67 twice, be taken from the Squares of 163 and 439. But this (because of 19) calls in the Squares of 7 and 11; and consequently, (because then we have 3 four times) the Squares of 37 and 67 already rejected. So that this first way succeeds not,

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 3, 67
191	7, 13, 13, 31

263 | 7, 7, 13, 109
 499 | 3, 7, 109, 109
 3 | 13
 29 | 13, 67
 37 | 3, 7, 67
 439 | 3, 31, 31, 67
 4 | 5 | 31
 2 | 7
 11 | 7, 19
 7 | 3, 19
 163 | 3, 7, 19, 67

263 | 7, 7, 13, 109
 499 | 3, 7, 109, 109
 3 | 13
 29 | 13, 67
 163 | 3, 7, 19, 67
 439 | 3, 31, 31, 67
 7 | 3, 19
 11 | 7, 19
 37 | 3, 7, 67
 67 | 3, 7, 7, 31

263 | 7, 7, 13, 109
 499 | 3, 7, 109, 109
 191 | 7, 13, 13, 31
 439 | 3, 31, 31, 67
 37 | 3, 7, 67
 163 | 3, 7, 19, 67
 7 | 3, 19
 11 | 7, 19
 67 | 3, 7, 7, 31
 4 | 31
 5 | 31

If we take the second way of supplying 13 twice, (which at the Squares of 263 and 499 were wanting) by the Square of 191 (omitting those of 3 and 29;) then, because here we have 31 once, which must therefore be supplied twice more: We will first try whether it may be done by the Square of 439 (where it comes twice;) and then whether it can be done without this.

If we supply it from the Square of 439; this brings in 67, which must therefore be Tripled: But not by the Square of 29 (as already rejected, and as bringing in a fourth 13;) therefore from those of 37 and 163. Where because we have 19 once, we must have it twice more, from the Squares of 7 and 11. And by this time we have 7 seven times, and must therefore have it twice more: And we have 3 five times, and must therefore have it once more. Both which we may have from the Square of 67 (and from thence only, because 3 is to be had no where else;) and now we have 31 a fourth time; which requires it twice more (that it may be six times;) and these we have at the Squares of 4 and 5. So that now we have a Cube compleated; whose

Components are, 7, nine times; 3 and 31, six times; 13, 67, and 109, three times. And the Square whence it ariseth, is that of $4 \times 5 \times 7 \times 11 \times 37 \times 67 \times 163 \times 191 \times 263 \times 439 \times 499$.
 The

The remaining Squares which are not ingredients into this, are those of 2, 3, 29.

Now if from these (without the other) we could form another Cube, such Cube would not only be another such Cube as is desired, but (being a Prime to that already found) might be Compounded with that found, to make a third. But this cannot be: Because (for these) we have no Prime that comes three times.

It remains to see, if (omitting the Square of 439) we can otherwise supply 31 twice, which at the Square of 191 were wanting. Where, first, it is manifest, that (the Square of 439 being laid aside) those of 37 and 163 (because of 67) must also be laid aside, unless we can have a third 67 from the Square of 29. Which cannot be, because this would introduce a fourth 13, and we have not two more to make up six. Then, having laid by that of 163, we must (because of 19) lay by those of 7 and 11. So that there remain only the Squares of 2, 4, 5, 67, to supply 31 twice (because we have it once) and 7 twice (because we have it four times) and 3 twice (because we have it once.) Now 31 might be supplied twice from the Squares of 4 and 5, (but then we could take no more, because that of 2 is included in 4; and 67 would bring in a fourth 31.) Or it might be supplied by one of those (suppose 5,) with that of 67. And thus we should have a supply of 31 twice, and of 7 twice, and of 3 once: But there wants another 3 (which the remaining Squares of 2 and 4 cannot supply) to compleat the Cube. So that this affords, not a Cube, but $\frac{1}{2}$ of a Cube. There is therefore no other Cube (but that before assigned) here to be had, retaining (as is hitherto supposed) the Numbers 109, 109, 109.

Let us therefore now leave out 109, and consequently the Squares of 263 and 499, where it is found; and see whether the

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
37	3, 7, 67
163	3, 7, 19, 67
29	13, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
4, 5	31
67	3, 7, 7, 31

the remaining Squares will afford such a Cube as is desired. Now these are,

2| 7 3|13 7|3, 19 29|13, 67 67|3, 7, 7, 31 191|7, 13, 13, 31
43| 5|31 11|7, 19 37|3, 7, 67 163|3, 7, 19, 67 439|3, 31, 31, 67

7|3, 19
11|7, 19
163|3, 7, 19, 67
37|3, 7, 67
439|3, 31, 31, 67

7|3, 19
11|7, 19
163|3, 7, 19, 67
37|3, 7, 67
29|13, 67
31|3

7|3, 19
11|7, 19
163|3, 7, 19, 67
37|3, 7, 67
29|13, 67
191|7, 13, 13, 31
67|3, 7, 7, 31
4|5|31

7|3, 19
11|7, 19
163|3, 7, 19, 67
29|13, 67
439|3, 31, 31, 67
191|7, 13, 13, 31

Of these, we will first begin with 19, which comes thrice (and but thrice) at the Squares of 7, 11, 163. Where we have 67 once, and therefore must have it twice more. Now if, for one of these, we take the Square of 37; we must, for the other, take either the Square of 439, or of 29. If that of 439; this brings in 3 a fourth time; which may not be, because it comes not twice more to make up six times. Therefore (if at all) it must be that of 29, (or else 37 must be laid aside;) But this brings in 13 once, for which we may have a second at the Square of 3, but then we cannot have a third without a fourth, at the Square of 191. Therefore (waving that at the Square of 3) we must take both (if at all) at the Square of 191. Now this brings in 7 a fourth time, which calls for a fifth and sixth: One of these we might have at the Square of 2; but then we cannot have a sixth without a seventh. Therefore (waving that at 2) we must (if at all) take both at the Square of 67. But here, beside a second 31 (for which we may have a third at the Square of 4, or of 5,) we have 3 a fourth time (which will make up, not a Cube, but the Triple of a Cube,) which is not to be admitted, because we cannot have a fifth and sixth. And consequently, the Square of 37 must be laid aside, (as not to be joined either with that

that of 439 or 29;) but (waving that) we must have recourse to the other two (at 29 and 439) for Tripling of 67. Now here we have 13 once; and therefore must have it twice more; not from the Square of 3, (because, as before, if we take a second here, we cannot have a third without a fourth;) but from that of 191. Which doth not only supply 13 twice; but also 7 and 31 which were also wanting: So that we have now a second Cube, such as was desired; whose Components are, 3, 7, 13, 19, 31, 67, thrice taken. And the Square whence it ariseth, is that of $7 \times 11 \times 29 \times 163 \times 191 \times 439$.

And if, from the remaining Square of 2, 4, 3, 5, 37, 67, we could form a third; this, Compounded with the last foregoing (as Prime to it) would form a fourth. But this cannot be, because no Prime doth here thrice occur, but only 7 and 31: And neither of these can be thrice taken, without being incumbered with 3, which cannot be Tripled. So that, retaining 19 (as is hitherto supposed) we can have (from thence) no other Cube than what is already found.

Let us now therefore lay by 19; and consequently the Squares of 7, 11, 163, wherein it is found. And we have then these only left for consideration.

$$\begin{array}{l} 2|7 \quad 3|13 \quad 29|13,67 \quad 67|3,7,7,31 \quad 439|3,31,31,67 \\ 4|31 \quad 5|31 \quad 37|3,7,67 \quad 191|7,13,13,31 \end{array}$$

We have here 67 three times, at the Squares of 29, 37, 439. And (with these) we have 3 twice; which calls for a third from the Square of 67. And we have 13 once, for which we might have a second at the Square of 3; but could not then have a third without a fourth; therefore (waving that) we take both from the Square of 191. And we have then 31 four times, and therefore must take it twice more from the Squares of 4, and of 5. But we have 7 four

$$\begin{array}{l} 29|13, 67 \\ 37|3, 7, 67 \\ 439|3, 31, 31, 67 \\ 67|3, 7, 7, 31 \\ 191|7, 13, 13, 31 \\ 4|31 \\ 5|31 \end{array}$$

2 Y

times;

times; yet cannot find it twice more to make it up six times; nor indeed once more, because we cannot here Compound the Square of 2, as being included in that of 4. So that, with 67, we may make up, not a Cube, but a Sextuple of a Cube.

Suppose we then that 67 be laid aside; and therefore the Squares of 29, 37, 439. Those that then remain are,

$$2|7 \quad 4|31 \quad 3|13 \quad 5|31 \quad 67|3, 7, 7, 31 \quad 191|7, 13, 13, 31$$

Of these, that of 67 must be laid aside (because 3 occurs but once,) and consequently (because 7 comes then but twice) that of 2 and 191. And for the other three (of 3, 4, 5,) the Number 13 comes but once; and 31 but twice. So that no further Cube can be hence expected.

We conclude therefore (having fully considered all) that (within the extent of this Table) we may have two Squares (and but two) such as are desired; whose Aggregate of Divisors shall be a Cube. Namely, the Square of $7 \times 11 \times 29 \times 163 \times 191 \times 439$, whose Aggregate of Divisors is the Cube of $3 \times 7 \times 13 \times 19 \times 31 \times 67$. And the Square of $4 \times 5 \times 7 \times 11 \times 37 \times 67 \times 163 \times 191 \times 263 \times 439 \times 499$; whose Aggregate of Divisors is the Cube of $3 \times 3 \times 7 \times 7 \times 7 \times 13 \times 19 \times 31 \times 31 \times 67 \times 109$.

And, if any think it worth the pains to seek out more; they must enlarge the Table, to take in more Primes, or more Quadratick Powers of these Primes.

It had been easy to have rendered this business more stupendous (as some other would have done,) if (concealing the methods whereby I came at them) I would have performed the Multiplications here directed; and then, in those great Numbers, exhibited

$$\begin{array}{l} 7|3, 19 \\ 11|7, 19 \\ 29|13, 67 \\ 163|3, 7, 19, 67 \\ 191|7, 13, 13, 31 \\ 439|3, 31, 31, 67 \\ 5 \end{array}$$

bited these two Squares, with the two Cubes thence arising; affirming, that (within such extent of Numbers) there is no other Square Number (beside these two, vastly great,) which added to all its Aliquot Parts will make a Cube: Or perhaps, having assigned those two, proposed a Challenge to all the Mathematicians in *France*,) to find a third within those limits. But this would serve only to amuse a Reader, not to instruct him. And I chuse rather (in what I publish) to inform my Reader, by what steps I come at those discoveries I make, and whereby he may (if he please) attain the like; designing more, the benefit of others, than ostentation.

I may here add (as is done after the former Question,) that the same method is to be used, if (instead of a Cube) it had been demanded, that such Aggregate should be the Triple (or other designed Multiple) of a Cube: (supposing such designed Multiple to be possible :) Of which I have given some instances as I passed along; and might have done more if it had been needful.

But we must not then demand the Duple, Quadruple, Sextuple of a Cube, or otherwise Multiple thereof by an *even* Number: For all such are impossible. For, since every Quadratick power of a Prime Number (be it the first, second, third, or further Square thereof,) hath, for its Divisors, (beside 1) all its Degrees or Powers so far; (as, for instance, a^6 hath for its Divisors 1, a , aa , a^3 , a^4 , a^5 , a^6 ,) and all these (because it is a Quadratick Power) are (excluding 1) in Number even; (and every of them either odd or even according as is the Prime a whence it ariseth;) and consequently, the Aggregate of all except 1, an even Number; (for an even Number of *odds*, as well as an even Number of *evens*, will still make an even Number;) to this even Number, if 1 be added (which is also an Aliquot Part, and therefore a Divisor,) this always makes the whole Aggregate an odd Number: Which therefore cannot be Duple of Cube, or its Multiple by an even Number. And the same will hold as well for the Quadratick Powers of any Compound Number: For (as was shewed before) the Ag-

gregate of Divisors of such Compound Square, is always Compounded of such Aggregates of Divisors of some Quadratick Powers of Primes; which, being (as is now shewed) odd Numbers, their Compound must be so too. For an odd Number, Multiplied by an odd Number (and so continually) will still produce an odd Number; and therefore, not the Duple (or otherwise Multiple by an even Number) of any Number whatsoever.

In the former Question, concerning Cubick Powers, whose Aggregate of Divisors should be equal to a Square, (or a designed Multiple of a Square,) this will not hold. For there the Aggregate may be either an odd or an even Number. Yet with this diversity: If the Prime a be 2, then all the Degrees thereof will be even Numbers, to which when 1 is added the Aggregate will be odd. If the Prime a be 3 (or other odd Prime,) and the Cube thence arising be the first, third, fifth Cube, (or other in odd places) whose Number of dimensions is 3, 9, 15, or other odd Number; the Number of Divisors, without 1, will be odd also; and therefore, with 1, it will become even. But if such Prime a , be odd, and the Cubick Power thereof be the second, fourth, sixth, or other in even places, whose Number of dimensions will therefore be 6, 12, 18, or other even Number (which will therefore be Quadratick as well as Cubick;) here the Number of Divisors without 1, will be even, and their Aggregate even; and therefore with 1, the Aggregate will be odd. And accordingly an estimate is to be made of the Compounds of such Aggregates: For, if all the Compounding Aggregates be odd, the Compound will be also odd; but if any one of them be even, the Compound Aggregate will be even. I forbear to pursue this to any nicer determination: But any who please may pursue it further,

*Of the Third Question mentioned above in pages 322, 323 ;
which was proposed by Dr. WALLIS to Monsieur FERMAT.*

III. A third Question I added to those two ; not as a new difficulty, but as a trial whether Monsieur *Fermat* did thoroughly understand the mystery of his own two Questions ; and did not only by chance light on them : For if he thoroughly understood those, he must needs be able to solve this with much ease ; which it seems, by Epist. 37, he did not find so easy ; and therefore, what solution he did find, he chose rather to conceal than let us know it. Nor doth any where let us know, whether he were able to solve his own Questions. But Monsieur *Frenicle* gives solutions both of this and those ; but without acquainting us by what methods he came at them ; which makes me think they are not better than mine.

The Question is this : *To find two Square Numbers, which added to their Aliquot Parts shall make the same Number (or, whose Aggregate of Divisors shall be the same;) As for instance* $16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1$; *Let two such other be found.*

Now 'tis manifest (by what hath been before delivered) that any Multiple of those two (16 and 25) by any other Square which is a Prime to both of them (as 9, 49, 121, &c,) will do what is desired. For the Multiple of 31, by the Aggregate of Divisors of any such other Square, will be the Aggregate of Divisors, both of 16, and of 25, Multiplied by such Square. As for instance, because $9 + 3 + 1 = 13$; therefore $31 \times 13 = 403$, is the Aggregate of the Divisors, as well of $16 \times 9 = 144$, as of $25 \times 9 = 225$.

But, if we would have others than the Equimultiples of 16 and 25 ; we may make use of the former Table of Squares ; wherein (because we do not meet with any single Squares,

Squares, (other than those of 4 and of 5,) whose Aggregate of Divisors is the same) we are so to Compound two or more of them in several parties, as that the Aggregates be the same. As, the Squares of

$$\begin{array}{l} 4 \} 31. \\ 5 \} \end{array} \quad \begin{array}{l} 29 \times 67 \\ 2 \times 3 \times 5 \times 37 \end{array} \} 3 \times 7 \times 7 \times 13 \times 31 \times 67.$$

$$\begin{array}{l} 2 \times 19 \times 29 \\ 3 \times 8 \times 37 \end{array} \} 3 \times 7 \times 13 \times 67 \times 127.$$

$$\begin{array}{l} 7 \times 8 \times 29 \times 67 \\ 3 \times 4 \times 11 \times 19 \times 37 \end{array} \} 3 \times 3 \times 7 \times 7 \times 13 \times 19 \times 31 \times 67 \times 127.$$

$$\begin{array}{l} 7 \times 8 \times 92 \times 76 \\ 3 \times 5 \times 11 \times 19 \times 37 \end{array} \} 3 \times 3 \times 7 \times 7 \times 13 \times 19 \times 31 \times 67 \times 127.$$

All which arise from Compounding the Squares of the Primes less than 100, taking into the Number the second and third Squares of 2.

And more Couples than these are not to be found within those limits, unless by Multiplying both the Numbers of some of these Couples by some common Square which is a Prime to both of them; which may be done at pleasure. But if we extend the limits, to other Primes, and other Powers of these Primes, we may have more without stint.

And by the same means we may have Three or more such Squares, whose Aggregate of Divisors shall make the same sum. As (amongst these) we have Three. Namely the Squares of

$$\begin{array}{l} 7 \times 8 \times 29 \times 67 \\ 3 \times 4 \times 11 \times 19 \times 37 \\ 3 \times 5 \times 11 \times 19 \times 37 \end{array} \} 3 \times 3 \times 7 \times 7 \times 13 \times 19 \times 31 \times 67 \times 127.$$

But if we enlarge the bounds, we may find others (Two's, Threes, Fours, &c,) in great Multitudes, whose Aggregate of

of Divisors shall be the same. As any man by experience, may find, who (without going farther) will give himself the trouble of pursuing the whole Table here given, as I have done those Primes which are smaller than 100.

I forbear to pursue more Questions of this nature ; but, according to the same method, any others of like kind may be dispatched.

F I N I S.

AN
APPENDIX

TO THE
ENGLISH TRANSLATION
OF

RHONIUS'S GERMAN TREATISE OF ALGEBRA,

MADE BY
MR. THOMAS BRANCKER, M. A.

And Published by him,
With the Advice and Assistance of Dr. JOHN PELL,
At London, in the Year 1668;

CONTAINING
A TABLE OF ODD NUMBERS
LESS THAN ONE HUNDRED THOUSAND,

SHewing,

First, Which of them are INCOMPOSIT, or PRIME, NUMBERS,
And, Secondly, The FACTORS, or CO-EFFICIENTS, by the Multi-
plication of which the others are produced; Supputated, or Computed,
by the same THOMAS BRANCKER,

** Recd^d Dr John Pell, mathematician, born 1610 Died 1685.*

THE

TRANSLATOR'S PREFACE.

THE *Title-Page* says that this Book was a Translation, but hath been much altered. If any man desire to know what the alterations are, and why they were made; he may do well to compare it with the Original: A Printed Copy whereof may be had at Francfort in Germany, by any that inquires there for it by this Title, Algebra Rhonii Germanicè; Tiguri * apud Bodmerum, 1659, in quarto. The Copy which I have, was given me anno 1662, by a good Friend, who then told me he much desired to read it in some Language that he understood; I then promised him to English it. As soon as my leisure permitted, I corrected it according to the Printed Catalogue of Errata, and then began the Translation. When it was finished, I desired to see it Printed, and got it Licensed May 18, 1665, with the name of An Introduction to Algebra. And so without any alteration either in the Precepts or Examples, save only the correction of many Mistakes: It was sent to the Press, with order to Re-print the six leaves of His Table of Incomposits precisely as they stand there.

A little after, I heard that there was at that time in London, a Person of Note † very worthy to be made acquainted with my design, before I made any farther progress in the Impression. Being admitted to speak with him, I found him not only able to direct me, but also very willing so to do, so far as his leisure would permit. He gave me divers cautions concerning the Work. He shewed me the way of making the Table of Incomposits, of examining it, and of continuing it as far as I would. He encouraged me to extend it to 100 thousand: Telling me that by that time that I had Calculated and Printed that Table, be

* That is, at Zurich, in Switzerland.

† Dr. John Pell.

hoped to be at leisure to review some of Monsieur Rhonius his Problemes, and to work them anew; and that he would send them to me, with leave to publish them or to keep them by me.

I had finished and Printed that Table, as also Twelve sheets of the Book itself, before he sent me his Alterations. They begin with Probl. 24, pag. 100. All from thence to the end is his Work: As also pag. 79, 80, 81, 82, which he sent last of all: So that instead of the first 124 pages of Rhonius, this hath just twice as many: Instead of those 8 or 9 sheets remaining in Rhonius, how much shall be hereafter published, I will not adventure to foretell, because of the uncertainty of life, health, leisure, and of the acceptance which this shall find amongst the Lovers of these Studies, to whom this might have been more acceptable, if it had been wholly void of Press-faults.

As for the Table of Incomposits, I was very sensible of the bad effects of perfunctoriness in Supputating, Transcribing, or Printing of it. My care therefore was not small: yet pag. 198, is almost filled with Errata, and I dare not warrant that none have escaped unseen: But seeing so few are fit to undertake to Supputate it anew, whosoever shall happen to discover any other fault in that Table, shall do well to signify it to the Book-seller, or to any other likely to be concerned in the next Impression.

The Errata in the rest of the Book are many, notwithstanding my care, and the diligence of a good friend, who Corrected part of it, after my removal to an abode so far from London. Most of them cannot trouble the more exercised sort of Readers. But fear of leaving any stumbling-block in the way of Beginners hath caused this larger Enumeration of them in the three next following pages.

White-gate in Cheshire,
April 22, 1668.

T. B.

From

From pages 34 and 35 of BRANCKER's Translation of
RHONIUS's Algebra.

BUT it is oftentimes very troublesome to find a Square, Cube, &c, whereby this *Abbreviation* may be performed. Find therefore all the *Partes aliquotæ*, or *just Dividers*, and these will tell us whether, and how often any Cube, Square, &c, is contained in the Quantity assigned.

Forasmuch then as the Discovery of the *Partes aliquotæ* is many waies useful in *Vulgar Arithmetick*, I have adjoyned a Table in the End of this Book, which discovers them in all uneven Numbers as far as 100,000.

In which Table [p] stands for a *Prime Number* throughout.

The Use of that Table is

To discover *at view* whether any given Quantity be compound or simple, *i. e.* be divisible or indivisible, and how many *Partes aliquotæ* it hath. On the left side you see, run down all the odd Numbers to 99, which must be set after the Numbers in the Head-Row, as Occasion is, thus. Let the Number given be 21449, seek 49 in the side, and the other 214 in the head, then run downward, and side-waies till their Rows meet in a Square, where we find 89, which is a *Part aliquota*, which dividing 21449, gives Quotient 241. With this 241 do as before (*i. e.* seek 41 on the side, and 2 in the head) and in its Square you find (P) which shews that it is an indivisible or *Prime Number*. Wherefore the aliquot Parts of this 21449 stand thus.

$$\begin{array}{r}
 \text{I} \\
 89 \cdot 241 \\
 \hline
 21449.
 \end{array}$$

If

$$\begin{array}{r}
 \\
 \\
 2.2.2 2.2.2 \\
 \hline
 4.8.16.32.64 \\
 \hline
 3.6.12.24.48.96.192 \\
 \hline
 113.226.452.904.1808.3616.7232.339.678.1356.2712.5424 \\
 10848.21696.
 \end{array}$$

Mr. THOMAS BRANCKER's Preface to his long Table of Incomposit, or Prime, Numbers; from pages 193, 194, 195, 196, 197, and 198, of his Translation of RHONIUS's Algebra, published in the Year 1668.

Each page hath 21 columels, whereof the *first* is filled with 40 odd numbers standing in their natural order. The *following twenty* columels are distinguished on their Tops, by numbers

numbers in their natural order 0, 1, 2, 3, to 998, 999. These *Top numbers* are *hundreds*; the 40 *marginal numbers* are *Unites adhering to the Centuries*. A line running from any marginal cross the page, shews, in any column, the *place* of the number made up of the *Top-number* and that marginal. In every such *place* of concourse you shall either find the letter *p*, or some *incomposit* less than 317. The letter *p* shews the number to be a *prime* or *incomposit*. (See Euclid, VII. def. 11 and 13.) If any number less than 100,000, do end in 1, 3, 7, or 9, you may find its place in one of those 50 pages, and then see whether it be a *prime* or no: If it be *composit*, you will there find its *least Divisor*. Thus in page 1, where the line marked with the marginal 67, crosseth the column whose *Top-number* is 16; there you find *p*, that is, 1667 is a prime. Where the same line crosseth the next column, you find 3; That is, 1767 is no prime, and 3 is the *least Divisor* of it. So in page 25, you see 49031, 49033, 49037 are primes; but 49039 is a *Composit*, and 19 is its smallest *Divisor*.

It may be of great use sometimes to have a *complete and orderly enumeration* of all *incomposits* between 0, and 100,000, without any mixture of *Composits*; thus 1. 2. 3. 5. 7. 11. 13. &c, leaving out 9, 21 and all other *composits*. The numbers 2 and 5 are primes, though they be left out of the long Table, because no other *incomposit* ends so. These two prime numbers 2 and 5 being duely placed, all the rest of the primes are taken out of the long Table as they there stand marked with *p*, from 7 in the first page to 99991 in the end of the 50th page.

If to each of these primes you set the Briggian Logarithm, you may find the Logarithms for all the rest of the numbers in the first 100 Chiliads, by addition of the Logarithms of their *incomposit* Factors.

The Resolving of a number into all its *incomposit* Factors [as 4620 into 2. 2. 3. 5. 7. 11.] is altogether necessary, for the determining how many *Divisors* that number hath, and which they be: As in pages 194, 195.

(29)	ab. acde	c. aaabd	aac. abb	aa. aaaa	bb. aac	a. abc
1. abcdef	ac. abde	d. aaabc	(23)	aaa. aaa	bc. aab	b. aac
a. bcdef	ad. abce	aa. abcd	1.aaaabc	(18)	(15)	c. aab
b. acdef	ae. abcd	ab. aacd	a. aaabc	1. abcde	1.aaabc	aa. bc
c. abdef	bc. aade	ac. aabd	b. aaaac	a. bcde	a. aabc	ab. ac
d. abcef	bd. aace	ad. aabc	c. aaaab	b. acde	b. aaac	(9)
e. abcdf	be. aacd	bc. aaad	aa. aabc	c. abde	c. aaab	1. aabb
f. abcde	cd. aabe	bd. aaac	ab. aaac	d. abce	aa. abc	a. abb
ab. cdef	ce. aabd	cd. aaab	ac. aaab	e. abcd	ab. aac	b. aab
ac. bdef	de. aabc	aaa. bcd	bc. aaaa	ab. cde	ac. aab	aa. bb
ad. bcef	aab. cde	aab. acd	aaa. abc	ac. bde	bc. aaa	ab. ab
ae. bcdf	aac. bde	aac. abd	aab. aac	ad. bce	(14)	(8)
af. bcde	aad. bce	aad. abc	(22)	ae. bcd	1.aaabb	1.aaab
bc. adef	aae. bcd	(25)	1.aaabbb	bc. ade	a. aabb	a. aab
bd. acef	abc. ade	1. aabbbc	a. aabbb	bd. ace	b. aab	b. aaa
be. acdf	abd. ace	a. abbcc	b. aaabb	be. acd	aa. abb	aa. ab
bf. acde	abe. acd	b. aabcc	aa. abbb	cd. abe	ab. aab	(7)
cd. abef	(27)	c. aabbc	ab. aabb	ce. abd	bb. aaa	1. aaaa
ce. abdf	1. aabbcd	aa. bbcc	bb. aabb	de. abc	(13)	a. aaa
cf. abde	a. abbcd	ab. abce	aaa. bbb	(17)	1. aaaab	aa. aa
de. abcf	b. aabcd	ac. abbc	aab. abb	1. aabcd	a. aab	(6)
df. abce	c. aabbd	bb. aacc	(21)	1. abcd	b. aaaa	1. abc
ef. abcd	d. aabbc	bc. aabc	1.aaaabb	b. aacd	aa. aab	a. bc
abc. def	aa. bbcd	cc. aabb	a. aaabb	c. aabd	ab. aaa	b. ac
abd. cef	ab. abcd	aab. bcc	b. aaaaab	d. aabc	(12)	c. ab
abe. cdf	ac. abbd	aac. bbc	aa. aabb	aa. bcd	1. aaaaa	(5)
abf. cde	ad. abbc	abb. acc	ab. aaab	ab. acd	a. aaaa	1. aab
acd. bef	bb. aacd	abc. abc	bb. aaaa	ac. abd	aa. aaa	a. ab
ace. bdf	bc. aabd	(24)	aaa. abb	ad. abc	(11)	b. aa
acf. bde	bd. aabc	1. aaabbc	aab. aab	bc. aad	1. abcd	(4)
ade. bcf	cd. aabb	a. aabbc	(20)	bd. aac	a. bcd	1. aaa
adf. bce	aab. bcd	b. aaabc	1.aaaaab	cd. aab	b. acd	a. aa
aef. bcd	aac. bbd	c. aaabb	a. aaaaab	(16)	c. abd	(3)
(28)	aad. bbc	aa. abbc	b. aaaaa	1. aabbc	d. abc	1. ab
1. aabcde	abb. acd	ab. aabc	aa. aaat	1. abbc	ab. cd	a. b
a. abcde	abc. abd	ac. aabb	ab. aaaa	b. aabc	ac. bd	(2)
b. aacde	(26)	bb. aaac	aaa. aal	c. aabb	ad. bc	1. aa
c. aabde	1. aaabcd	bc. aaab	(19)	aa. bbc	(10)	a. a
d. aabce	a. aabcd	aaa. bbc	1. aaaaaa	ab. abc	(10)	(1)
e. aabcd	b. aaacd	aab. abc	a. aaaaa	ac. abb	1. aabc	1. a
aa. bcde						

(29)	6.770	5.168	20.18	4. 16	9.20	2.30	Sort	Forme	Div.
1.30030	10.462	7.120	(23)	8. 8	15.12	3.20			
2.15015	14.330	4.210	1.240	(18)	(15)	5.12	29	abc def	64
3.10010	22.210	6.140	2.120	1.2310	1.120	4.15	28	abcde	48
5.6006	15.308	10. 84	3. 80	2.1155	2. 60	6.10	27	aabbcd	36
7.4290	21.220	14. 60	5. 48	3. 770	3. 40	(9)	26	aaabcd	32
11.2730	33.140	15. 56	4. 60	5. 462	5. 24	1.36	25	abbbcc	27
13.2310	35.132	21. 40	6.40	7. 330	4. 30	2.18	24	aaabbc	24
6.5005	55. 84	35. 24	10.24	11.210	6. 20	3.12	23	aaaaabc	20
10.3003	77. 60	8. 105	15.16	6.385	10.12	4. 9	22	aabbbb	16
14.2145	12.385	12. 70	8.30	10.231	15. 8	6. 6	21	aaabbb	15
22.1365	20.231	20. 42	12.20	14.165	(14)	(8)	20	aaaaab	12
26.1155	28.165	28. 30	(22)	22.105	1.72	1.24	19	aaaaaa	7
15.2002	44.105	(25)	1.216	15.154	2.36	2.12	18	abcde	32
21.1430	30.154	1.900	2.108	21.110	3.24	3. 8	17	aabcd	24
33. 910	42.110	2.450	3.72	33. 70	4.18	4. 6	16	aabbc	18
39. 770	66. 70	3.300	4.54	35. 66	6.12	(7)	15	aaabc	16
35. 858	(27)	5.180	6.36	55. 42	9. 8	1.16	14	aaabbb	12
55. 546	1.1260	4.225	9.24	77. 30	(13)	2. 8	13	aaaaab	10
65. 462	2.630	6.150	8.27	(17)	1.48	4. 4	12	aaaaa	6
77. 390	3.420	10. 90	12.18	1.420	2.24	(6)	11	abcd	16
91. 330	5.252	9.100	(21)	2.210	3.16	1.30	10	aabc	12
143.210	7.180	15. 60	1.144	3.140	4.12	2.15	9	aabb	9
30.1001	4.315	25. 36	2.72	5. 84	6. 8	3.10	8	aaab	8
42.715	6.210	12. 75	3.48	7. 60	(12)	5. 6	7	aaaa	5
66.455	10.126	20. 45	4.36	4.105	1.32	(5)	6	abc	8
78.385	14. 90	18. 50	6.24	6. 70	2.16	1.12	5	aab	6
70.429	9. 140	30. 30	9.16	10. 42	4. 8	2. 6	4	aaa	4
110.273	15. 84	(24)	8.18	14. 30	(11)	3. 4	3	ab	4
130.231	21. 60	1.360	12.12	15. 28	1.210	(4)	2	aa	3
154.195	35. 36	2.180	(20)	21. 20	2.105	1. 8	1	a	2
182.165	12.105	3.120	1.96	35. 12	3.70	2. 4	That is, the 29th fort hath 64 Divisors; the 18th hath but 32, &c.		
286.105	20. 63	5. 72	2.48	(16)	5.42	(3)			
(28)	28. 45	4. 90	3.32	1.180	7.30	1. 6			
1.4620	18. 70	6. 60	4.24	2. 90	6.35	2. 3			
2.2310	30. 42	10. 36	6.16	3. 60	10.21	(2)			
3.1540	(26)	9. 40	8.12	5. 36	14.15	1. 4			
5.924	1.840	15. 24	(19)	4. 45	(10)	2. 2			
7.660	2.420	8. 45	1.64	6. 30	1.60	(1)			
11.420	3.280	12. 30	2.32	10. 18		1. 2			
4.1155									

*Use of the Long Table of Numbers, ending in
1, 3, 7, or 9.*

Every *Aliquot part* of a Number is one of the just Divisors of it. The greatest Divisor being equal to the *whole* Dividend, must not be called a *Part*: Wherefore, subtract 1 from every number in the last column of page 195, you shall have the number of *aliquot* parts belonging to every one of those 29 forts.

*Having the least Divisor of any Number of the long Table,
to find all its other incompisit Co-efficients.*

If that Divisor end in 1 or 9, and have a black stroke *under* it in the Dividend's place in the long table; or if the Divisor end in 3 or 7, and have such a stroke *over* it in the Dividend's place; the Dividend is the *square of an incompisit*, and the Quotient is given, for it is equal to the Divisor.

If the least Divisor have no such stroke by it, let it divide the proposed number, the Quotient shall be the greatest *aliquot part* of that Dividend: Seek that Quotient in the same long Table; if it be there *marked with p*, your inquiry is at an end; the Dividend is of the form AB. If it be *not so marked*, by the Prime there found, divide your *first* Quotient, deal with the *second* Quotient as you had done with the *first*, repeating such Divisions, till the Quotient be incompisit. Thus 53191 is found in page 27, with its
smallest

Smallest Divisor 43. Now 53191 divided by 43 gives 1237. Page 1 says, this 1237 is a prime. Inquire no farther.

But, desiring the incompofit factors of 93611, I find it in page 47 of the long Table, with 7 for its least Divisor. The Quotient 13373 is found in page 7, with its least Divisor 43. This 43 gives a second quotient 311. Page 1 says, this 311 is an incompofit. So the prime Co-efficients of 93611 are 7. 43. 311. (Hence infer that 53191 is to 93611, as 1237 to 2177 = $7 * 311$, or 7×311).

If you divide *any odd* number by *all the primes in order*, beginning with 3, The first Divisor that finds a Quotient without fraction, is the least Divisor that the Dividend can have. Thus, 239 is the least number that measures 1111111. Try 3, 7, 11, &c. No prime can divide 1111111 till you come to 239. If no fuch Divisor find an Integer Quotient, before the Quotient is less than the Divisor, pronounce your Dividend to be incompofit, and that last Divisor to be greater than the Dividend's square root. Frequent occasion of Dividing by Incompofits calls for a *Tariffa* of as many *primes* as shall be needful. For resolving of numbers less than 100,000, it fufficeth if it be extended to 313, as in the next page.

A Tariffa, or Table, of all Incomposits, or Prime, Numbers, less than $\sqrt{100,000}$, multiplied by 2, 3, 4, 5, 6, 7, 8, 9.

1	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73
2	4	6	10	14	22	26	34	38	46	58	62	74	82	86	94	106	118	122	134	142	146
3	6	9	15	21	33	39	51	57	69	87	93	111	123	129	141	159	177	183	201	213	219
4	8	12	20	28	44	52	68	76	92	116	124	148	164	172	188	212	236	244	268	284	292
5	10	15	25	35	55	65	85	95	115	145	155	185	205	215	235	265	295	305	335	355	365
6	12	18	30	42	66	78	102	114	138	172	186	222	246	258	282	318	354	366	402	426	438
7	14	21	35	49	77	91	119	133	161	203	217	259	287	301	329	371	413	427	469	497	511
8	16	24	40	56	88	104	136	152	184	232	248	296	328	344	376	424	472	488	536	568	584
9	18	27	45	63	99	117	153	171	207	261	279	333	369	387	423	477	531	549	603	639	657
1	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157					
2	158	166	178	194	202	206	214	218	226	254	262	274	278	298	302	314					
3	237	249	267	291	303	309	321	327	339	381	393	411	417	447	453	471					
4	316	332	356	388	404	412	428	436	452	508	524	548	556	596	604	628					
5	395	415	445	485	505	515	535	545	565	635	655	685	695	745	755	785					
6	474	498	534	582	606	618	642	654	678	762	786	822	834	894	906	942					
7	553	581	623	679	707	721	749	763	791	889	917	959	973	1043	1057	1099					
8	632	664	712	776	808	824	856	872	904	1016	1048	1096	1112	1192	1208	1256					
9	711	747	801	873	909	927	963	981	1017	1143	1179	1233	1251	1341	1359	1413					

1	163	167	173	179	181	191	193	197	199	211	223	227	229	233
2	326	334	346	358	362	382	386	394	398	422	446	454	458	466
3	489	501	519	537	543	573	579	591	597	633	669	681	687	699
4	652	668	692	716	724	764	772	788	796	844	892	908	916	932
5	815	835	865	895	905	955	965	985	995	1055	1115	1135	1145	1165
6	978	1002	1038	1074	1086	1146	1158	1182	1194	1266	1338	1362	1374	1398
7	1141	1169	1211	1255	1267	1337	1351	1379	1393	1477	1561	1589	1603	1631
8	1304	1336	1384	1432	1448	1528	1544	1576	1592	1688	1784	1816	1832	1864
9	1467	1503	1557	1611	1629	1719	1737	1773	1791	1899	2007	2043	2061	2097

1	239	241	251	257	263	269	271	277	281	285	293	307	311	313
2	478	482	502	514	526	538	542	554	562	566	586	614	622	626
3	717	725	753	771	789	807	813	831	843	849	879	921	933	939
4	956	964	1004	1028	1052	1076	1084	1108	1124	1132	1172	1228	1244	1252
5	1195	1205	1255	1285	1315	1345	1355	1385	1405	1415	1465	1535	1555	1565
6	1434	1446	1506	1542	1578	1614	1626	1662	1686	1698	1758	1842	1866	1878
7	1673	1687	1757	1799	1841	1883	1897	1939	1967	1981	2051	2145	2177	2191
8	1912	1928	2008	2056	2104	2152	2168	2216	2248	2264	2344	2456	2488	2504
9	2151	2169	2259	2313	2367	2421	2439	2493	2529	2547	2637	2763	2799	2817

Amendments to the following Table.

P. Guldin says, 149 is divisible by 7, and 229 by 31. *Schooten* leaves 809 out of his *Catalogue of Incomposits*. *Rbonius* makes 1209 and 1673 incomposits, and says 11833 is divisible by 19. But *this Table* says more truly, that 149:229.809.11833 are Incomposits: and that 1209 is divisible by 3, and 1673 by 7. Yet trust it not, before you have amended *these faults* in it.

Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.
5	9211	19	61	21	40277	13	p		60779	63	163		72381	p	3
	9799	40	41		40591	3	p		61779	p	3		72383	3	p
6	10199	p	7		40593	p	3	32	62011	3	p		72557	73	37
	10813	13	11		40597	3	p		62013	p	3		72601	97	79
7	13201	23	43	23	44659	11	17		62017	3	p		73023	p	3
9	17563	3	7		45353	p	7		62019	p	3		73051	7	11
	17981	41	p		45837	p	3		63839	71	p		73481	179	197
10	18903	7	3	24	466..	470	466		63883	191	193		73493	p	7
	18907	3	7		46089	7	3	33	641..	541	641		73913	p	7
	18909	7	3		46457	3	p		659..	569	659	40	78199	p	11
12	23203	3	p		47201	11	7		64237	61	p	41	80333	67	11
	23381	193	103		47577	7	3		64693	3	p		80663	p	11
13	24011	3	13		47579	3	7		64973	23	43	42	83123	103	101
	25093	13	23		47663	p	7		65959	17	71	43	84311	57	59
	25873	23	p	25	48601	53	7	34	66239	19	p	44	86699	281	181
14	27233	31	113	27	53361	7	3		66561	7	3	46	91189	p	7
	27517	p	7		53791	3	p		66563	3	7		91707	p	3
15	28201	3	p	28	54507	7	3		66567	p	3		91793	23	17
	28203	p	3		54509	3	7		66569	3	p	47	92701	3	7
	29599	blank	p		54589	71	79		66761	191	101		92703	7	3
17	32297	71	p	29	56323	157	151		66951	2	3		92773	163	113
	33259	97	79	30	58123	11	13	35	68809	53	13		93101	151	157
	33591	7	3		58181	71	73	36	70313	167	p		93161	52	59
	33593	3	7		58301	137	173		70981	167	p	48	95371	281	283
18	34089	7	3		58901	blank	p		71113	3	7		95797	p	13
	34209	23	3		59901	7	3		71603	p	7	49	97903	3	13
	35089	3	p		59909	137	139		71983	167	p	50	98099	26	263
20	39263	p	7	31	60079	63	73	37	72357	7	3		98551	39	139
	39589	p	11		60293	7	p		72359	3	7		99443	17	271

THOMAS

Mr. THOMAS BRANCKER's TABLE of INCOMPOSIT, or
PRIME, NUMBERS, less than 100,000.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
01	1	P	P	3	7	P	3	P	P	3	17	7	3	P	P	3	19	P	3	P
03	P	P	7	3	13	P	3	19	11	3	17	P	3	P	23	3	7	13	3	11
07	P	P	3	P	11	3	P	7	3	P	19	3	17	P	3	11	P	3	13	P
09	3	P	11	3	P	P	3	P	P	3	P	P	3	7	P	3	P	P	3	23
11	P	3	P	P	3	7	13	3	P	P	3	11	7	3	17	P	3	29	P	3
13	P	P	3	P	7	3	P	23	3	11	P	3	P	13	3	17	P	3	7	P
17	P	3	7	P	3	11	P	3	19	7	3	P	P	3	13	37	3	17	23	3
19	P	7	3	11	P	3	P	P	3	P	P	3	23	P	3	7	P	3	17	19
21	3	11	13	3	P	P	3	7	P	3	P	19	3	P	7	3	P	P	3	17
23	P	3	P	17	3	P	7	3	P	13	3	P	P	3	P	P	3	P	P	3
27	3	P	P	3	7	17	3	P	P	3	13	7	3	P	P	3	P	11	3	41
29	P	3	P	7	3	23	17	3	P	P	3	P	P	3	P	11	3	7	31	3
31	P	P	3	P	P	3	P	17	3	7	P	3	P	11	3	P	7	3	P	P
33	3	7	P	3	P	13	3	P	7	3	P	11	3	31	P	3	23	P	3	P
37	P	P	3	P	19	3	7	11	3	P	17	3	P	7	3	29	P	3	11	13
39	3	P	P	3	P	7	3	P	P	3	P	17	3	13	P	3	11	37	3	7
41	P	3	P	11	3	P	P	3	29	P	3	7	17	3	11	23	3	P	7	3
43	P	11	3	7	P	3	P	P	3	23	7	3	11	17	3	P	31	3	19	29
47	P	3	13	P	3	P	P	3	7	P	3	31	29	3	P	7	3	P	P	3
49	7	P	3	P	P	3	11	7	3	13	P	3	P	19	3	P	17	3	43	P
51	3	P	P	3	11	19	3	P	23	3	P	P	3	7	P	3	13	17	3	P
53	P	3	11	P	3	7	P	3	P	P	3	P	7	3	P	P	3	P	17	3
57	3	P	P	3	P	P	3	P	P	3	7	13	3	23	31	3	P	7	3	19
59	P	3	7	P	3	13	P	3	P	7	3	19	P	3	P	P	3	P	11	3
61	P	7	3	19	P	3	P	P	3	31	P	3	13	P	3	7	11	3	P	37
63	3	P	P	3	P	P	3	7	P	3	P	P	3	29	7	3	P	41	3	13
67	P	P	3	P	P	3	23	13	3	P	11	3	7	P	3	P	P	3	P	7
69	3	13	P	3	7	P	3	P	11	3	P	7	3	37	13	3	P	29	3	11
71	P	3	P	7	3	P	11	3	13	P	3	P	31	3	P	P	3	7	P	3
73	P	P	3	P	11	3	P	P	3	7	29	3	19	P	3	11	7	3	P	P
77	7	3	P	13	3	P	P	3	P	P	3	11	P	3	7	19	3	P	P	3
79	P	P	3	P	P	3	7	19	3	11	13	3	P	7	3	P	23	3	P	P
81	3	P	P	3	13	7	3	11	P	3	23	P	3	P	P	3	41	13	3	7
83	P	3	P	P	3	11	P	3	P	P	3	7	P	3	P	P	3	P	7	3
87	3	11	7	3	P	P	3	P	P	3	P	P	3	19	P	3	7	P	3	P
89	P	3	17	P	3	19	13	3	7	23	3	29	P	3	P	7	3	P	P	3
91	7	P	3	17	P	3	P	7	3	P	P	3	P	13	3	37	19	3	31	11
93	3	P	P	3	17	P	3	13	19	3	P	P	3	7	P	3	P	11	3	P
97	P	P	3	P	7	3	17	P	3	P	P	3	P	11	3	P	P	3	7	P
99	3	P	13	3	P	P	3	17	29	3	7	11	3	P	P	3	P	7	3	P

	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
01	3	11	31	3	7	41	3	37	P	3	P	7	3	P	19	3	13	7	3	47
03	P	3	P	7	3	P	19	3	P	P	3	29	P	3	41	31	3	7	P	3
07	3	7	P	3	29	23	3	P	7	3	31	13	3	P	P	3	P	11	3	P
09	7	3	47	P	3	13	P	3	53	P	3	P	P	3	7	11	3	P	13	3
11	P	P	3	P	P	3	7	P	3	41	P	3	13	7	3	P	23	3	37	P
13	3	P	P	3	19	7	3	P	29	3	23	11	3	P	P	3	P	47	3	7
17	P	29	3	7	P	3	P	11	3	P	7	3	P	31	3	P	P	3	11	P
19	3	13	7	3	41	11	3	P	P	3	P	P	2	P	13	3	7	P	3	P
21	43	3	P	11	3	P	P	3	7	23	3	P	P	3	11	7	3	61	P	3
23	7	11	3	23	P	3	43	7	3	37	P	3	11	P	3	13	P	3	P	P
27	P	3	17	13	3	7	37	3	11	P	3	53	7	3	23	P	3	P	43	3
29	P	P	3	17	7	3	11	P	3	29	13	3	P	P	3	P	19	3	7	P
31	3	P	23	3	11	P	3	P	19	3	7	31	3	P	47	3	P	7	3	P
33	19	3	7	P	3	17	P	3	P	7	3	13	53	3	P	P	3	P	P	3
37	3	P	P	3	P	43	3	7	P	3	P	P	3	47	7	3	P	37	3	31
39	P	3	P	P	3	P	7	3	17	P	3	43	41	3	19	P	3	P	11	3
41	13	P	3	P	P	3	19	P	3	17	P	3	7	13	3	P	11	3	23	7
43	3	P	P	3	7	P	3	13	P	3	17	7	3	P	11	3	P	19	3	P
47	23	19	3	P	P	3	P	41	3	7	11	3	17	P	3	P	7	3	P	P
49	3	7	13	3	31	P	3	P	7	3	P	47	3	17	P	3	41	23	3	11
51	7	3	P	P	3	P	11	3	P	13	3	23	P	3	7	53	3	11	P	3
53	P	P	3	13	11	3	7	P	3	P	43	3	P	7	3	11	13	3	P	59
57	11	3	37	P	3	P	P	3	P	P	3	7	P	3	P	P	3	13	7	3
59	29	17	3	7	P	3	P	31	3	11	7	3	P	P	3	P	P	3	17	37
61	3	P	7	3	23	13	3	11	P	3	P	29	3	P	P	3	7	P	3	17
63	P	3	31	17	3	11	P	3	7	P	3	P	13	3	P	7	3	53	P	3
67	3	11	P	3	P	17	3	P	47	3	P	3	7	P	3	19	P	3	P	
69	P	3	P	23	3	7	17	3	19	P	3	P	7	3	P	43	3	P	53	3
71	19	13	3	P	7	3	P	17	3	P	37	3	P	P	3	P	P	3	7	11
73	3	41	P	3	P	31	3	47	13	3	7	19	3	P	23	3	P	7	3	29
77	31	7	3	P	P	3	P	P	3	13	17	3	29	11	3	7	P	3	P	41
79	3	P	43	3	37	P	3	7	P	3	P	11	3	31	7	3	13	P	3	23
81	P	3	P	P	3	29	7	3	43	11	5	P	17	3	59	P	3	19	P	3
83	P	37	3	P	13	3	P	11	3	19	P	3	7	17	3	P	29	3	11	7
87	P	3	P	7	3	13	P	3	P	29	3	P	19	3	11	17	3	7	13	3
89	P	11	3	P	19	3	P	P	3	7	P	3	11	P	3	37	7	3	P	P
91	3	7	29	3	47	P	3	P	7	3	11	P	3	P	P	3	P	17	3	13
93	7	3	P	P	3	P	P	3	11	41	3	31	37	3	7	P	3	P	17	3
97	3	13	P	3	11	7	3	P	P	3	19	23	3	43	13	3	P	P	3	7
99	P	3	11	P	3	23	P	3	13	P	3	7	P	3	P	59	3	29	7	3

	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
01	P	3	P	11	3	7	43	3	P	13	3	P	7	3	11	P	3	P	P	3
03	P	11	3	13	7	3	P	P	3	P	3	P	11	P	3	P	13	3	7	P
07	P	3	7	59	3	P	17	3	11	7	3	P	41	3	P	P	3	13	P	3
09	19	7	3	31	P	3	11	17	3	P	P	3	P	P	3	7	71	3	37	19
11	3	P	P	3	11	13	3	7	17	3	P	19	3	47	7	3	31	P	3	23
13	P	3	11	19	3	P	7	3	P	17	3	P	13	3	P	37	3	29	P	3
17	3	23	P	3	7	P	3	53	P	3	29	7	3	13	P	3	41	P	3	61
19	P	3	P	7	3	P	31	3	61	P	3	P	17	3	P	P	3	7	11	3
21	P	43	3	29	P	3	P	P	3	7	P	3	23	17	3	P	7	3	P	31
23	3	7	41	3	P	P	3	P	7	3	P	47	3	P	11	3	P	59	3	P
27	P	P	3	P	19	3	7	29	3	13	11	3	P	7	3	P	17	3	P	P
29	3	P	P	3	43	7	3	P	11	3	47	23	3	73	61	3	13	17	3	7
31	29	3	P	61	3	23	11	3	P	P	3	7	P	3	P	P	3	11	7	3
33	37	P	3	7	11	3	41	P	3	P	7	3	P	P	3	11	43	3	19	17
37	11	3	19	P	3	13	P	3	7	P	3	11	P	3	P	7	3	P	13	3
39	7	P	3	P	23	3	P	7	3	11	P	3	13	19	3	29	P	3	P	P
41	3	41	P	3	P	19	3	11	47	3	71	53	3	7	P	3	P	P	3	13
43	13	3	P	43	3	7	P	3	29	P	3	37	7	3	P	23	3	P	P	3
47	3	11	31	3	P	P	3	47	37	3	7	P	3	P	13	3	P	7	3	19
49	P	3	7	P	3	P	P	3	13	7	3	19	29	3	P	31	3	P	P	3
51	P	7	3	19	P	3	P	P	3	P	P	3	59	P	3	7	P	3	P	11
53	3	P	P	3	61	29	3	7	23	3	31	P	3	53	7	3	P	11	3	P
57	P	P	3	P	P	3	P	67	3	P	13	3	7	11	3	P	P	3	P	7
59	3	P	P	3	7	47	3	P	43	3	P	7	3	23	53	3	P	13	3	59
61	31	3	P	7	3	P	59	3	P	11	3	13	P	3	43	67	3	7	P	3
63	17	23	3	P	P	3	P	11	3	7	61	3	19	31	3	P	7	3	11	67
67	7	3	17	11	3	P	13	3	31	P	3	P	23	3	7	19	3	73	P	3
69	13	11	3	17	41	3	7	19	3	P	37	3	11	7	3	P	P	3	P	47
71	3	43	P	3	17	7	3	13	P	3	11	P	3	41	P	3	53	29	3	7
73	P	3	P	P	3	17	P	3	11	P	3	7	P	3	13	P	3	23	7	3
77	3	P	7	3	11	23	3	17	P	3	P	31	3	19	P	3	7	53	3	43
79	P	3	11	29	3	19	P	3	7	13	3	P	P	3	P	7	3	P	P	3
81	7	37	3	13	P	3	31	7	3	17	P	3	P	3	P	3	13	3	P	P
83	3	47	P	3	P	3	P	3	19	3	13	71	3	7	P	3	P	3	31	P
87	61	53	3	41	7	3	43	P	3	P	P	3	17	P	3	37	11	3	7	P
89	3	59	P	3	67	13	3	P	P	3	7	P	3	17	11	3	P	7	3	53
91	P	3	7	P	3	P	P	3	67	7	3	29	11	3	17	P	3	P	43	3
93	P	7	3	23	P	3	13	P	3	P	11	3	67	P	3	7	P	3	71	13
97	17	3	P	P	3	P	7	3	59	19	3	P	P	3	23	29	3	11	P	3
99	P	13	3	53	11	3	37	P	3	P	P	3	7	P	3	11	41	3	17	7

	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
01	17	P	3	P	37	3	7	P	3	67	P	3	19	7	3	13	11	3	29	P
03	3	17	P	3	19	7	3	P	P	3	47	P	3	67	11	3	P	P	3	7
07	P	31	3	7	43	3	P	19	3	P	7	3	P	P	3	P	P	3	37	P
09	3	41	7	3	13	23	3	P	11	3	43	P	3	P	31	3	7	13	3	11
11	P	3	P	P	3	17	11	3	7	P	3	13	P	3	P	7	3	11	73	3
13	7	P	3	59	11	3	17	7	3	31	P	3	P	71	3	11	23	3	13	41
17	11	3	P	P	3	7	13	3	17	P	3	11	7	3	P	P	3	P	P	3
19	13	29	3	71	7	3	P	P	3	11	P	3	P	13	3	73	19	3	7	P
21	3	P	P	3	P	P	3	11	19	3	7	P	3	P	41	3	P	7	3	89
23	19	3	7	P	3	11	37	3	P	7	3	17	31	3	13	P	3	P	P	3
27	3	11	13	3	P	61	3	7	P	3	P	P	3	17	7	3	29	P	3	P
29	P	3	P	P	3	P	7	3	P	13	3	P	P	3	17	P	3	59	P	3
31	37	P	3	13	59	3	19	53	3	29	79	3	7	P	3	17	13	3	41	7
33	3	P	23	3	7	47	3	P	P	3	13	7	3	P	P	3	17	11	3	P
37	P	17	3	P	41	3	P	P	3	7	31	3	P	11	3	P	7	3	17	P
39	3	7	17	3	47	13	3	23	7	3	P	11	3	41	43	3	P	71	3	17
41	7	3	79	17	3	31	29	3	P	11	3	37	13	3	7	P	3	P	P	3
43	P	P	3	P	17	3	7	11	3	53	P	3	P	7	3	19	P	3	11	13
47	P	3	P	11	3	P	17	3	41	P	3	7	P	3	11	P	3	61	7	3
49	23	11	3	7	P	3	61	17	3	P	7	3	11	P	3	P	P	3	47	P
51	3	P	7	3	P	P	3	43	13	3	11	P	3	P	P	3	7	23	3	P
53	P	3	13	P	3	P	P	3	7	17	3	23	P	3	29	7	3	P	P	3
57	3	47	P	3	11	79	3	29	P	3	P	17	3	7	P	3	13	P	3	73
59	73	3	11	P	3	7	P	3	19	P	3	P	7	3	P	P	3	P	29	3
61	11	61	3	P	7	3	P	P	3	P	23	3	53	17	3	P	47	3	7	19
63	3	P	P	3	23	P	3	P	P	3	7	13	3	37	17	3	79	7	3	P
67	P	7	3	P	29	3	59	67	3	P	37	3	13	53	3	7	11	3	P	31
69	3	31	P	3	P	P	3	7	P	3	P	67	3	P	7	3	P	17	3	13
71	13	3	P	23	3	P	7	3	P	P	3	71	11	3	31	67	3	19	17	3
73	P	P	3	P	P	3	P	13	3	19	11	3	7	73	3	P	P	3	P	7
77	59	3	P	7	3	P	11	3	13	P	3	P	19	3	P	P	3	7	P	3
79	P	37	3	P	11	3	P	P	3	7	P	3	29	47	3	11	7	3	P	79
81	3	7	11	3	P	P	3	P	7	3	73	43	3	11	P	3	P	31	3	23
83	7	3	61	13	3	29	41	3	P	P	3	11	P	3	7	P	3	43	P	3
87	3	23	P	3	13	7	3	11	71	3	19	P	3	83	P	3	P	13	3	7
89	P	3	19	P	3	11	P	3	83	29	3	7	37	3	P	P	3	P	7	3
91	P	41	3	7	P	3	P	P	3	P	7	3	23	19	3	P	P	3	13	61
93	3	11	7	3	43	19	3	P	61	3	41	P	3	P	59	3	7	P	3	P
97	7	P	3	P	73	3	37	7	3	P	47	3	P	13	3	71	43	3	53	11
99	3	P	P	3	67	P	3	13	P	3	31	23	3	7	P	3	P	11	3	19

	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
01	3	P	59	3	31	P	3	7	13	3	P	19	3	71	7	3	P	89	3	P
03	53	3	13	19	3	11	7	3	P	29	3	P	P	3	P	13	3	31	P	3
07	3	11	29	3	7	47	3	P	P	3	P	7	3	41	23	3	13	17	3	P
09	P	3	P	7	3	67	P	3	23	59	3	P	P	3	97	37	3	7	1	3
11	P	P	3	P	13	3	79	31	3	7	P	3	61	P	3	P	7	3	P	11
13	3	7	43	3	47	P	3	P	7	3	P	13	3	67	P	3	P	11	3	23
17	P	P	3	P	19	3	7	23	3	37	71	3	13	7	3	31	59	3	P	47
19	3	23	P	3	P	7	3	P	P	3	29	11	3	P	P	3	P	P	3	7
21	13	3	P	53	3	P	37	3	P	11	3	7	P	3	P	P	3	P	7	3
23	71	P	3	7	P	3	P	11	3	P	7	3	23	P	3	89	P	3	11	P
27	23	3	19	11	3	P	P	3	7	79	3	P	P	3	11	7	3	71	31	3
29	7	11	3	P	P	3	P	7	3	P	P	3	11	19	3	13	P	3	P	P
31	3	47	P	3	P	19	3	P	P	3	11	23	3	7	P	3	P	3	3	P
33	29	3	P	13	2	7	89	3	11	P	3	P	7	3	P	P	3	P	P	3
37	3	79	P	3	11	P	3	P	P	3	7	P	3	P	P	3	23	7	3	19
39	P	3	7	31	3	P	53	3	P	7	3	13	P	3	P	P	3	P	P	3
41	11	7	3	19	23	3	P	P	3	P	P	3	P	P	3	7	31	3	13	P
43	3	17	P	3	P	P	3	7	37	3	P	41	3	P	7	3	P	P	3	61
47	13	P	3	17	P	3	P	3	23	83	3	7	13	3	P	11	3	43	7	P
49	3	29	73	3	7	83	3	13	P	3	P	7	3	P	11	3	P	3	3	P
51	83	3	37	7	3	17	41	3	53	P	3	P	11	3	13	P	3	7	P	3
53	P	31	3	P	79	3	17	P	3	7	11	3	19	47	3	41	7	3	59	37
57	7	3	23	61	3	43	11	3	17	13	3	P	P	3	7	19	3	11	P	3
59	P	41	3	13	11	3	7	19	3	17	P	3	47	7	3	11	13	3	P	23
61	3	P	11	3	P	7	3	P	P	3	13	P	3	11	P	3	P	43	3	7
63	11	3	P	P	3	P	P	3	P	3	7	59	3	P	73	3	13	7	3	P
67	3	P	7	3	P	13	3	11	P	3	P	89	3	17	P	3	7	P	3	P
69	P	3	P	P	3	11	P	3	7	P	3	53	13	3	17	7	3	P	71	3
71	7	P	3	11	43	3	13	7	3	P	47	3	73	P	3	17	19	3	P	13
73	3	11	P	3	37	P	3	31	19	3	43	P	3	7	P	3	17	29	3	P
77	41	13	3	P	7	3	P	67	3	47	29	3	P	P	3	61	P	3	7	11
79	3	P	17	3	61	23	3	P	13	3	7	67	3	83	P	3	P	7	3	17
81	P	3	7	17	3	P	P	3	83	7	3	P	P	3	19	11	3	P	41	3
83	59	7	3	83	17	3	19	P	3	13	31	3	P	11	3	7	23	3	P	67
87	P	3	P	P	3	31	7	3	P	11	3	P	7	3	53	P	3	P	P	3
89	P	19	3	P	13	3	P	11	3	89	61	3	7	41	3	43	P	3	11	7
91	3	P	P	3	7	11	3	59	17	3	P	7	3	P	P	3	11	P	3	97
93	P	3	P	7	3	13	P	3	P	17	3	29	P	3	11	53	3	7	13	3
97	3	7	P	3	29	P	3	19	7	3	11	47	3	P	P	3	P	97	3	13
99	7	3	43	37	3	P	P	3	11	P	3	P	17	3	7	29	3	41	19	3

	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119
01	73	3	101	P	3	P	P	3	7	11	3	17	23	3	13	7	3	P	P	3
03	7	P	3	P	101	3	23	7	3	P	P	3	17	89	3	P	41	3	11	P
09	P	3	59	11	3	7	P	3	101	13	3	29	7	3	11	37	3	23	P	3
09	P	11	3	13	7	3	103	P	3	P	101	3	11	43	3	17	13	3	7	P
11	3	P	P	3	29	23	3	P	19	3	7	41	3	P	P	3	17	7	3	43
13	17	3	7	P	3	P	P	3	11	7	3	P	P	3	101	29	3	13	P	3
17	3	67	17	3	11	13	3	7	29	3	23	P	3	P	7	3	P	P	3	17
19	43	3	11	17	3	67	7	3	31	61	3	P	13	3	19	P	3	P	53	3
21	11	29	3	P	17	3	13	71	3	67	103	3	7	P	3	41	P	3	P	7
23	3	53	P	3	7	17	3	P	79	3	73	7	3	13	P	3	59	29	3	P
27	37	13	3	23	P	3	P	17	3	7	P	3	103	47	3	P	7	3	P	P
29	3	7	53	3	P	P	3	P	7	3	41	31	3	P	11	3	29	37	3	79
31	7	3	13	P	3	P	P	3	P	17	3	P	11	3	7	13	3	P	P	3
33	79	P	3	P	P	3	7	P	3	13	11	3	47	7	3	19	P	3	P	P
37	P	3	29	P	3	41	11	3	P	3	7	17	3	P	83	3	11	7	3	
39	P	P	3	7	11	3	P	P	3	P	7	3	P	17	3	11	103	3	P	P
41	3	P	7	3	53	83	3	23	37	3	61	13	3	11	17	3	7	59	3	P
43	11	3	P	P	3	13	29	3	7	31	3	11	P	3	P	7	3	P	13	3
47	3	73	P	3	31	53	3	11	P	3	P	71	3	7	P	3	19	17	3	13
49	13	3	37	79	3	7	23	3	19	P	3	P	7	3	107	P	3	31	17	3
51	19	P	3	11	7	3	P	13	3	47	43	3	P	P	3	P	61	3	7	17
53	3	11	P	3	P	61	3	P	P	3	7	19	3	P	13	3	43	7	3	P
57	89	7	3	P	P	3	P	31	3	P	3	P	3	41	3	7	P	3	71	11
59	3	P	P	3	P	P	3	7	P	3	P	P	3	37	7	3	89	11	3	P
61	P	3	31	13	3	59	7	3	P	97	3	P	P	3	73	11	3	19	29	3
63	29	P	3	43	P	3	P	47	3	19	13	3	7	11	3	31	107	3	P	7
67	P	3	P	7	3	P	P	3	P	11	3	13	19	3	P	43	3	7	P	3
69	P	P	3	P	19	3	47	11	3	7	P	3	59	P	3	23	7	3	11	P
71	3	7	P	3	37	11	3	P	7	3	P	P	3	83	P	3	11	79	3	P
73	7	3	P	11	3	97	13	3	83	P	3	P	P	3	7	71	3	61	31	3
77	3	P	43	3	P	7	3	13	73	3	11	P	3	31	23	3	P	P	3	7
79	P	3	19	97	3	71	59	3	11	P	3	7	P	3	13	P	3	P	7	3
81	17	P	3	7	47	3	11	P	3	79	7	3	29	19	3	37	P	3	109	P
83	3	17	7	3	11	19	3	41	P	3	P	53	3	P	P	3	7	P	3	23
87	7	61	3	13	P	3	P	7	3	P	P	3	P	59	3	P	13	3	P	P
89	3	23	P	3	17	P	3	P	P	3	13	67	3	7	P	3	P	P	3	19
91	P	3	41	P	3	7	P	3	P	29	3	19	7	3	P	67	3	13	11	3
93	P	P	3	19	7	3	17	43	3	P	3	P	3	23	P	3	11	3	7	67
97	23	3	7	37	3	P	19	3	17	3	P	P	11	3	P	P	3	47	P	3
99	P	7	3	P	P	3	13	P	3	17	11	3	P	P	3	7	P	3	73	13

Incomposita, or Prime, Numbers, less than 100,000.

373

	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139
01	11	P	3	P	P	3	P	13	3	7	P	3	43	47	3	23	7	3	37	P
03	3	7	P	3	79	P	3	P	7	3	P	P	3	53	13	3	61	71	3	P
07	P	P	3	31	19	3	7	97	3	P	P	3	47	7	3	13	11	3	P	P
09	3	P	29	3	P	7	3	71	P	3	P	P	3	P	11	3	31	P	3	7
11	P	3	P	13	3	P	P	3	23	P	3	7	11	3	P	59	3	P	7	3
13	41	P	3	7	P	3	P	P	3	37	7	3	73	P	3	P	P	3	19	P
17	61	3	19	109	3	P	11	3	7	P	3	13	P	3	P	7	3	11	41	3
19	7	P	3	97	11	3	P	7	3	P	47	3	P	19	3	P	P	3	13	31
21	3	17	11	3	P	19	3	P	P	3	49	P	3	7	P	3	53	P	3	P
23	11	3	17	P	3	7	13	3	P	P	3	11	7	3	31	P	3	P	23	3
27	3	67	P	3	17	P	3	11	101	3	7	P	3	P	29	3	P	7	3	19
29	23	3	7	P	3	11	73	3	P	7	3	19	P	3	13	83	3	P	P	3
31	53	7	3	11	31	3	17	29	3	67	83	3	101	P	3	7	43	3	P	P
33	3	11	13	3	P	83	3	7	41	3	P	23	3	67	7	3	P	31	3	P
37	P	53	3	13	P	3	P	47	3	17	P	3	7	P	3	P	13	3	101	7
39	3	61	P	3	7	P	3	P	37	3	13	7	3	P	89	3	23	11	3	53
41	P	3	P	7	3	P	P	3	P	P	3	17	P	3	P	11	3	7	P	3
43	P	P	3	P	23	3	47	P	3	7	P	3	17	11	3	29	7	3	109	73
47	7	3	37	P	3	P	P	3	29	11	3	P	13	3	7	19	3	59	61	13
49	P	P	3	53	59	3	7	11	3	23	P	3	P	7	3	17	P	3	11	13
51	3	29	P	3	P	7	3	41	71	3	31	P	3	13	P	3	11	P	3	7
53	17	3	P	11	3	P	P	3	P	3	7	29	3	11	P	3	17	7	3	3
57	3	P	7	3	P	29	3	P	13	3	11	59	3	19	P	3	7	P	3	17
59	31	3	13	17	3	19	P	3	7	P	3	P	P	3	43	7	3	P	P	3
61	7	P	3	47	17	3	11	7	3	13	37	3	89	31	3	71	19	3	83	23
63	3	P	P	3	11	17	3	P	19	3	P	P	3	7	P	3	13	P	3	P
67	11	23	3	83	7	3	53	17	3	P	73	3	P	3	P	3	79	3	7	P
69	3	43	P	3	37	P	3	113	17	3	7	13	3	29	P	3	P	7	3	61
71	P	3	7	89	3	13	P	3	61	7	3	P	23	3	19	41	3	47	11	3
73	P	7	3	P	P	3	19	53	3	P	17	3	13	43	3	7	11	3	P	89
77	13	3	P	P	3	P	7	3	79	19	3	P	11	3	P	P	3	23	P	3
79	47	19	3	P	P	3	31	13	3	P	11	3	7	17	3	37	P	3	P	7
81	3	13	P	3	7	23	3	P	11	3	103	7	3	P	13	3	P	P	3	17
83	43	3	71	7	3	P	11	3	13	P	3	P	37	3	97	17	3	7	P	3
87	3	7	11	3	P	41	3	19	7	3	23	P	3	11	P	3	P	17	3	71
89	7	3	P	13	3	P	P	3	P	31	3	11	97	3	7	107	3	P	17	3
91	107	73	3	P	P	3	7	P	3	11	13	3	P	7	3	P	P	3	29	17
93	3	89	19	3	13	7	3	11	P	3	P	79	3	59	103	3	P	13	3	7
97	P	P	3	7	P	3	P	67	3	41	7	3	P	P	3	P	P	3	13	P
99	3	11	7	3	29	43	3	P	P	3	P	67	3	P	P	3	7	P	3	P

	140	141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159
01	3	59	11	3	P	17	3	61	19	3	7	P	3	11	P	3	P	7	3	P
03	11	3	7	P	3	P	17	3	113	7	3	11	23	3	73	37	3	41	P	3
07	3	P	P	3	P	19	3	7	13	3	43	P	3	P	7	3	P	113	3	P
09	P	3	13	41	3	11	7	3	59	17	3	29	67	3	19	13	3	23	P	3
11	P	103	3	11	P	3	19	47	3	13	17	3	7	61	3	P	67	3	97	7
13	3	11	61	3	7	23	3	P	P	3	P	7	3	P	P	3	13	19	3	P
17	107	19	3	103	13	3	47	P	3	7	P	3	P	17	3	59	7	3	P	11
19	3	7	59	3	P	P	3	41	7	3	23	13	3	P	17	3	P	11	3	P
21	7	3	P	P	3	13	P	3	P	43	3	P	31	3	7	11	3	79	13	3
23	37	29	3	P	P	3	7	P	3	P	83	3	13	7	3	19	17	3	P	P
27	13	3	41	P	3	73	P	3	P	11	3	7	P	P	P	P	3	P	7	3
29	P	71	3	7	47	3	P	11	3	P	7	3	97	P	3	53	P	3	11	17
31	3	13	7	3	P	11	3	P	P	3	P	P	3	P	13	3	7	P	3	89
33	P	3	43	11	3	P	P	3	7	109	3	37	P	3	11	7	3	P	71	3
37	3	67	23	3	P	P	3	P	37	3	11	P	3	7	43	3	19	P	3	P
39	101	3	29	13	3	7	P	3	11	P	3	P	7	3	P	41	3	P	47	3
41	19	79	3	P	7	3	11	P	3	67	13	3	P	23	3	P	P	3	7	19
43	3	P	P	3	11	P	3	23	P	3	7	19	3	67	P	3	P	7	3	107
47	11	7	3	P	P	3	97	P	3	P	41	3	79	103	3	7	P	3	13	37
49	3	P	P	3	P	P	3	7	31	3	101	P	3	P	7	3	P	P	3	41
51	P	3	P	113	3	P	7	3	P	P	3	109	101	3	P	P	3	19	11	3
53	13	P	3	31	97	3	P	P	3	19	P	3	7	13	3	103	11	3	83	7
57	P	3	53	7	3	P	P	3	83	P	3	23	11	3	13	47	3	7	101	3
59	17	P	3	83	19	3	107	P	3	7	11	3	P	P	3	P	7	3	P	P
61	3	7	13	3	P	P	3	29	7	3	P	P	3	P	P	3	P	P	3	11
63	7	3	17	53	3	P	11	3	89	13	3	59	P	3	7	79	3	11	29	3
67	3	31	11	3	17	7	3	P	P	3	13	29	3	11	P	3	P	P	3	7
69	11	3	19	P	3	17	P	3	P	P	3	7	P	3	31	P	3	13	7	3
71	P	37	3	7	29	3	17	P	3	11	7	3	P	19	3	23	P	3	59	P
73	3	P	7	3	41	13	3	11	73	3	P	P	3	P	P	3	7	P	3	P
77	7	P	3	11	31	3	13	7	3	17	P	3	P	P	3	37	61	3	P	13
79	3	11	109	3	P	61	3	P	3	17	43	3	7	23	3	P	31	3	19	
81	P	3	P	73	3	7	53	3	23	71	3	17	7	3	113	P	3	43	P	3
83	P	13	3	19	7	3	P	P	3	P	P	3	17	P	3	P	3	7	11	
87	P	3	7	P	3	29	19	3	P	7	3	P	P	3	17	11	3	P	P	3
89	73	7	3	P	P	3	37	23	3	13	79	3	P	11	3	7	29	3	P	59
91	3	23	31	3	43	P	3	7	P	3	P	11	3	P	7	3	13	P	3	P
93	17	3	P	37	3	P	7	3	53	11	3	P	41	3	P	31	3	17	23	3
97	3	P	17	3	7	11	3	P	P	3	31	7	P	3	89	P	3	11	P	3
99	23	3	79	7	3	13	P	3	47	53	3	P	3	P	3	11	19	3	7	13

	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179
01	P	3	17	P	3	29	13	3	53	P	3	7	103	3	P	11	3	31	7	3
03	13	P	3	7	47	3	P	P	3	P	7	3	P	11	3	23	29	3	19	P
07	P	3	19	23	3	17	P	3	7	11	3	P	P	3	13	7	3	P	P	3
09	7	89	3	47	61	3	17	7	3	37	73	3	P	19	3	P	P	3	11	P
11	3	P	13	3	P	11	3	17	P	3	P	71	3	7	23	3	11	89	3	P
13	67	3	31	11	3	7	37	3	17	13	3	109	7	3	11	83	3	P	47	3
17	3	71	P	3	P	83	3	73	67	3	7	P	3	P	P	3	79	7	3	19
19	83	3	7	P	3	P	P	3	11	7	3	17	67	3	P	P	3	13	103	3
21	37	7	3	19	P	3	11	23	3	P	P	3	17	P	3	7	67	3	71	P
23	3	23	P	3	11	13	3	7	P	3	29	P	3	17	7	3	P	37	3	P
27	11	P	3	29	P	3	13	43	3	P	P	3	7	P	3	17	P	3	P	7
29	3	127	P	3	7	P	3	P	3	P	7	3	13	29	3	17	P	3	3	P
31	17	3	P	7	3	61	P	P	P	P	3	37	P	3	P	47	3	7	11	3
33	P	13	3	P	P	3	P	29	3	7	P	3	19	P	3	89	7	3	17	79
37	7	3	13	17	3	23	127	3	113	P	3	P	11	3	7	13	3	P	P	3
39	43	P	3	P	17	3	7	19	3	13	11	3	P	7	3	P	31	3	P	P
41	3	P	109	3	41	7	3	P	11	3	P	61	3	P	107	3	13	113	3	7
43	61	3	37	59	3	71	11	3	P	P	3	7	43	3	P	53	3	11	7	3
47	3	67	7	3	P	P	3	P	17	3	P	13	3	11	73	3	7	3	131	3
49	11	3	P	P	3	13	P	3	7	17	3	11	47	3	P	7	3	P	13	3
51	7	31	3	83	P	3	P	7	3	11	17	3	13	P	3	P	19	3	P	29
53	3	29	P	3	P	P	3	11	19	3	P	17	3	7	31	3	127	41	3	13
57	P	107	3	11	7	3	P	13	3	31	37	3	P	17	3	97	P	3	7	P
59	3	11	71	3	109	29	3	P	23	3	7	P	3	P	13	3	P	7	3	P
61	P	3	7	P	3	P	P	3	13	7	3	131	41	3	19	17	3	P	53	3
63	P	7	3	P	101	3	19	P	3	P	113	3	61	97	3	7	17	3	P	11
67	P	3	P	13	3	P	7	3	101	19	3	P	31	3	P	11	3	109	17	3
69	P	19	3	P	43	3	79	41	3	71	13	3	7	11	3	P	P	3	107	7
71	3	103	53	3	7	73	3	31	P	3	43	7	3	29	P	3	41	13	3	P
73	P	3	P	7	3	P	P	3	47	11	3	13	23	3	101	P	3	7	61	3
77	3	7	41	3	P	11	3	19	7	3	P	89	3	P	P	3	11	29	3	P
79	7	3	73	11	3	59	13	3	P	P	3	41	37	3	7	P	3	23	19	3
81	13	11	3	P	P	3	7	97	3	P	19	3	11	7	3	P	P	3	P	P
83	3	P	19	3	53	7	3	13	P	3	11	P	3	P	P	3	P	P	3	7
87	P	P	3	7	P	3	11	P	3	P	7	3	59	P	3	43	23	3	31	P
89	3	P	7	3	11	53	3	103	P	3	23	P	3	P	P	3	7	P	3	P
91	P	3	11	37	3	47	P	3	7	13	3	P	P	3	P	7	3	P	P	3
93	7	P	3	13	P	3	P	7	3	P	P	3	P	P	3	73	13	3	29	19
97	P	3	43	19	3	7	59	3	61	23	3	29	7	3	P	P	3	13	11	3
99	17	97	3	23	7	3	P	107	3	89	P	3	P	127	3	P	11	3	7	41

	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199
01	47	23	3	P	P	3	11	P	3	41	P	3	7	P	3	P	17	3	P	7
03	3	43	109	3	7	P	3	59	P	3	21	7	3	97	P	3	P	17	3	13
07	11	19	3	P	79	3	23	13	3	7	83	3	P	43	3	P	7	3	29	17
09	3	7	13	3	41	83	3	53	7	3	P	97	3	P	13	3	P	3	43	
11	7	3	P	P	3	107	37	3	13	P	3	29	P	3	7	109	3	23	11	3
13	P	59	3	P	P	3	7	P	3	P	3	P	3	P	7	3	13	11	3	P
17	43	3	P	13	3	P	P	3	31	P	3	7	11	3	P	29	3	P	7	3
19	37	P	3	7	113	3	43	P	3	P	7	3	P	P	3	131	23	3	P	P
21	3	P	7	3	13	P	3	97	11	3	23	P	3	139	P	3	7	13	3	11
23	67	3	P	73	3	P	11	3	7	127	3	13	47	3	P	7	3	11	43	3
27	3	P	11	3	P	97	3	61	67	3	53	31	3	7	P	3	19	P	3	P
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31	13	P	3	23	7	3	81	P	3	11	P	3	P	13	3	P	67	3	7	19
33	3	P	P	3	P	43	3	11	37	3	7	19	3	P	3	P	3	29	7	3
37	17	7	3	11	103	3	P	41	3	29	P	3	P	61	3	7	73	3	83	P
39	3	11	13	3	P	P	3	7	P	3	79	P	3	83	7	3	41	P	3	127
41	P	9	17	P	9	P	7	3	83	13	3	P	71	3	P	P	3	19	P	3
43	P	P	3	13	P	3	103	P	3	19	137	3	7	23	3	P	13	3	P	7
45	P	3	71	7	3	17	29	3	47	P	3	41	19	3	P	11	3	7	89	3
49	P	P	3	59	19	3	17	P	3	7	43	3	P	11	3	113	7	3	23	P
51	3	7	P	3	P	13	3	17	7	3	P	11	3	37	53	3	43	P	3	71
53	7	3	P	P	3	P	23	3	17	11	3	107	13	3	7	P	3	P	P	3
57	3	67	P	8	P	7	3	P	109	3	17	P	3	13	P	3	11	23	3	7
59	P	3	29	11	3	67	47	3	P	P	3	7	P	3	11	P	3	P	7	3
61	P	11	3	7	P	3	P	73	3	67	7	3	11	19	3	31	P	3	P	P
63	3	41	7	3	37	19	3	29	13	3	11	P	3	17	P	3	7	P	3	P
67	7	37	3	P	59	3	11	7	3	13	23	3	P	107	3	17	71	3	P	41
69	3	P	P	3	11	31	3	137	P	3	P	29	3	7	P	3	13	53	3	19
71	17	3	11	P	3	7	P	3	113	61	3	19	7	3	P	P	3	17	31	3
73	11	17	3	19	7	3	71	P	3	P	P	3	P	3	P	3	23	103	3	7
77	P	3	7	17	3	13	19	3	43	7	3	127	37	3	P	P	3	P	11	3
79	101	7	3	P	17	3	P	89	3	P	P	3	13	P	2	7	11	3	103	P
81	3	P	101	3	P	17	3	7	79	3	P	P	3	P	7	3	P	131	3	13
83	13	3	47	31	3	P	7	3	23	41	3	P	11	3	P	P	3	73	59	3
87	3	13	P	3	7	P	3	P	11	3	P	7	3	P	13	3	P	47	3	11
89	P	3	P	7	3	29	11	3	13	17	3	31	P	3	P	19	3	7	P	3
91	79	P	3	53	11	3	P	19	3	7	17	3	101	P	3	11	7	3	P	P
93	3	7	11	3	P	P	3	P	7	3	61	17	3	11	101	3	47	P	3	P
97	P	31	3	P	53	3	7	P	3	11	13	3	23	7	3	P	P	3	101	P
99	3	P	19	3	13	7	3	11	P	3	71	73	3	19	17	3	P	13	3	7

	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219
01	3	P	P	3	23	13	3	127	11	3	P	P	3	7	P	3	P	P	3	11
03	83	3	89	79	3	7	11	3	71	P	3	47	7	3	17	P	3	11	P	3
07	3	P	11	3	P	P	3	P	P	3	7	P	3	11	P	3	17	7	3	19
09	11	3	7	23	3	P	37	3	P	7	3	11	127	3	79	137	3	17	113	3
11	P	7	3	19	P	3	P	139	3	11	P	3	P	101	3	7	P	3	17	P
13	3	P	17	3	137	73	3	7	13	3	P	43	3	P	7	3	P	P	3	17
17	37	P	3	11	17	3	53	P	3	13	P	3	7	P	3	P	P	3	P	7
19	3	11	P	3	7	17	3	P	109	3	P	7	3	P	P	3	13	37	3	23
21	P	3	73	7	3	P	17	3	47	P	3	P	P	3	31	P	3	7	P	3
23	P	P	3	P	13	3	41	17	3	7	P	3	19	P	3	P	7	3	139	11
27	7	3	113	P	3	13	P	3	59	17	3	37	P	3	7	11	3	P	13	3
29	P	P	3	29	31	3	7	19	3	P	17	3	13	7	3	P	43	3	83	P
31	3	41	P	3	P	7	3	P	37	3	P	11	3	83	29	3	97	31	3	7
33	13	3	P	P	3	P	47	3	83	11	3	7	17	3	P	61	3	103	7	3
37	3	13	7	3	107	11	3	89	67	3	109	23	3	19	13	3	7	P	3	P
39	29	3	37	11	3	19	P	3	7	P	3	P	67	3	11	7	3	P	P	3
41	7	11	3	P	P	3	P	7	3	43	53	3	11	P	3	13	17	3	P	37
43	3	P	31	3	P	P	3	P	19	3	11	P	3	7	41	3	23	17	3	P
47	P	P	3	P	7	3	11	P	3	P	13	3	P	P	3	29	P	3	7	17
49	3	P	P	3	11	P	3	P	3	7	P	3	37	89	3	P	7	3	47	
51	P	3	7	47	3	P	107	3	29	7	3	13	79	3	19	23	3	P	P	3
53	11	7	3	P	113	3	19	P	3	23	37	3	53	131	3	7	59	3	13	29
57	31	3	47	P	3	61	7	3	P	19	3	P	29	3	43	P	3	P	11	3
59	13	19	3	P	41	3	73	P	3	P	P	3	7	13	3	P	11	3	P	7
61	3	P	P	3	7	29	3	13	23	3	P	7	3	41	11	3	P	47	3	P
63	P	3	23	7	3	P	P	3	31	P	3	P	11	3	13	P	3	7	P	3
67	3	7	13	3	97	131	3	19	7	3	P	61	3	23	P	3	47	P	3	11
69	7	3	P	P	3	67	11	3	41	13	3	P	P	3	7	P	3	11	19	3
71	P	23	3	13	11	3	7	P	3	67	19	3	89	7	3	11	13	3	P	127
73	3	P	11	3	59	7	3	P	P	3	13	31	3	11	109	3	P	P	3	7
77	17	P	3	7	P	3	23	79	3	11	7	3	P	P	3	P	53	3	131	P
79	3	17	7	3	P	13	3	11	P	3	107	P	3	P	47	3	7	29	3	31
81	43	3	17	89	3	11	P	3	7	P	3	59	13	3	P	7	3	23	P	3
83	7	P	3	11	P	3	13	7	3	P	29	3	P	3	113	P	3	79	13	
87	53	3	P	19	3	7	137	3	P	31	3	P	7	3	P	P	3	P	43	3
89	P	13	3	P	7	3	17	P	3	139	P	3	61	73	3	P	23	3	7	11
91	3	61	103	3	31	59	3	17	13	3	7	P	3	P	P	3	109	7	3	P
93	7	3	7	P	3	P	P	3	17	7	3	P	107	3	P	11	3	19	P	3
97	3	19	P	3	103	43	3	7	P	3	17	11	3	P	7	3	13	71	3	P
99	101	3	53	P	3	P	7	3	P	11	3	17	19	3	P	P	3	P	61	3

	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239
01	7	3	149	29	3	P	97	3	151	P	3	13	P	3	7	71	3	127	P	3
03	P	23	3	P	43	3	7	73	3	37	P	3	P	7	3	19	P	3	13	11
07	59	3	53	P	3	71	13	3	P	P	3	7	23	3	89	11	3	151	7	3
09	13	P	3	7	P	3	23	P	3	31	7	3	P	11	3	P	P	3	29	P
11	3	P	7	3	73	P	3	13	P	3	P	11	3	P	41	3	7	131	3	P
13	P	3	97	53	3	47	P	3	7	11	3	29	139	3	13	7	3	23	P	3
17	3	17	13	3	29	11	3	P	P	3	P	P	3	7	P	3	11	37	3	P
19	97	3	17	11	3	7	P	3	19	13	3	61	7	3	11	29	3	P	P	3
21	19	11	3	13	7	3	P	P	3	P	P	3	11	P	3	43	13	3	7	19
23	3	P	71	3	17	101	3	31	29	3	7	19	3	83	59	3	P	7	3	47
27	P	7	3	83	41	3	11	P	3	101	P	3	P	P	3	7	P	3	P	71
29	3	P	P	3	11	13	3	7	37	3	P	101	3	41	7	3	P	61	3	P
31	P	3	11	137	3	P	7	3	17	23	3	P	13	3	P	P	3	19	P	3
33	11	P	3	23	P	3	13	127	3	17	31	3	7	P	3	101	P	3	P	7
37	P	3	37	7	3	31	P	3	41	P	3	17	19	3	23	P	3	7	11	3
39	P	13	3	89	19	3	P	P	3	7	P	3	17	P	3	P	7	3	31	37
41	3	7	23	3	P	P	3	P	7	3	P	73	3	17	11	3	47	P	3	89
43	7	3	13	P	3	P	P	3	53	P	3	P	11	3	7	13	3	P	113	3
47	3	P	P	3	P	7	3	23	11	3	19	79	3	37	P	3	13	P	3	7
49	17	3	19	P	3	P	11	3	73	53	3	7	67	3	131	P	3	11	7	3
51	P	17	3	7	11	3	P	P	3	59	7	3	P	19	3	11	67	3	17	43
53	3	P	7	3	P	19	3	61	P	3	P	13	3	11	47	3	7	P	3	17
57	7	P	3	79	17	3	139	7	3	11	P	3	13	P	3	P	41	3	P	P
59	3	P	P	3	37	17	3	11	P	3	P	P	3	7	P	3	59	23	3	13
61	13	3	113	59	3	7	17	3	P	P	3	19	7	3	29	P	3	P	107	3
63	P	37	3	11	7	3	131	13	3	P	P	3	43	61	3	P	P	3	7	31
67	P	3	7	P	3	P	19	3	13	7	3	P	53	3	31	P	3	P	29	3
69	29	7	3	P	P	3	P	P	3	103	17	3	P	P	3	7	P	3	P	11
71	3	P	P	3	23	P	3	7	P	3	P	17	3	P	7	3	P	11	3	P
73	P	P	P	13	3	P	7	3	89	P	3	P	17	3	P	11	3	P	P	3
77	3	67	P	3	7	107	3	P	P	3	47	7	3	97	17	3	P	13	3	P
79	P	3	P	7	3	67	P	3	137	11	3	13	P	3	53	17	3	7	P	3
81	71	41	3	P	3	37	11	3	7	P	3	31	103	3	P	7	3	11	P	P
83	3	7	P	3	P	11	3	P	7	3	41	97	3	67	23	3	11	17	3	29
87	13	11	3	61	113	3	7	P	3	127	P	3	11	7	3	103	P	3	P	17
89	3	P	31	3	43	7	3	13	47	3	11	P	3	19	83	3	P	P	3	7
91	P	3	P	P	3	19	P	3	11	83	3	7	P	3	13	31	3	37	7	3
93	P	P	3	7	83	3	11	23	3	P	7	3	P	149	3	P	19	3	P	P
97	19	3	11	P	3	59	P	3	7	13	3	P	P	3	P	7	3	53	23	3
99	7	79	3	13	149	3	P	7	3	109	P	3	23	P	3	P	13	3	P	103

	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259
01	P	7	3	19	13	3	73	17	3	37	23	3	11	P	3	7	P	3	P	59
03	3	P	P	3	23	107	3	7	17	3	11	13	3	P	7	3	P	P	3	P
07	P	P	3	109	P	3	11	31	3	P	17	3	7	P	3	23	29	3	131	7
09	3	P	43	3	7	P	3	P	P	3	89	7	3	P	P	3	P	47	3	13
11	13	3	11	7	3	127	P	3	43	29	3	P	17	3	P	97	3	7	53	3
13	11	P	3	41	P	3	151	13	3	7	P	3	19	17	3	31	7	3	83	P
17	7	3	61	P	3	P	103	3	13	P	3	P	151	3	7	17	3	P	11	3
19	P	89	3	83	P	3	7	19	3	P	127	3	P	7	3	13	11	3	P	P
21	3	P	53	3	P	7	3	59	P	3	131	P	3	P	11	3	P	17	3	7
23	P	3	P	13	3	137	P	3	103	P	3	7	11	3	P	P	3	29	7	3
27	3	23	7	3	13	P	3	79	11	3	29	P	3	19	47	3	7	13	3	11
29	P	3	P	P	3	19	11	3	97	3	13	P	3	59	7	3	11	23	3	3
31	7	59	3	29	11	3	P	7	3	107	P	3	23	73	3	11	19	3	13	P
33	3	P	11	3	53	P	3	P	19	3	P	41	3	7	29	3	P	P	3	P
37	13	P	3	P	7	3	71	29	3	11	P	3	P	13	3	P	31	3	7	37
39	3	101	P	3	P	53	3	11	59	3	7	23	3	P	P	3	P	7	3	P
41	29	3	7	101	3	11	41	3	P	7	3	31	43	3	13	P	3	P	P	3
43	P	7	3	11	P	3	19	109	3	P	79	3	P	P	3	7	P	3	43	P
47	139	3	P	97	3	P	7	3	P	13	3	P	P	3	P	59	3	P	P	3
49	P	19	3	13	23	3	157	P	3	61	37	3	7	P	3	29	13	3	P	7
51	3	P	P	3	7	P	3	53	P	3	13	7	3	101	31	3	113	11	3	P
53	67	3	79	7	3	43	89	3	29	P	3	P	3	P	11	3	7	103	3	3
57	3	7	127	3	37	13	3	19	7	3	P	11	3	P	P	3	P	43	3	101
59	7	3	17	P	3	41	P	3	P	11	3	139	13	3	7	61	3	P	19	3
61	P	37	3	17	61	3	7	11	3	109	19	3	P	7	3	P	67	3	11	13
63	3	73	19	3	17	7	3	P	23	3	71	P	3	13	P	3	11	P	3	7
67	41	11	3	7	43	3	17	P	3	P	7	3	11	P	3	37	P	3	P	23
69	3	P	7	3	P	79	3	17	13	3	11	P	3	23	P	3	7	73	3	P
71	P	3	13	P	3	P	P	3	7	P	3	P	37	3	P	7	3	P	41	3
73	7	23	3	P	P	3	11	7	3	13	P	3	127	P	3	107	P	3	P	19
77	P	3	11	19	3	7	P	3	P	P	3	17	7	3	73	P	3	149	113	3
79	11	P	3	P	7	3	23	71	3	P	31	3	17	41	3	P	P	3	7	83
81	3	P	P	3	P	47	3	P	139	3	7	13	3	17	83	3	61	7	3	P
83	P	3	7	37	3	13	P	3	149	7	3	P	131	3	17	P	3	19	11	3
87	3	19	149	3	47	23	3	7	41	3	P	89	3	53	7	3	17	107	3	13
89	13	3	107	29	3	67	7	3	P	P	3	P	11	3	71	P	3	17	P	3
91	P	17	3	P	19	3	P	13	3	67	11	3	7	P	3	157	23	3	17	7
93	3	13	17	3	7	P	3	P	11	3	23	7	3	67	13	3	P	P	3	11
97	P	P	3	31	11	3	P	137	3	7	P	3	41	109	3	11	7	3	19	P
99	3	7	11	3	P	17	3	P	7	3	19	113	3	11	43	3	31	P	3	P

	260	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279
01	3	43	7	3	17	P	3	P	P	3	13	41	3	23	11	3	7	P	3	P
03	P	3	P	29	3	17	37	3	7	P	3	P	11	3	67	7	3	13	P	3
07	3	P	73	3	P	13	3	17	11	3	113	P	3	7	P	3	19	103	3	11
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13	3	P	11	3	61	P	3	P	P	3	7	19	3	11	79	3	53	7	3	103
17	P	7	3	P	P	3	43	P	3	11	P	3	17	59	3	7	P	3	P	P
19	3	P	157	3	29	23	3	7	13	3	41	47	3	17	7	3	71	53	3	P
21	P	3	13	P	3	11	7	3	P	P	3	37	163	3	17	13	3	19	43	3
23	53	151	3	11	P	3	79	P	3	13	61	3	7	89	3	17	23	3	P	7
27	17	3	P	7	3	41	P	3	139	P	3	P	19	3	P	P	3	7	P	3
29	P	17	3	113	13	3	31	P	3	7	151	3	73	P	3	P	7	3	17	11
31	3	7	17	3	P	43	3	P	7	3	P	13	3	151	P	3	P	11	3	17
33	7	3	37	17	3	13	P	3	P	23	3	43	113	3	7	11	3	P	13	3
37	3	59	P	3	P	7	3	P	47	3	19	11	3	P	P	3	29	P	3	7
39	13	3	19	P	3	P	17	3	P	11	3	7	P	3	23	P	3	P	7	3
41	P	P	3	7	137	3	P	11	3	29	7	3	P	19	3	P	131	3	11	P
43	3	13	7	3	31	11	3	47	17	3	P	P	3	37	13	3	7	P	3	P
47	7	11	3	P	53	3	P	7	3	P	17	3	11	23	3	13	P	3	P	P
49	3	79	P	3	P	139	3	23	P	3	11	17	3	7	P	3	43	P	3	19
51	109	3	P	13	3	7	29	3	11	P	3	19	7	3	97	P	3	P	P	3
53	P	P	3	19	7	3	11	31	3	P	13	3	P	17	3	59	P	3	7	P
57	71	3	7	P	3	P	19	3	107	7	3	13	97	3	P	17	3	41	89	3
59	11	7	3	43	P	3	53	P	3	P	P	3	P	109	3	7	17	3	13	73
61	3	P	P	3	47	P	3	7	P	3	P	157	3	P	7	3	139	17	3	P
63	67	3	P	41	3	101	7	3	P	59	3	23	137	3	29	43	3	P	11	3
67	3	137	P	3	7	31	3	13	67	3	P	7	3	P	11	3	73	P	3	P
69	131	3	109	7	3	163	P	3	97	149	3	101	11	3	13	19	3	7	29	3
71	29	P	3	P	103	3	149	19	3	7	11	3	P	101	3	79	7	3	47	83
73	3	7	13	3	23	P	3	41	7	3	P	29	3	31	83	3	P	P	3	11
77	89	P	3	13	11	3	7	P	3	53	P	3	P	7	3	11	13	3	61	101
79	3	47	11	3	F	7	3	61	P	3	13	P	3	11	P	3	89	P	3	7
81	11	3	41	23	3	19	P	3	P	P	3	7	P	3	P	P	3	13	7	3
83	P	P	3	7	71	3	P	P	3	11	7	3	P	139	3	P	19	3	P	P
87	19	3	97	P	3	11	P	3	7	P	3	31	13	3	P	7	3	37	79	3
89	7	P	3	11	P	3	13	7	3	137	103	3	29	61	3	47	P	3	167	13
91	3	11	61	3	59	P	3	73	P	3	P	P	3	7	37	3	P	P	3	23
93	97	3	P	P	3	7	P	3	P	3	P	3	71	7	3	19	41	3	P	3
97	3	17	P	3	P	P	3	127	13	3	7	P	3	P	31	3	P	7	3	P
99	P	3	7	P	3	67	P	3	37	7	3	59	P	3	107	11	3	P	23	3

	280	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299
01	P	3	P	7	3	11	37	3	83	P	3	P	P	3	P	P	3	7	17	3
03	41	157	3	11	P	3	P	P	3	7	13	3	19	P	3	163	7	3	P	17
07	7	3	67	P	3	29	P	3	P	137	3	13	P	3	7	19	3	61	41	3
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13	109	3	89	23	3	P	13	3	P	29	3	7	131	3	67	11	3	43	7	3
17	3	31	7	3	157	P	3	13	P	3	P	11	3	19	23	3	7	P	3	P
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21	7	61	3	127	97	3	P	7	3	P	P	3	P	109	3	53	19	3	11	P
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27	P	11	3	13	7	3	P	23	3	P	P	3	11	P	3	P	13	3	7	P
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31	P	3	7	41	3	103	P	3	11	7	3	P	P	3	19	P	3	13	23	3
33	17	7	3	29	P	3	11	59	3	P	P	3	23	P	3	7	P	3	P	37
37	23	3	11	43	3	P	7	3	P	19	3	P	13	3	P	P	3	131	P	3
39	11	19	3	17	P	3	13	29	3	43	71	3	7	P	3	109	107	3	53	7
41	3	107	31	3	7	P	3	41	151	3	113	7	3	13	59	3	P	P	3	79
43	29	3	61	7	3	17	P	3	P	103	3	151	P	3	P	31	3	7	11	3
47	3	7	47	3	P	P	3	17	7	3	31	P	3	P	11	3	23	151	3	P
49	7	3	13	P	3	P	P	3	17	P	3	103	11	3	7	13	3	71	19	3
51	P	P	3	P	23	3	7	P	3	13	11	3	P	7	3	29	149	3	P	61
53	3	47	19	3	37	7	3	P	11	3	17	P	3	149	P	3	13	P	3	7
57	P	37	3	7	11	3	P	149	3	23	7	3	17	31	3	11	47	3	73	29
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63	7	P	3	113	P	3	P	7	3	11	P	3	13	P	3	17	P	3	P	19
67	13	3	23	19	3	7	109	3	P	83	3	P	7	3	79	P	3	17	P	3
69	P	17	3	41	7	3	P	13	3	59	41	3	P	43	3	P	P	3	7	23
71	3	11	17	3	71	P	3	P	P	3	7	31	3	23	13	3	P	7	3	17
73	67	3	7	17	3	P	53	3	13	7	3	P	73	3	P	P	3	19	P	3
77	3	19	P	3	P	17	3	7	67	3	P	163	3	29	7	3	59	11	3	31
79	43	3	P	13	3	P	7	3	P	P	3	P	19	3	41	11	3	97	P	3
81	P	P	3	101	19	3	23	17	3	73	13	3	7	11	3	P	67	3	P	7
83	3	P	P	3	7	101	3	107	17	3	127	7	3	P	P	3	P	13	3	P
87	P	71	3	P	61	3	P	11	3	7	17	3	P	3	P	3	7	3	11	157
89	3	7	P	3	31	11	3	P	7	3	19	17	3	P	37	3	11	P	3	P
91	7	3	19	11	3	P	13	3	167	53	3	P	17	3	7	127	3	31	71	3
93	13	11	3	P	P	3	7	P	3	79	47	3	11	7	3	101	23	3	167	89
97	P	3	P	73	3	P	P	3	11	107	3	7	P	3	13	17	3	83	7	3
99	P	163	3	7	P	3	11	31	3	47	7	3	83	P	3	P	17	3	29	131

	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319
01	19	31	3	157	7	3	71	11	3	13	29	3	41	113	3	17	P	3	7	19
03	3	P	P	3	P	11	3	P	P	3	7	19	3	23	31	3	11	7	3	61
07	37	7	3	P	13	3	127	P	3	31	101	3	11	P	3	7	P	3	17	P
09	3	P	17	3	47	P	3	7	P	3	11	13	3	131	7	3	73	37	3	17
11	P	3	P	17	3	13	7	3	11	P	3	53	23	3	101	P	3	19	13	3
13	P	P	3	P	17	3	11	P	3	19	P	3	7	173	3	101	3	29	7	
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19	11	P	3	P	19	3	67	13	3	7	P	3	P	P	3	43	7	3	47	59
21	3	7	47	3	29	23	3	31	7	3	67	P	3	P	13	3	103	P	3	137
23	7	3	P	P	3	131	113	3	13	17	3	P	P	3	7	29	3	P	11	3
27	3	47	167	3	P	7	3	29	3	19	17	3	P	11	3	P	3	P	3	7
29	P	3	19	13	3	P	109	3	P	157	3	7	11	3	53	41	3	P	7	3
31	59	29	3	7	P	3	P	79	3	P	7	3	P	17	3	P	47	3	139	37
33	3	P	7	3	13	19	3	73	11	3	P	163	3	P	17	3	7	13	3	11
37	7	P	3	23	11	3	P	7	3	P	41	3	P	P	3	11	17	3	13	109
39	3	P	11	3	61	P	3	59	P	3	P	P	3	7	149	3	29	17	3	19
41	11	3	P	P	3	7	13	3	P	P	3	11	7	3	23	P	3	P	17	3
43	13	43	3	19	7	3	P	71	3	11	37	3	157	13	3	P	P	3	7	17
47	P	3	7	P	3	11	19	3	109	7	3	P	P	3	13	P	3	53	P	3
49	151	7	3	11	P	3	P	97	3	P	61	3	P	23	3	7	P	3	P	43
51	3	11	13	3	37	137	3	7	P	3	P	P	3	107	7	3	31	P	3	89
53	41	3	P	127	3	P	7	3	P	13	3	P	P	3	71	139	3	113	53	3
57	3	53	79	3	7	P	3	P	59	3	13	7	3	P	83	3	P	11	3	P
59	P	3	P	7	3	P	23	3	P	83	3	P	3	163	11	3	7	P	3	
61	23	P	3	97	83	3	P	19	3	7	89	3	43	11	3	37	7	3	151	31
63	3	7	53	3	41	13	3	P	7	3	P	11	3	79	73	3	P	23	3	P
67	107	97	3	P	P	3	7	11	3	173	47	3	P	7	3	P	P	3	11	13
69	3	P	P	3	P	7	3	29	P	3	P	71	3	13	P	3	11	P	3	7
71	P	3	P	11	3	19	P	3	P	P	3	7	P	3	11	131	3	P	7	3
73	17	11	3	7	31	3	37	P	3	47	7	3	11	137	3	P	19	3	P	P
77	19	3	13	37	3	P	P	3	7	P	3	P	P	3	P	7	3	43	127	3
79	7	193	3	17	29	3	11	7	3	13	P	3	31	P	3	23	79	3	71	113
81	3	P	107	3	11	53	3	P	P	3	P	P	3	7	P	3	13	61	3	P
83	67	3	11	23	3	7	61	3	89	P	3	P	7	3	19	P	3	37	P	3
87	3	P	31	3	43	73	3	17	67	3	7	13	3	P	23	3	P	7	3	29
89	P	3	7	P	3	13	P	3	17	7	3	P	67	3	P	31	3	83	11	3
91	P	7	3	P	P	3	47	41	3	17	P	3	13	P	3	7	11	3	P	P
93	3	109	P	3	P	P	3	P	7	3	17	P	3	P	P	7	3	41	P	13
97	P	P	3	113	P	3	P	13	3	139	11	3	7	P	3	19	29	3	167	7
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	320	321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339
61	3	47	13	3	P	7	3	53	P	3	61	79	3	P	127	3	P	67	3	7
63	P	3	P	P	3	P	P	3	P	13	3	7	P	3	P	P	3	P	7	3
67	3	97	7	3	23	P	3	P	53	3	13	P	3	19	11	3	7	37	3	41
69	P	3	31	P	3	19	P	3	7	P	3	113	11	3	P	7	3	13	P	3
71	7	163	3	79	P	3	P	7	3	P	11	3	P	P	3	23	19	3	P	P
73	3	17	P	3	P	13	3	P	11	3	P	P	3	7	P	3	P	P	3	11
77	101	P	3	17	7	3	13	P	3	P	137	3	59	P	3	11	P	P	3	7
79	3	P	11	3	17	31	3	P	37	3	7	P	3	11	23	3	P	7	3	107
81	11	3	7	P	3	17	P	3	23	7	3	11	139	3	19	P	3	P	31	3
83	31	7	3	P	P	3	17	43	3	11	P	3	P	47	3	7	P	3	149	P
87	P	3	13	P	3	11	7	3	17	19	3	157	149	3	P	13	3	29	P	3
89	P	19	3	11	P	3	67	23	3	13	P	3	7	P	3	P	P	3	P	7
91	3	11	16	3	7	P	3	71	P	3	17	7	3	P	101	3	13	89	3	P
93	103	3	P	7	3	P	P	3	P	P	3	17	167	3	67	P	3	7	23	3
97	3	7	P	3	163	P	3	19	7	3	P	13	3	17	29	3	P	11	3	P
99	7	3	103	73	3	13	127	3	P	P	3	31	43	3	7	11	3	P	13	3
41	179	P	3	P	P	3	7	29	3	P	19	3	13	7	3	17	P	3	43	P
43	3	P	19	3	P	7	3	137	P	3	173	11	3	P	53	3	17	41	3	7
47	73	17	3	7	71	3	P	11	3	47	7	3	P	P	3	P	P	3	11	83
49	3	13	7	3	37	11	3	P	107	3	P	P	3	P	13	3	7	P	3	17
51	P	3	P	11	3	43	103	3	7	83	3	P	41	3	11	7	3	P	P	3
53	7	11	3	P	17	3	P	7	3	31	P	3	11	P	3	13	73	3	97	19
57	P	3	P	13	3	7	17	3	11	P	3	71	7	3	P	23	3	P	P	3
59	P	P	3	P	7	3	11	17	3	23	13	3	79	P	3	37	97	3	7	29
61	3	29	P	3	11	P	3	181	17	3	7	P	3	73	P	3	41	7	3	P
63	P	3	7	P	3	P	89	3	59	7	3	13	29	3	109	P	3	19	P	3
67	3	19	41	3	P	29	3	7	23	3	43	17	3	61	7	3	131	P	3	P
69	P	3	23	P	3	P	7	3	P	P	3	41	17	3	P	P	3	P	11	3
71	13	53	3	P	19	3	37	P	3	P	P	3	7	13	3	59	11	3	P	7
73	3	P	59	3	7	P	3	13	71	3	P	7	3	23	11	3	151	P	3	53
77	P	23	3	P	47	3	41	73	3	7	11	3	107	P	3	P	7	3	19	61
79	3	7	13	3	P	P	3	P	7	3	19	P	3	29	P	3	P	17	3	11
81	7	3	19	P	3	31	11	3	131	13	3	P	23	3	7	P	3	11	17	3
83	P	P	3	13	11	3	7	P	3	P	P	3	83	7	3	11	13	3	31	17
87	11	3	83	139	3	P	P	3	P	P	3	7	P	3	P	P	3	13	7	3
89	P	P	3	7	53	3	97	P	3	11	7	3	P	173	3	P	59	3	P	41
91	3	P	7	3	P	13	3	11	31	3	P	P	3	P	107	3	7	P	3	19
93	67	3	43	29	3	11	P	3	7	P	3	19	13	3	P	7	3	47	P	3
97	3	11	P	3	37	3	P	67	3	23	89	3	7	19	3	31	P	3	P	P
99	P	3	P	179	3	7	19	3	167	P	3	P	7	3	139	P	3	73	109	3

	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359
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03	37	67	3	P	P	3	P	P	3	11	17	3	7	43	3	13	P	3	P	7
07	31	3	79	7	3	11	P	3	P	67	3	P	17	3	P	P	3	7	61	3
09	71	23	3	11	19	3	53	61	3	7	13	3	137	17	3	P	7	3	P	149
11	3	7	P	3	13	P	3	103	7	3	157	P	3	P	17	3	149	13	3	P
13	7	3	P	P	3	P	P	3	31	P	3	13	23	3	7	17	3	71	59	3
17	3	109	P	3	127	7	3	149	37	3	19	P	3	P	107	3	P	11	3	7
19	P	3	19	P	3	P	13	3	P	P	3	7	41	3	P	11	3	23	7	3
21	13	149	3	7	P	3	89	P	3	47	7	3	P	11	3	P	179	3	113	17
23	3	P	7	3	29	19	3	13	97	3	P	11	3	P	P	3	7	139	3	P
27	7	P	3	P	173	3	31	7	3	53	P	3	P	P	3	P	23	3	11	37
29	3	P	13	3	P	11	3	P	29	3	23	P	3	7	71	3	11	P	3	19
31	P	3	P	11	3	7	P	3	61	15	3	19	7	3	11	P	3	P	P	3
33	P	11	3	13	7	3	59	47	3	181	53	3	11	89	3	P	13	3	7	P
37	101	3	7	P	3	P	19	3	11	7	3	41	167	3	P	P	3	13	P	3
39	P	7	3	23	P	3	11	P	3	P	37	3	131	P	3	7	157	3	P	83
41	3	P	97	3	11	13	3	7	P	3	67	P	3	59	7	3	29	103	3	127
43	59	3	11	61	3	P	7	3	P	83	3	113	13	3	23	P	3	31	73	3
47	3	P	23	3	7	179	3	P	P	3	101	7	3	13	P	3	43	P	3	103
49	79	3	29	7	3	P	P	3	P	P	3	P	101	3	P	19	3	7	11	3
51	17	13	3	P	47	3	P	19	3	7	P	3	P	23	3	73	7	3	P	P
53	3	7	P	3	131	109	3	23	7	3	P	P	3	P	11	3	101	P	3	157
57	P	P	3	17	P	3	7	P	3	13	11	3	P	7	3	31	181	3	23	41
59	3	P	P	3	17	7	3	P	11	3	P	P	3	19	59	3	13	P	3	7
61	P	3	P	P	3	17	11	3	71	P	3	7	37	3	P	43	3	11	7	3
63	23	127	3	7	11	3	17	P	3	P	7	3	179	P	3	11	19	3	P	P
67	11	3	P	P	3	13	P	3	7	73	3	11	P	3	29	7	3	47	13	3
69	7	47	3	P	P	3	37	7	3	11	P	3	13	113	3	P	53	3	P	P
71	3	P	43	3	P	181	3	11	P	3	17	P	3	7	79	3	P	P	3	13
73	13	3	P	37	3	7	P	3	43	41	3	17	7	3	19	P	3	83	29	3
77	3	11	151	3	23	71	3	83	P	3	7	29	3	17	13	3	P	7	3	P
79	53	3	7	31	3	151	P	3	13	7	3	127	P	3	17	47	3	37	P	3
81	173	7	3	P	29	3	79	P	3	P	P	3	P	P	3	7	31	3	53	11
83	3	P	P	3	P	P	3	7	P	3	P	151	3	41	7	3	17	11	3	P
87	89	17	3	137	P	3	P	43	3	59	13	3	7	11	3	19	127	3	17	7
89	3	179	17	3	7	P	3	19	139	3	P	7	3	43	23	3	89	13	3	17
91	73	3	53	7	3	P	113	3	23	11	3	13	P	3	P	P	3	7	19	3
93	103	31	3	163	17	3	11	3	11	3	19	3	29	P	P	P	7	3	11	P
97	7	3	P	11	3	29	13	3	P	79	3	61	47	3	7	P	3	P	P	3
99	13	11	3	41	P	3	7	17	3	31	P	3	11	7	3	97	29	3	P	P

	360	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379
01	7	13	3	31	89	3	17	7	3	P	163	3	P	11	3	P	19	3	103	151
03	3	79	41	3	59	173	3	17	13	3	P	11	3	7	113	3	31	37	3	29
07	P	P	3	P	7	3	P	11	3	13	23	3	29	P	3	P	P	3	7	P
09	3	P	P	3	23	11	3	P	P	3	7	43	3	P	P	3	11	7	3	167
11	P	3	7	11	3	29	31	3	131	7	3	17	127	3	11	P	3	43	P	3
13	P	7	3	P	13	3	19	P	3	P	3	11	P	3	7	49	3	P	3	1
17	P	3	P	23	3	13	7	3	11	19	3	P	P	3	17	P	3	F	13	3
19	181	19	3	P	79	3	11	73	3	P	P	3	7	67	3	17	P	3	59	7
21	3	41	29	3	7	59	3	P	P	3	P	7	3	P	23	3	17	67	3	13
23	13	3	11	7	3	P	53	3	23	P	3	P	P	3	P	157	3	7	109	3
27	3	7	17	3	73	P	3	19	7	3	61	137	3	163	13	3	191	31	3	17
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31	137	P	3	47	17	3	7	23	3	P	19	3	31	7	3	13	11	3	P	83
33	3	23	19	3	P	7	3	109	P	3	29	71	3	37	11	3	P	97	3	7
37	P	P	3	7	83	3	P	17	3	43	7	3	23	P	3	P	61	3	157	59
39	3	71	7	3	13	61	3	P	11	3	P	P	3	P	29	3	7	13	3	11
41	23	3	P	P	3	P	11	3	7	17	3	13	167	3	P	7	3	11	79	3
43	7	47	3	P	11	3	P	7	3	P	17	3	P	107	3	11	P	3	13	19
47	11	3	67	19	3	7	13	3	P	3	11	7	3	P	3	P	3	P	P	3
49	13	37	3	163	7	3	67	P	3	11	P	3	193	13	3	P	P	3	7	137
51	3	P	P	3	P	3	11	43	3	7	97	3	41	17	3	23	7	3	P	P
53	31	3	7	P	3	11	P	3	137	7	3	53	P	3	13	17	3	19	P	3
57	3	11	13	3	P	139	3	7	P	3	P	73	3	P	7	3	P	17	3	P
59	107	3	101	103	3	P	7	3	29	13	3	P	19	3	47	23	3	61	17	3
61	P	P	3	13	19	3	61	P	3	23	P	3	7	P	3	P	13	3	P	7
63	3	29	P	3	7	P	3	97	191	3	13	7	3	P	P	3	P	11	3	P
67	P	59	3	41	P	3	37	P	3	7	101	3	83	11	3	P	7	3	19	P
69	3	7	P	3	P	13	3	83	7	3	19	11	3	P	89	3	139	179	3	43
71	7	3	19	37	3	P	P	3	P	11	3	P	13	3	7	P	3	107	P	3
73	P	61	3	P	P	3	7	11	3	P	131	3	P	7	3	P	101	3	11	13
77	43	3	P	11	3	79	P	3	P	103	3	7	P	3	11	53	3	37	7	3
79	109	11	3	7	P	3	43	P	3	P	7	3	11	P	3	P	41	3	P	163
81	3	97	7	3	191	157	3	P	13	3	11	P	3	29	37	3	7	P	3	19
83	P	3	13	P	3	P	P	3	7	31	3	19	23	3	P	7	3	P	43	3
87	3	P	131	3	11	P	3	P	3	P	3	41	3	7	19	3	13	29	3	P
89	151	3	11	P	3	7	19	3	37	47	3	P	7	3	P	P	3	23	P	3
91	11	P	3	151	7	3	P	P	3	71	29	3	89	139	3	P	P	3	7	P
93	3	17	P	3	P	23	3	P	79	3	7	13	3	61	P	3	P	7	3	P
97	P	7	3	17	P	3	P	31	3	P	P	3	13	P	3	7	11	3	P	P
99	3	53	P	3	17	P	3	7	P	3	23	P	3	149	7	3	P	P	3	13

	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399
01	3	7	P	3	11	P	3	13	7	3	43	61	3	P	31	3	199	29	3	P
03	7	3	11	P	3	139	P	3	P	P	3	P	197	3	7	P	3	P	53	3
07	3	53	13	3	193	7	3	P	151	3	19	P	3	23	157	3	P	59	3	7
09	191	3	19	29	3	97	P	3	197	13	3	7	P	3	P	P	3	P	7	3
11	P	23	3	7	71	3	P	P	3	167	7	3	113	19	3	P	11	3	41	107
13	3	P	7	3	107	19	3	P	37	3	13	P	3	P	11	3	7	151	3	167
17	7	47	3	P	41	3	23	7	3	P	11	3	P	P	3	43	173	3	29	179
19	3	P	P	3	103	13	3	31	11	3	P	P	3	7	P	3	P	P	3	11
21	193	3	37	P	3	7	11	3	P	P	3	19	7	3	79	P	3	11	P	3
23	47	67	3	19	7	3	13	P	3	P	3	61	P	3	11	P	3	7	13	
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29	17	7	3	P	83	3	P	P	3	11	31	3	P	67	3	7	23	3	P	P
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33	73	3	13	P	3	11	7	3	P	P	3	P	P	3	47	13	3	P	61	3
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39	P	3	P	7	3	17	P	3	P	23	3	P	P	3	P	19	3	7	P	3
41	109	43	3	23	13	3	17	19	3	7	P	3	P	P	3	P	7	3	P	11
43	3	7	167	3	37	P	3	17	7	3	P	13	3	P	P	3	29	11	3	59
47	P	37	3	31	P	3	7	P	3	17	P	3	13	7	3	71	41	3	P	43
49	3	P	23	3	P	7	3	P	53	3	17	11	3	19	103	3	31	P	3	7
51	18	3	29	P	3	19	P	3	P	11	3	7	P	3	P	P	3	127	7	3
53	P	P	3	7	P	3	P	11	3	P	7	3	17	23	3	37	19	3	11	P
57	19	3	67	11	3	P	29	3	7	163	3	P	37	3	11	7	3	83	P	3
59	7	11	3	89	P	3	67	7	3	P	139	3	11	P	3	13	P	3	23	31
61	3	31	P	3	P	P	3	83	P	P	3	11	P	3	7	P	3	17	P	3
63	17	3	83	13	3	7	23	3	11	47	3	P	7	3	19	P	3	17	P	3
67	3	P	17	3	11	P	3	P	P	3	7	53	3	P	61	3	P	7	3	17
69	P	3	7	17	3	P	P	3	47	7	3	13	107	3	29	P	3	P	P	3
71	11	7	3	P	17	3	P	137	3	P	89	3	173	P	3	7	P	3	13	P
73	3	59	P	3	79	17	3	7	P	3	41	43	3	P	7	3	97	31	3	71
77	13	P	3	P	109	3	P	17	3	P	23	3	7	13	3	19	11	3	P	7
79	3	73	101	3	7	173	3	13	17	3	P	7	3	53	11	3	P	P	3	P
81	113	3	P	7	3	41	47	3	59	17	3	P	11	3	13	P	3	7	19	3
83	P	P	3	131	29	3	101	P	3	7	11	3	163	P	3	23	7	3	P	P
87	7	3	P	23	3	47	11	3	37	13	3	149	17	3	7	31	3	11	P	3
89	41	P	3	13	11	3	7	79	3	127	P	3	101	7	3	11	13	3	113	P
91	3	181	11	3	61	7	3	P	P	3	13	P	3	11	17	3	19	P	3	7
93	11	3	149	P	3	P	3	P	3	19	P	3	7	P	3	73	17	3	13	7
97	3	P	7	3	137	13	3	11	97	3	P	19	3	P	127	3	7	17	3	23
99	31	3	P	19	3	11	P	3	7	59	3	P	13	3	P	7	3	P	17	3

	400	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419
01	13	3	7	19	3	101	11	3	P	7	3	23	P	3	19	47	3	11	P	3
03	109	7	3	41	11	3	19	13	3	P	131	3	P	103	3	7	P	3	17	P
07	11	3	31	17	3	P	7	3	13	19	3	11	89	3	47	P	3	179	97	3
09	P	19	3	173	17	3	P	P	3	11	23	3	7	101	3	13	P	3	P	7
11	3	P	79	3	7	17	3	11	37	3	P	7	3	109	P	3	F	53	3	P
13	P	3	P	7	3	11	71	3	P	163	3	P	P	3	P	P	3	7	P	3
17	3	7	131	3	13	31	3	19	7	3	P	P	3	79	83	3	P	13	3	167
19	7	3	37	23	3	P	151	3	P	17	3	13	47	3	7	P	3	P	19	3
21	31	53	3	61	83	3	7	43	3	151	17	3	F	7	3	P	P	3	13	11
23	3	P	19	3	P	7	3	193	P	3	P	17	3	31	23	3	107	11	3	7
27	13	P	3	7	P	3	P	139	3	P	7	3	P	11	3	131	P	3	151	P
29	3	P	7	3	P	P	3	13	P	3	89	11	3	37	17	3	7	P	3	23
31	P	3	P	31	3	P	41	3	7	11	3	P	P	3	13	7	3	29	59	3
33	7	67	3	53	P	3	179	7	3	P	37	3	P	P	3	41	17	3	11	19
37	P	3	P	11	3	7	P	3	97	13	3	31	7	3	11	73	3	P	17	3
39	P	11	3	13	7	3	P	P	3	P	P	3	11	67	3	P	13	3	7	17
41	3	137	P	3	37	71	3	131	P	3	7	P	3	P	29	3	P	7	3	P
43	23	3	P	P	3	P	97	3	11	7	3	P	P	3	P	P	3	13	P	3
47	3	19	167	3	11	13	3	7	P	3	P	23	3	173	7	3	P	109	3	P
49	29	3	11	157	3	23	7	3	P	P	3	P	13	3	181	P	3	83	P	3
51	11	P	3	P	19	3	13	P	3	31	P	3	7	P	3	37	P	3	P	7
53	3	P	P	3	7	107	3	83	P	3	61	7	3	13	P	3	23	43	3	P
57	41	13	3	P	23	3	109	53	3	7	P	3	P	P	3	29	7	3	19	P
59	3	7	127	3	P	P	3	P	7	3	19	79	3	59	11	3	P	P	3	P
61	7	3	13	P	3	47	73	3	29	P	3	P	11	3	7	13	3	P	41	3
63	P	P	3	181	43	3	7	P	3	13	11	3	F	7	3	89	61	3	P	29
67	103	3	67	37	3	113	11	3	P	71	3	7	29	3	P	197	3	11	7	3
69	17	P	3	7	11	3	67	59	3	53	7	3	P	41	3	11	P	3	149	P
71	3	17	7	3	P	29	3	P	23	3	67	13	3	11	113	3	7	P	3	19
73	11	3	17	47	3	13	89	3	7	P	3	11	149	3	67	7	3	37	13	3
77	3	P	P	3	17	P	3	11	41	3	P	P	3	7	19	3	71	P	3	13
79	13	3	47	149	3	7	19	3	P	43	3	P	7	3	P	P	3	41	P	3
81	149	23	3	11	7	3	17	13	3	107	P	3	P	P	3	43	P	3	7	P
83	3	11	P	3	P	P	3	17	P	3	7	P	3	29	13	3	73	7	3	P
87	P	7	3	P	P	3	23	P	3	17	181	3	19	P	3	7	P	3	P	11
89	3	P	P	3	19	37	3	7	31	3	17	P	3	P	7	3	47	11	3	199
91	47	3	43	13	P	3	7	3	103	179	3	17	157	3	P	11	3	23	163	3
93	P	P	3	31	3	P	P	19	3	P	13	3	7	11	3	P	173	3	P	7
97	101	3	59	7	P	3	P	3	P	3	13	3	61	3	17	P	3	7	P	3
99	P	61	3	71	P	3	P	11	3	7	73	3	P	P	3	17	7	3	11	P

	420	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439
01	97	P	3	7	109	3	13	P	3	P	7	3	P	19	3	41	59	3	P	11
03	3	71	7	3	P	19	3	P	23	3	P	P	3	13	P	3	7	11	3	43
07	7	13	3	P	P	3	137	7	3	107	29	3	P	11	3	139	P	3	71	23
09	3	17	P	3	P	P	3	P	13	3	41	11	3	7	83	3	P	109	3	19
11	43	3	13	29	3	7	P	3	31	11	3	19	7	3	P	13	3	P	193	3
13	P	23	3	17	7	3	43	11	3	13	P	3	79	P	3	53	P	3	7	P
17	P	3	7	11	3	17	19	3	47	7	3	P	23	3	11	P	3	P	43	3
19	P	7	3	101	13	3	17	P	3	167	P	3	11	P	3	7	53	3	29	37
21	3	73	P	3	59	101	3	7	P	3	11	13	3	P	7	3	181	P	3	107
23	P	3	P	P	3	13	7	3	11	P	3	29	P	3	173	71	3	23	13	3
27	3	103	P	3	7	23	3	P	113	3	17	7	3	37	P	3	P	73	3	13
29	13	3	11	7	3	71	47	3	P	P	3	17	139	3	137	19	3	7	41	3
31	11	P	3	P	151	3	89	13	3	7	37	3	17	P	3	101	7	3	53	197
33	3	7	157	3	P	P	3	151	7	3	23	P	3	17	13	3	P	101	3	P
37	127	29	3	P	P	3	7	P	3	P	P	3	P	7	3	13	11	3	59	53
39	3	P	P	3	31	7	3	79	P	3	193	179	3	19	11	3	17	191	3	7
41	17	3	53	13	3	19	P	3	P	23	3	7	11	3	P	P	3	17	7	3
43	P	17	3	7	P	3	P	P	3	P	7	3	83	89	3	P	19	3	17	P
47	19	3	83	17	3	157	11	3	7	67	3	13	59	3	23	7	3	11	163	3
49	7	113	3	P	11	3	P	7	3	29	P	3	61	67	3	11	P	3	13	71
51	3	61	11	3	P	17	3	P	73	3	P	P	3	7	P	3	P	67	3	P
53	11	3	29	41	3	7	13	3	P	3	11	7	3	19	97	3	P	3	P	3
57	3	P	P	3	P	P	3	11	17	3	7	103	3	191	P	3	149	7	3	113
59	137	3	7	P	3	11	29	3	P	7	3	P	181	3	13	43	3	P	61	3
61	P	7	3	11	P	3	37	61	3	P	17	3	P	131	3	7	P	3	23	P
63	3	11	13	3	P	31	3	7	P	3	P	17	3	103	7	3	47	107	3	P
67	23	149	3	13	P	3	P	P	3	P	P	3	7	17	3	19	13	3	P	7
69	3	P	43	3	7	P	3	19	163	3	13	7	3	31	17	3	P	11	3	P
71	P	3	41	7	3	P	71	3	43	97	3	23	P	3	29	11	3	7	19	3
73	P	181	3	P	P	3	139	P	3	7	19	3	109	11	3	P	7	3	73	P
77	7	3	67	31	3	P	P	3	53	11	3	P	13	3	7	P	3	P	17	3
79	29	P	3	P	107	3	7	11	3	P	23	3	113	7	3	P	31	3	11	13
81	3	P	P	3	23	7	3	179	137	3	67	29	3	13	P	3	11	P	3	7
83	P	3	P	11	3	97	P	3	19	53	3	7	P	3	11	41	3	P	7	3
87	3	P	7	3	1	37	P	3	13	3	11	19	3	43	P	3	7	P	3	P
89	P	3	13	19	3	P	P	3	7	P	3	P	73	3	157	7	3	P	P	3
91	7	31	3	P	P	3	11	7	3	13	41	3	P	P	3	P	P	3	P	P
93	3	P	P	3	11	191	3	P	59	3	P	47	3	7	23	3	13	P	3	29
97	11	P	3	P	7	3	P	P	3	19	71	3	29	P	3	P	37	3	7	P
99	3	19	P	3	P	41	3	127	P	3	7	13	3	P	P	3	89	7	3	23

	440	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459
01	3	P	P	3	7	P	3	P	71	3	11	7	3	89	83	3	31	23	3	197
03	79	3	P	7	3	191	13	3	11	83	3	23	17	3	P	P	3	7	163	3
07	3	7	P	3	11	P	3	13	7	3	P	43	3	P	17	3	59	P	3	29
09	7	3	11	59	3	47	31	3	P	P	3	79	53	3	7	17	3	43	19	3
11	11	P	3	73	89	3	7	P	3	97	19	3	29	7	3	71	17	3	61	31
13	3	31	13	3	23	7	3	61	41	3	P	197	3	113	P	3	P	17	3	7
17	P	157	3	7	P	3	P	97	3	P	7	3	103	P	3	23	11	3	P	17
19	3	P	7	3	43	P	3	197	P	3	13	P	3	P	11	3	7	131	3	47
21	P	3	P	23	3	211	P	3	7	29	3	P	11	3	53	7	3	13	P	3
23	7	P	3	127	31	3	P	7	3	167	11	3	41	61	3	P	43	3	P	19
27	P	3	47	19	3	7	11	3	23	P	3	P	7	3	P	53	3	11	P	3
29	P	P	3	97	7	3	13	P	3	179	37	3	31	P	3	11	109	3	7	13
31	3	P	11	3	157	P	3	41	127	3	7	P	3	11	181	3	P	7	3	23
33	11	3	7	43	3	P	P	3	107	7	3	11	P	3	P	P	3	19	P	3
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39	47	3	13	101	3	11	7	3	P	P	3	P	19	3	P	13	3	53	23	3
41	P	37	3	11	19	3	P	P	3	13	73	3	7	P	3	P	P	3	P	7
43	3	11	151	3	7	P	3	101	P	3	31	7	3	P	29	3	13	149	3	P
47	17	131	3	61	13	3	P	29	3	7	107	3	P	137	3	37	7	3	19	11
49	3	7	P	3	P	P	3	73	7	3	19	13	3	101	47	3	191	11	3	P
51	7	3	17	P	3	13	P	3	P	79	3	163	37	3	7	11	3	P	13	3
53	P	67	3	17	P	3	7	P	3	P	P	3	13	7	3	P	71	3	P	P
57	13	3	P	P	3	17	P	3	31	11	3	7	167	3	131	P	3	P	7	3
59	P	P	3	7	23	3	17	11	3	P	7	3	P	67	3	29	P	3	11	P
61	3	13	7	3	173	11	3	17	113	3	P	P	3	P	13	3	7	67	3	19
63	139	3	P	11	3	P	59	3	7	P	3	19	P	3	11	7	3	P	P	3
67	3	29	P	3	53	41	3	89	P	3	11	31	3	7	19	3	P	P	3	43
69	127	3	P	13	3	7	19	3	11	193	3	17	7	3	41	P	3	37	P	3
71	P	P	3	P	7	3	11	P	3	P	13	3	17	59	3	199	109	3	7	P
73	3	163	P	3	11	29	3	P	23	3	7	199	3	17	37	3	P	7	3	31
77	11	7	3	199	79	3	43	P	3	41	P	3	19	P	3	7	P	3	13	23
79	3	P	P	3	19	P	3	7	P	3	61	P	3	23	7	3	17	P	3	P
81	17	3	P	P	3	109	7	3	37	31	3	P	P	3	P	19	3	17	11	3
83	13	17	3	P	P	3	P	19	3	P	P	3	7	13	3	79	11	3	17	7
87	P	3	67	7	3	P	P	3	P	P	3	73	11	3	13	P	3	7	P	3
89	P	P	3	P	17	3	23	P	3	7	11	3	P	P	3	P	7	3	109	P
91	3	7	13	3	P	17	3	47	7	3	67	P	3	19	P	3	P	29	3	11
93	7	3	P	103	3	19	11	3	P	13	3	43	P	3	7	127	3	11	P	3
97	3	193	11	3	P	7	3	P	17	3	13	P	3	11	P	3	P	41	3	7
99	11	3	31	29	3	103	P	3	59	17	3	7	97	3	173	P	3	13	7	3

	460	461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479
01	157	3	47	P	3	7	P	3	17	P	3	19	7	3	107	P	3	P	13	3
03	179	P	3	19	7	3	29	P	3	17	11	3	13	P	3	67	181	3	7	P
07	13	3	7	P	3	P	11	3	P	7	3	17	P	3	P	P	3	11	P	3
09	139	7	3	P	11	3	127	13	3	61	29	3	17	P	3	7	P	3	P	23
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13	11	3	37	29	3	193	7	3	13	43	3	11	31	3	17	P	3	P	137	3
17	3	107	113	3	7	181	3	11	P	3	P	7	3	P	P	3	17	P	3	P
19	17	3	P	7	3	11	P	3	P	P	3	P	23	3	P	19	3	7	P	3
21	P	17	3	11	61	3	23	19	3	7	13	3	P	79	3	P	7	3	17	173
23	3	7	17	3	13	P	3	P	7	3	59	P	3	37	47	3	P	13	3	17
27	P	193	3	P	17	3	7	P	3	167	31	3	83	7	3	P	97	3	13	11
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33	13	P	3	7	59	3	P	17	3	P	7	3	149	11	3	P	19	3	31	P
37	19	3	P	P	3	173	149	3	7	11	3	P	P	3	13	7	3	P	P	3
39	7	29	3	149	P	3	P	7	3	73	17	3	97	P	3	137	P	3	11	P
41	3	P	13	3	P	11	3	43	31	3	P	17	3	7	P	3	11	P	3	191
43	41	3	131	11	3	7	P	3	139	13	3	P	7	3	11	P	3	P	P	3
47	3	P	103	3	P	89	3	P	79	3	7	P	3	113	17	3	29	7	3	P
49	P	3	7	P	3	P	P	3	11	7	3	P	37	3	23	17	3	13	59	3
51	P	7	3	P	P	3	11	P	3	29	P	3	P	P	3	7	17	3	109	P
53	3	P	23	3	11	13	3	7	P	3	211	61	3	P	7	3	P	17	3	79
57	11	101	3	151	P	3	13	P	3	P	P	3	7	23	3	19	P	3	P	7
59	3	31	167	3	7	P	3	19	47	3	P	7	3	13	P	3	P	163	3	197
61	P	3	P	7	3	101	29	3	P	151	3	P	167	3	31	199	3	7	11	3
63	73	13	3	71	97	3	P	101	3	7	19	3	151	P	3	P	7	3	23	P
67	7	3	13	199	3	P	23	3	P	67	3	101	11	3	7	13	3	37	151	3
69	23	137	3	89	31	3	7	P	3	13	11	3	P	7	3	P	73	3	P	1
71	3	P	P	3	P	7	3	P	11	3	103	43	3	127	37	3	13	23	3	7
73	P	3	P	79	3	P	11	3	19	107	3	7	41	3	29	113	3	11	7	3
77	3	61	7	3	P	47	3	29	P	3	179	13	3	11	107	3	7	P	3	P
79	11	3	P	19	3	13	P	3	7	109	3	11	P	3	79	7	3	P	13	3
81	7	P	3	P	53	3	P	7	3	11	23	3	13	P	3	P	P	3	P	P
83	3	P	31	3	23	37	3	11	173	3	197	29	3	7	103	3	41	7	3	13
87	17	P	3	11	7	3	P	13	3	19	P	3	P	3	23	43	3	7	47	3
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91	P	3	7	23	3	P	P	3	13	7	3	41	19	3	P	P	3	P	83	3
93	P	7	3	17	19	3	53	73	3	P	P	3	P	83	3	7	37	3	47	11
97	31	3	67	13	3	17	7	3	23	P	3	109	P	3	P	11	3	P	211	3
99	P	P	3	P	P	3	17	53	3	43	13	3	7	11	3	P	P	3	197	7

	480	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499
01	23	103	3	11	29	3	7	31	3	79	19	3	P	7	3	59	193	3	P	139
03	3	11	19	3	97	7	3	113	37	3	P	P	3	47	127	3	P	23	3	7
07	61	73	3	7	P	3	13	53	3	P	7	3	P	P	3	31	113	3	P	11
09	3	P	7	3	P	179	3	67	P	3	P	P	3	13	P	3	7	11	3	29
11	41	3	37	P	3	139	P	3	7	59	3	67	P	3	P	7	3	P	P	3
13	P	13	3	P	P	3	173	7	3	41	23	3	29	11	3	67	P	3	109	19
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19	31	P	3	211	7	3	P	11	3	13	P	3	83	149	3	23	29	3	7	P
21	3	P	P	3	41	11	3	83	P	3	7	P	3	31	73	3	11	7	3	P
23	P	3	7	11	3	P	P	3	P	7	3	P	P	3	11	P	3	19	P	3
27	3	17	29	3	79	P	3	7	157	3	11	13	3	107	7	3	P	P	3	P
29	P	3	17	31	3	13	7	3	11	113	3	73	19	3	P	P	3	223	13	3
31	43	P	3	17	19	3	11	P	3	167	P	3	7	P	3	P	31	3	P	7
33	3	127	139	3	7	P	3	P	47	3	P	7	3	P	P	3	P	41	3	13
37	11	37	3	P	P	3	17	13	3	7	P	3	53	103	3	P	7	3	19	P
39	3	7	P	3	59	P	3	17	7	3	19	P	3	P	13	3	P	P	3	P
41	7	3	19	P	3	P	127	3	13	109	3	157	41	3	7	107	3	P	11	3
43	107	31	3	29	193	3	7	79	3	17	P	3	23	7	3	13	11	3	P	P
47	23	3	P	13	3	43	P	3	P	P	3	7	11	3	197	P	3	P	7	3
49	P	89	3	7	P	3	P	29	3	31	7	3	17	61	3	P	131	3	79	199
51	3	179	7	3	13	47	3	P	11	3	181	23	3	17	P	3	7	13	3	11
53	29	3	73	P	3	23	11	3	7	P	3	13	P	3	17	7	3	11	P	P
57	3	P	11	3	47	59	3	P	P	3	P	P	3	7	19	3	17	P	3	P
59	11	3	P	37	3	7	13	3	P	173	3	11	7	3	P	P	3	17	73	3
61	13	17	3	137	7	3	P	P	3	11	71	3	P	13	3	29	53	3	7	47
63	3	P	17	3	P	P	3	11	131	3	7	211	3	P	P	3	P	7	3	17
67	71	7	3	11	17	3	41	P	3	23	139	3	19	P	3	7	P	3	47	29
69	3	11	13	3	19	17	3	7	P	3	P	3	P	3	7	3	P	157	3	107
71	53	3	P	P	3	P	7	3	P	13	3	P	29	3	61	19	3	71	P	3
73	P	67	3	13	P	3	P	17	3	P	31	3	7	97	3	89	13	3	53	7
77	131	3	23	7	3	31	P	3	37	17	3	P	P	3	P	11	3	7	P	3
79	P	P	3	101	P	3	P	P	3	7	17	3	P	11	3	43	7	3	31	23
81	3	7	P	P	P	13	3	P	7	3	P	11	3	19	P	3	P	67	3	151
83	7	3	53	P	3	19	89	3	P	11	3	137	13	3	7	179	3	P	83	3
87	3	P	109	3	P	7	3	P	19	3	191	101	3	13	17	3	11	P	3	7
89	19	3	43	11	3	P	181	3	P	P	3	7	23	3	11	17	3	P	7	3
91	P	11	3	7	P	3	23	97	3	P	7	3	11	P	3	101	17	3	P	P
93	3	P	7	3	71	P	3	59	13	3	11	P	3	P	43	3	7	17	3	P
97	7	P	3	P	P	3	11	7	3	13	29	3	P	47	3	P	P	3	41	17
99	3	157	P	3	11	23	3	P	107	3	37	P	3	7	P	3	13	19	3	P

	500	501	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519
01	3	P	17	3	13	11	3	7	37	3	P	137	3	29	7	3	11	13	3	17
03	31	3	61	11	3	P	7	3	101	109	3	13	P	3	11	P	3	149	P	3
07	3	89	P	3	7	17	3	P	23	3	11	7	3	P	P	3	P	29	3	F
09	43	3	23	7	3	53	13	3	11	P	3	P	41	3	101	19	3	7	103	3
11	13	P	3	P	P	3	11	17	3	7	29	3	83	13	3	P	7	3	197	23
13	3	7	149	3	11	P	3	13	7	3	139	79	3	23	P	3	P	P	3	P
17	11	38	3	67	P	3	7	41	3	59	17	3	P	7	3	P	71	3	P	193
19	3	P	13	3	127	7	3	67	89	3	163	17	3	19	P	3	41	P	3	7
21	P	3	P	P	3	19	223	3	P	13	3	7	17	3	P	P	3	P	7	3
23	P	P	3	7	P	3	23	P	3	P	7	3	181	17	3	67	11	3	29	137
27	19	3	P	59	3	P	3	7	127	3	29	11	3	P	7	3	13	P	3	3
29	7	P	3	P	211	3	197	7	3	P	11	3	P	P	3	227	17	3	P	P
31	3	P	P	3	29	13	3	97	11	3	P	P	3	7	P	3	P	17	3	11
33	P	3	191	P	3	7	11	3	P	31	3	P	7	3	19	29	3	11	17	3
37	3	181	11	3	31	97	3	113	29	3	7	P	3	11	P	3	P	7	3	167
39	11	3	7	71	3	P	79	3	P	7	3	11	P	3	P	P	3	31	P	3
41	163	7	3	P	P	3	89	P	3	11	43	3	P	P	3	7	113	3	47	P
43	3	41	47	3	73	P	3	7	13	3	P	199	3	P	7	3	43	59	3	127
47	P	P	3	11	61	3	P	31	3	13	P	3	7	P	3	19	P	3	139	7
49	3	11	109	3	7	P	3	19	P	3	71	7	3	P	P	3	13	P	3	P
51	P	3	31	7	3	P	P	3	211	P	3	P	53	3	23	P	3	7	19	3
53	P	P	3	43	13	3	37	P	3	7	19	3	107	89	3	31	7	3	P	11
57	7	3	29	37	3	13	179	3	P	3	P	P	3	7	11	3	73	13	3	3
59	113	P	3	P	P	3	7	193	3	131	P	3	13	7	3	47	P	3	P	223
61	3	103	P	3	P	7	3	23	181	3	P	11	3	P	P	3	19	191	3	7
63	13	3	P	P	3	59	29	3	19	11	3	7	P	3	53	P	3	37	7	3
67	3	13	7	3	109	11	3	P	P	3	223	19	3	31	13	3	7	P	3	157
69	P	3	17	11	3	61	23	3	7	P	3	P	167	3	11	7	3	P	P	3
71	7	11	3	17	41	3	P	7	3	P	P	3	11	47	3	13	163	3	P	P
73	3	131	P	3	17	103	3	P	P	3	11	73	3	7	P	3	P	23	3	P
77	P	P	3	P	7	3	11	P	3	19	13	3	47	83	3	P	31	3	7	P
79	3	19	137	3	11	37	3	17	83	3	7	61	3	191	P	3	P	7	3	59
81	61	3	7	83	3	P	59	3	17	7	3	13	19	3	P	P	3	53	29	3
83	11	7	3	P	19	3	P	43	3	17	23	3	P	P	3	7	P	3	13	227
87	P	3	P	P	3	P	7	3	151	67	3	17	P	3	P	79	3	P	11	3
89	13	31	3	41	29	3	173	P	3	P	47	3	7	13	3	23	11	3	19	7
91	3	53	P	3	7	P	3	13	P	3	19	7	3	17	11	3	P	67	3	P
93	P	3	19	7	3	P	163	3	P	3	P	3	P	11	3	13	P	3	7	P
97	3	7	13	3	P	19	3	79	7	3	37	P	3	103	23	3	17	P	3	11
99	7	3	179	101	3	P	11	3	23	13	3	P	43	3	7	P	3	11	F	3

	520	521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539
01	149	3	P	P	3	P	23	3	7	P	3	P	P	3	P	7	3	83	11	3
03	7	P	3	193	13	3	41	7	3	P	P	3	83	151	3	P	11	3	173	19
07	131	3	17	19	3	7	31	3	P	191	3	23	7	3	P	P	3	43	13	3
09	P	107	3	17	7	3	P	P	3	157	11	3	13	P	3	73	P	3	7	31
11	3	31	109	3	17	P	3	P	11	3	7	173	3	89	P	3	P	7	3	11
13	13	3	7	P	3	17	11	3	P	7	3	P	127	3	31	59	3	11	P	3
17	3	13	11	3	23	P	3	7	P	3	P	P	3	11	7	3	P	P	3	P
19	11	3	79	11	3	29	7	3	13	P	3	11	19	3	P	109	3	P	P	3
21	P	P	3	P	19	3	101	P	3	11	37	3	7	71	3	13	29	3	107	7
23	3	47	P	3	7	53	3	11	101	3	17	7	3	P	41	3	P	31	3	P
27	P	P	3	11	103	3	P	P	3	7	13	3	17	P	3	P	7	3	19	P
29	3	7	29	3	13	P	3	67	7	3	19	P	3	17	23	3	P	13	3	199
31	7	3	19	43	3	131	P	3	23	41	3	13	P	3	7	199	3	P	P	3
33	61	37	3	59	P	3	7	P	3	43	181	3	P	7	3	17	P	3	13	11
37	17	3	P	199	3	107	13	3	P	P	3	7	139	3	P	11	3	17	7	3
39	13	17	3	7	41	3	P	23	3	167	7	3	P	11	3	37	P	3	17	P
41	3	23	7	3	229	P	3	13	53	3	29	11	3	41	P	3	7	61	3	17
43	71	3	89	17	3	P	61	3	7	11	3	19	37	3	13	7	3	223	23	3
47	3	P	13	3	179	11	3	P	43	3	P	P	3	7	19	3	11	71	3	73
49	23	3	P	11	3	7	17	3	41	13	3	P	7	3	11	P	3	59	P	3
51	P	11	3	13	7	3	37	17	3	P	P	3	11	31	3	P	13	3	7	P
53	3	P	P	3	P	P	3	71	17	3	7	23	3	P	P	3	P	7	3	163
57	P	7	3	41	P	3	11	P	3	P	17	3	19	229	3	7	P	3	P	79
59	3	43	P	3	11	13	3	7	P	3	97	17	3	P	7	3	23	P	3	P
61	79	3	11	P	3	P	7	3	P	211	3	P	13	3	193	19	3	37	P	3
63	11	P	3	P	23	3	13	19	3	P	47	3	7	17	3	9	103	3	61	7
67	P	3	P	7	3	P	P	3	29	P	3	79	P	3	127	17	3	7	11	3
69	P	13	3	P	71	3	31	P	3	7	P	3	P	83	3	P	7	3	103	29
71	3	7	167	3	137	P	3	113	7	3	73	P	3	19	11	3	191	17	3	31
73	7	3	13	83	3	19	P	3	37	P	3	P	11	3	7	13	3	P	17	3
77	3	P	61	3	97	7	3	89	11	3	P	41	3	P	53	3	13	P	3	7
79	19	3	23	P	3	P	11	3	P	31	3	7	P	3	P	131	3	11	7	3
81	P	P	3	7	11	3	139	47	3	P	7	3	P	P	3	11	P	3	P	23
83	3	P	7	3	31	P	3	P	3	109	13	3	11	79	3	7	P	3	3	37
87	7	23	3	P	73	3	19	7	3	11	P	3	13	97	3	41	37	3	P	P
89	3	P	P	3	P	43	3	11	P	3	P	3	7	89	3	53	19	3	13	
91	13	3	P	P	3	7	P	3	227	19	3	43	7	3	149	P	3	P	P	3
93	113	19	3	11	7	3	23	13	3	197	P	3	137	107	3	P	P	3	7	P
97	59	3	7	151	3	149	P	3	13	7	3	P	223	3	61	P	3	23	P	3
99	53	7	3	61	47	3	151	37	3	P	29	3	P	67	3	7	P	3	P	11

	540	541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559
01	P	P	3	13	P	3	P	19	3	7	P	3	P	17	3	P	7	3	41	P
03	3	7	67	3	P	P	3	11	7	3	13	P	3	29	17	3	P	53	3	P
07	53	61	3	11	41	7	7	227	3	P	67	3	P	7	3	47	17	3	P	17
09	3	11	151	3	P	3	3	P	23	3	P	P	3	19	67	3	P	17	3	7
11	P	3	23	P	3	19	97	3	59	43	3	7	13	3	P	P	3	P	7	3
13	P	53	3	7	P	3	13	P	3	89	7	3	P	P	3	43	19	3	P	11
17	19	3	P	29	3	P	P	3	7	P	3	P	P	3	151	7	3	P	P	3
19	7	13	3	P	P	3	193	7	3	P	37	3	P	11	3	59	P	3	P	199
21	3	P	59	3	P	P	3	P	13	3	P	11	3	7	157	3	P	P	3	P
23	89	3	13	P	3	7	P	3	73	11	3	199	7	3	19	13	3	103	P	3
27	3	113	211	3	37	11	3	P	109	3	7	P	3	61	43	3	11	7	3	P
29	97	3	7	11	3	3	P	3	P	7	3	29	P	3	11	P	3	23	P	3
31	71	7	3	P	13	3	P	229	3	163	113	3	11	P	3	7	P	3	31	P
33	3	P	193	3	29	23	3	7	P	3	11	13	3	P	7	3	P	P	3	P
37	P	43	3	67	P	3	11	127	3	137	47	3	7	P	3	19	23	3	P	7
39	3	P	73	3	7	P	3	19	29	3	23	7	3	P	P	3	P	139	3	13
41	13	3	11	7	3	P	101	3	173	P	3	67	37	3	P	P	3	7	19	3
43	11	29	3	31	P	3	53	13	3	7	19	3	P	P	3	67	7	3	P	43
47	7	3	17	P	3	P	P	3	13	23	3	101	3	7	3	7	3	107	11	3
49	P	173	3	17	P	3	7	53	3	P	P	3	P	7	3	13	11	3	P	P
51	3	P	P	3	17	7	3	P	P	3	P	131	3	P	11	3	19	197	3	7
53	191	3	227	13	3	17	31	3	19	179	3	7	11	3	23	73	3	127	7	3
57	3	31	7	3	13	89	3	17	11	3	P	19	3	197	P	3	7	13	3	14
59	P	3	29	19	3	P	11	3	7	P	3	13	P	3	31	7	3	11	83	3
61	7	41	3	P	11	3	47	7	3	17	P	3	73	23	3	11	P	3	13	107
63	3	P	11	3	107	P	3	23	83	3	17	P	3	7	37	3	P	P	3	191
67	13	P	3	P	7	3	P	P	3	11	53	3	17	13	3	181	P	3	7	P
69	3	19	P	3	P	197	3	11	P	3	7	43	3	17	P	3	179	7	3	97
71	139	3	7	P	3	11	23	3	37	7	3	P	19	3	13	61	3	43	P	3
73	23	7	3	11	19	3	P	P	3	P	P	3	31	P	3	7	P	3	59	23
77	17	3	P	P	3	P	7	3	P	13	3	23	167	3	29	149	3	17	71	3
79	41	17	3	13	157	3	P	P	3	P	3	7	79	3	7	9	13	3	17	7
81	3	P	17	3	7	P	3	29	P	3	13	7	3	P	109	3	P	11	3	17
83	P	3	19	7	3	P	149	3	71	P	3	139	59	3	113	11	3	7	29	3
87	3	7	P	3	23	13	3	P	7	3	31	11	3	97	P	3	233	P	3	P
89	7	3	233	137	3	71	17	3	131	11	3	229	13	3	7	P	3	47	P	3
91	P	47	3	109	29	3	7	11	3	127	89	3	P	7	3	23	P	3	11	13
93	3	P	P	3	P	7	3	157	17	3	37	97	3	13	211	3	11	P	3	7
97	47	11	3	7	P	3	83	37	3	43	7	3	11	31	3	53	P	3	P	P
99	3	83	7	3	P	71	3	P	13	3	11	17	3	P	19	3	7	P	3	29

	560	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576	577	578	579
01	3	P	43	3	P	P	3	P	79	3	7	11	3	P	61	3	P	7	3	P
03	P	3	7	13	3	P	23	3	43	7	3	17	P	3	137	P	3	19	P	3
07	3	19	P	3	13	11	3	7	P	3	109	P	3	17	7	3	11	13	3	79
09	P	3	P	11	3	P	7	3	P	P	3	13	19	3	11	131	3	P	P	3
11	79	11	3	P	19	3	P	P	3	P	47	3	7	223	3	17	53	3	13	7
13	3	P	67	3	7	31	3	P	P	3	11	7	3	37	P	3	17	P	3	29
17	13	17	3	199	P	3	11	43	3	7	23	3	29	13	3	113	7	3	17	P
19	3	7	17	3	11	P	3	13	7	3	19	P	3	31	67	3	157	P	3	17
21	7	3	11	17	3	29	41	3	P	P	3	239	P	3	7	97	3	197	67	3
23	11	P	3	157	17	3	7	131	3	P	127	3	P	7	3	23	29	3	53	P
27	179	3	59	23	3	P	17	3	P	13	3	7	89	3	P	P	3	P	7	3
29	43	37	3	7	73	3	P	17	3	P	7	3	151	P	3	P	11	3	P	53
31	3	P	7	3	P	P	3	P	17	3	13	P	3	P	11	3	7	P	3	19
33	137	3	53	P	3	P	P	3	7	17	3	19	11	3	79	7	3	13	151	3
37	3	73	P	3	P	13	3	P	11	3	P	17	3	7	19	3	P	3	P	11
39	P	3	P	53	3	7	11	3	113	97	3	P	7	3	71	163	3	11	P	3
41	P	31	3	103	7	3	13	23	3	P	P	3	P	17	3	11	P	3	7	13
43	3	23	11	3	P	P	3	179	P	3	7	P	3	11	17	3	59	7	3	P
47	41	7	3	29	47	3	37	P	3	11	P	3	19	P	3	7	17	3	P	P
49	3	P	P	3	19	193	3	7	13	3	89	P	3	P	7	3	P	17	3	167
51	23	3	13	37	3	11	7	3	139	P	3	67	P	3	73	13	3	P	17	3
53	P	233	3	11	P	3	181	19	3	13	59	3	7	83	3	67	P	3	P	7
57	29	3	101	7	3	23	53	3	P	P	3	61	31	3	P	P	3	7	47	3
59	61	89	3	P	13	3	P	211	3	7	P	3	P	41	3	P	7	3	P	11
61	3	7	127	3	131	163	3	31	7	3	43	13	3	19	37	3	23	11	3	149
63	7	3	P	157	3	13	P	3	101	P	3	P	173	3	7	11	3	47	13	3
67	3	P	P	3	P	7	3	P	19	3	149	11	3	P	P	3	P	61	3	7
69	13	3	P	P	3	P	61	3	29	17	3	P	3	101	23	3	41	7	3	3
71	47	P	3	7	149	3	P	11	3	23	7	3	P	103	3	P	101	3	11	29
73	3	13	7	3	P	11	3	P	P	3	P	P	3	P	13	3	7	P	3	P
77	7	11	3	P	P	3	19	7	3	227	P	3	11	181	3	13	137	3	31	P
79	3	P	167	3	P	29	3	P	23	3	11	P	3	7	229	3	P	19	3	37
81	P	3	23	13	3	7	P	3	11	19	3	211	7	3	47	71	3	P	P	3
83	17	19	3	P	7	3	11	P	3	P	13	3	P	P	3	89	37	3	7	23
87	P	3	7	113	3	71	P	3	163	7	3	13	P	P	3	P	3	P	107	3
89	11	7	3	17	P	3	83	107	3	P	P	3	59	P	3	7	P	3	13	103
91	3	83	181	3	17	P	3	7	P	3	37	P	3	29	7	3	31	P	3	P
93	P	3	41	P	3	17	7	3	P	P	3	P	23	3	P	P	3	P	11	3
97	3	P	19	3	7	P	3	13	P	3	P	7	3	P	11	3	P	29	3	59
99	P	3	P	7	3	P	31	3	17	P	3	47	11	3	13	239	3	7	P	3

	580	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599
01	31	3	11	137	3	19	P	3	127	P	3	7	53	3	191	13	3	227	7	7
03	11	97	3	7	P	3	P	47	3	13	7	3	73	31	3	157	19	3	79	37
07	19	3	P	199	3	41	103	3	7	P	3	P	P	3	P	7	3	P	11	3
09	7	P	3	P	13	3	29	7	3	P	P	3	P	127	3	P	11	3	P	137
11	3	P	P	3	P	P	3	P	23	3	P	13	3	7	11	3	P	29	3	181
13	P	3	23	P	3	7	P	3	103	P	3	P	7	3	19	P	3	211	13	3
17	3	89	P	3	P	163	3	71	11	3	7	31	3	23	P	3	P	7	3	11
19	13	3	7	29	3	139	11	3	131	7	3	P	3	P	3	P	53	3	11	41
21	17	7	3	P	11	3	31	13	3	P	3	P	3	P	137	3	7	P	3	103
23	3	11	11	3	37	43	3	7	59	3	P	P	3	11	7	3	109	P	3	31
27	P	37	3	17	P	3	23	P	3	11	67	3	7	41	3	13	P	3	27	7
29	3	P	P	3	7	107	3	11	89	3	P	7	3	79	67	3	P	P	3	P
31	P	3	P	7	3	11	P	3	P	31	3	29	61	3	103	59	3	7	19	3
33	131	61	3	11	71	3	17	P	3	7	13	3	P	P	3	37	7	3	P	73
37	7	3	P	P	3	P	191	3	17	P	3	13	37	3	7	29	3	31	53	3
39	127	47	3	227	P	3	7	151	3	17	43	3	P	7	3	P	23	3	13	11
41	3	53	139	3	P	7	3	P	29	3	17	P	3	P	P	3	19	11	3	7
43	P	3	P	41	3	P	13	3	19	P	3	7	P	3	P	11	3	P	7	3
47	3	P	7	3	211	127	3	13	83	3	137	11	3	17	P	3	7	P	3	151
49	P	3	31	19	3	P	223	3	7	11	3	P	179	3	13	7	3	149	97	3
51	7	P	3	23	P	3	89	7	3	167	P	3	193	P	3	17	P	3	11	P
53	3	P	13	3	P	11	3	41	229	3	P	149	3	7	P	3	11	P	3	167
57	P	11	3	13	7	3	P	P	3	19	73	3	11	P	3	P	13	3	7	P
59	3	19	17	3	53	31	3	67	71	3	7	P	3	P	37	3	P	7	3	17
61	P	3	7	17	3	157	P	3	11	7	8	67	19	3	97	P	3	13	31	3
63	31	7	3	P	17	3	11	P	3	P	P	3	P	23	3	7	P	3	P	61
67	P	3	11	P	3	P	7	3	37	P	3	P	13	3	P	P	3	59	131	3
69	11	P	3	P	59	3	13	17	3	109	P	3	7	P	3	71	P	3	19	7
71	3	P	P	3	7	37	3	P	17	3	19	7	3	13	P	3	P	P	3	P
73	P	3	19	7	3	P	23	3	113	17	3	47	P	3	P	41	3	7	11	3
77	3	7	101	3	P	19	3	53	7	3	P	17	3	P	11	3	83	23	3	37
79	7	3	13	P	3	P	P	3	97	P	3	23	11	3	7	13	3	P	P	3
81	241	71	3	79	P	3	7	43	3	13	11	3	P	7	3	P	37	3	233	P
83	3	83	167	3	233	7	3	29	11	3	P	P	3	43	17	3	13	191	3	7
87	29	31	3	7	11	3	P	P	3	61	7	3	101	P	3	11	17	3	P	223
89	3	P	7	3	23	41	3	P	P	3	37	13	3	11	19	3	7	17	3	239
91	11	3	71	P	3	13	19	3	7	P	3	11	211	3	41	7	3	P	13	3
93	7	P	3	P	29	3	P	7	3	11	P	3	13	P	3	23	P	3	101	17
97	13	3	97	23	3	7	79	3	P	P	3	P	7	3	P	61	3	P	89	3
99	P	P	3	11	7	3	P	13	3	41	113	3	19	P	3	107	P	3	7	P

	600	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619
01	29	P	3	47	11	3	P	101	3	P	P	3	7	59	3	11	229	3	23	7
03	3	P	11	3	7	17	3	P	41	3	53	7	3	11	P	3	P	P	3	103
07	23	P	3	13	29	3	P	17	3	7	P	3	97	101	3	P	7	3	19	31
09	3	7	P	3	193	P	3	11	7	3	13	53	3	37	P	3	P	23	3	P
11	7	3	19	41	3	11	P	3	P	17	3	23	P	3	7	P	3	13	113	3
13	P	47	3	11	P	3	7	109	3	P	17	3	41	7	3	137	P	3	P	101
17	P	3	P	P	3	73	P	3	61	P	3	7	13	3	P	227	3	P	7	3
19	47	79	3	7	31	3	13	P	3	P	7	3	29	17	3	P	43	3	P	11
21	3	59	7	3	23	P	3	41	P	3	139	P	3	13	17	3	7	11	3	19
23	193	3	P	179	3	29	P	3	7	P	3	19	P	3	239	7	3	P	211	3
27	3	P	229	3	P	P	3	P	13	3	P	11	3	7	19	3	P	17	3	P
29	P	3	13	23	3	7	19	3	59	11	3	P	7	3	47	13	3	P	17	3
31	173	157	3	P	7	3	P	11	3	13	P	3	P	P	3	37	P	3	7	17
33	3	P	29	3	223	11	3	P	127	3	7	113	3	P	23	3	11	7	3	P
37	P	7	3	P	13	3	P	P	3	P	67	3	11	83	3	7	P	3	P	241
39	3	P	59	3	19	P	3	7	83	3	11	13	3	P	7	3	53	107	3	23
41	P	3	107	83	3	13	7	3	11	149	3	P	47	3	P	19	3	29	13	3
43	97	137	3	P	3	11	19	3	P	3	P	7	3	P	3	P	P	3	P	7
47	13	3	11	7	3	191	P	3	71	59	3	47	73	3	43	P	3	7	23	3
49	11	P	3	29	P	3	P	13	3	7	41	3	23	31	3	61	7	3	127	P
51	3	7	P	3	61	151	3	79	7	3	P	P	3	19	13	3	P	P	3	41
53	7	3	89	P	3	19	131	3	13	P	3	P	P	3	7	P	3	37	11	3
57	3	43	P	3	P	7	3	P	19	3	P	23	3	P	11	3	P	P	3	7
59	19	3	P	13	3	23	P	3	P	47	3	7	11	3	41	P	3	151	7	3
61	17	P	3	7	103	3	P	P	3	P	7	3	P	43	3	P	197	3	P	P
63	3	17	7	3	13	71	3	P	11	3	227	31	3	P	P	3	7	13	3	11
67	7	P	3	17	11	3	19	7	3	41	79	3	197	109	3	11	P	3	13	P
69	3	P	11	3	17	37	3	67	P	3	173	P	3	7	P	3	83	19	3	31
71	11	3	P	73	3	7	13	3	29	19	3	11	7	3	P	23	3	223	P	3
73	13	19	3	P	7	3	17	P	3	11	157	3	71	13	3	67	P	3	7	29
77	P	3	7	173	3	11	47	3	17	7	3	131	29	3	13	139	3	163	43	3
79	63	7	3	11	197	3	P	P	3	17	103	3	233	P	3	7	37	P	P	P
81	3	11	13	3	31	29	3	7	23	3	17	193	3	P	7	3	P	P	3	P
83	P	3	23	P	3	47	7	3	107	13	3	17	P	3	P	P	3	31	19	3
87	3	139	19	3	7	43	3	89	P	3	13	7	3	17	P	3	P	11	3	P
89	P	3	P	7	3	P	P	3	P	71	3	43	167	3	17	11	3	7	199	3
91	P	23	3	131	241	3	137	31	3	7	P	3	P	11	3	17	7	3	59	P
93	3	7	7	3	P	13	3	P	7	3	199	11	3	29	P	3	17	61	3	47
97	19	17	3	P	P	3	7	11	3	181	107	3	P	7	3	3	103	3	11	13
99	3	37	17	3	101	7	3	163	P	3	P	19	3	13	89	3	11	29	3	7

	620	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639
01	3	13	P	3	P	P	3	P	P	3	251	89	3	7	13	3	P	11	3	P
03	P	3	17	P	3	7	P	3	13	P	3	P	7	3	19	11	3	P	P	3
07	3	173	P	3	17	P	3	73	181	3	7	11	3	29	163	3	P	7	3	P
09	59	3	7	13	3	17	137	3	107	7	3	223	31	3	P	41	3	P	P	3
11	3	7	3	P	139	3	17	11	3	53	13	3	P	P	3	7	P	3	11	79
13	P	179	P	3	13	11	3	7	23	3	61	P	3	P	7	3	11	13	3	P
17	3	11	3	101	P	3	P	59	3	17	29	3	7	P	3	19	P	3	13	7
19	P	P	P	3	7	101	3	19	P	3	11	7	3	23	P	3	113	P	3	41
21	109	3	43	7	3	103	13	3	11	P	3	17	191	3	P	P	3	7	19	3
23	13	23	3	P	P	3	11	P	3	7	19	3	17	13	3	139	7	3	P	97
27	7	3	11	P	3	31	P	3	P	P	3	P	23	3	7	P	3	P	83	3
29	11	P	3	157	163	3	7	149	3	P	P	3	53	7	3	17	P	3	29	P
31	3	P	13	3	149	7	3	P	83	3	P	P	3	P	137	3	17	101	3	7
33	17	3	P	83	3	P	P	3	19	13	3	7	37	3	229	P	3	17	7	3
37	3	P	7	3	29	23	3	43	31	3	13	19	3	P	11	3	7	P	3	17
39	P	3	109	17	3	P	P	3	7	P	3	103	11	3	P	7	3	13	71	3
41	7	P	3	31	17	3	37	7	3	113	11	3	P	97	3	P	23	3	P	43
43	3	P	67	3	41	13	3	P	11	3	23	233	3	7	P	3	31	P	3	11
47	P	29	3	P	7	3	13	17	3	19	67	3	P	P	3	11	P	3	7	13
49	3	19	11	3	197	P	3	131	17	3	7	P	3	11	67	3	P	7	3	P
51	11	3	7	P	3	71	31	3	P	7	3	11	19	3	107	103	3	37	67	3
53	P	7	3	23	19	3	P	P	3	11	17	3	43	P	3	7	53	3	P	31
57	P	3	13	127	3	11	7	3	239	157	3	137	17	3	23	13	3	103	P	3
59	229	61	3	11	P	3	P	9	3	13	P	3	7	17	3	P	P	3	19	7
61	3	11	23	3	7	73	3	P	P	3	9	7	3	P	17	3	13	P	3	167
63	53	3	19	7	3	P	223	3	37	79	3	83	41	3	P	17	3	7	P	3
67	3	7	71	3	P	19	3	23	7	3	P	13	3	P	P	3	P	11	3	47
69	7	3	73	47	3	13	29	3	P	P	3	181	151	3	7	11	3	43	13	3
71	P	P	3	97	179	3	7	41	3	P	59	3	13	7	3	151	P	3	23	17
73	3	79	P	3	P	7	3	P	P	3	P	11	3	127	P	3	41	P	3	7
77	23	97	3	7	P	3	233	11	3	71	7	3	P	P	3	P	37	3	11	P
79	3	13	7	3	43	11	3	67	227	3	P	P	3	61	13	3	7	23	3	137
81	P	3	61	11	3	P	19	3	7	P	3	23	P	3	11	7	3	P	127	3
83	7	11	3	P	P	3	P	7	3	P	199	3	11	241	3	13	43	3	191	109
87	47	3	199	13	3	7	P	3	11	P	3	179	7	3	P	P	3	227	29	3
89	29	P	3	89	7	3	11	37	3	P	13	3	19	P	3	P	P	3	7	61
91	3	P	167	3	11	P	3	P	61	3	7	29	3	P	173	3	P	7	3	89
93	31	3	7	43	3	53	71	3	109	7	3	13	167	3	P	19	3	P	181	3
97	P	37	3	P	3	P	7	3	P	3	P	P	3	P	7	3	P	131	3	P
99	3	3	P	23	3	59	7	3	31	73	3	P	P	3	P	3	P	3	11	3

	640	641	642	643	644	645	646	647	648	649	650	651	652	653	654	655	656	657	658	659
01	7	3	19	P	3	53	P	3	11	P	3	P	113	3	7	17	3	P	29	3
03	29	13	3	P	P	3	7	89	3	41	P	3	P	7	3	31	17	3	23	59
07	P	3	11	107	3	251	23	3	229	47	3	7	197	3	P	13	3	P	7	3
09	11	P	3	7	29	3	P	P	3	13	7	3	61	P	3	109	P	3	P	17
11	3	61	7	3	41	31	3	163	P	3	P	P	3	241	149	3	7	23	3	19
13	P	3	157	73	3	P	P	3	7	139	3	19	P	3	P	7	3	P	11	3
17	3	97	P	3	37	149	3	P	P	3	79	13	3	7	11	3	P	P	3	29
19	P	149	P	3	7	19	3	53	P	3	P	7	3	P	P	3	P	13	3	
21	73	37	3	131	7	3	P	61	3	P	11	3	13	83	3	P	211	3	7	P
23	3	P	P	3	23	113	3	59	11	3	7	P	3	P	P	3	137	7	3	11
27	43	7	3	P	11	3	P	13	3	P	P	3	19	P	3	7	29	3	P	P
29	3	13	11	3	19	173	3	7	241	3	P	P	3	11	7	3	P	P	3	P
31	11	3	P	23	3	47	7	3	13	29	3	11	37	3	59	19	3	P	P	3
33	P	59	3	P	3	P	19	3	11	P	3	7	79	3	13	P	3	43	7	
37	P	3	61	7	3	11	109	3	23	P	3	53	89	3	P	P	3	7	P	3
39	17	31	3	11	P	3	37	41	3	7	13	3	P	223	3	P	7	3	P	233
41	3	7	227	3	13	233	3	101	7	3	193	P	3	19	31	3	41	13	3	23
43	7	3	17	37	3	19	127	3	61	101	3	13	53	3	7	P	3	29	P	3
47	3	23	41	3	17	7	3	P	19	3	29	P	3	101	P	3	P	11	3	7
49	19	3	47	229	3	17	13	3	P	107	3	7	7	3	P	11	3	37	7	3
51	13	P	3	7	P	3	17	73	3	P	7	3	23	11	3	P	P	3	P	P
53	3	P	7	3	P	P	3	13	P	3	P	11	3	P	29	3	7	47	3	101
57	7	P	3	139	43	3	19	7	3	17	67	3	P	P	3	P	P	3	11	P
59	3	83	13	3	7	11	3	31	79	3	17	23	3	7	67	3	11	19	3	17
61	29	3	179	11	3	7	P	3	37	13	3	11	7	3	11	53	3	P	67	3
63	P	11	3	13	7	3	P	P	3	167	P	3	11	163	3	P	13	3	7	P
67	P	3	7	191	3	P	P	3	11	7	3	P	P	3	17	173	3	13	P	3
69	79	7	3	59	23	3	11	239	3	P	31	3	P	131	3	7	97	3	199	41
71	3	P	P	3	11	13	3	7	P	3	P	P	3	P	7	3	17	89	3	37
73	17	3	11	P	3	31	7	3	29	23	3	P	13	3	233	23	3	17	19	3
77	3	29	17	3	7	P	3	211	P	3	59	7	3	13	41	3	P	P	3	17
79	139	3	P	7	3	P	P	3	P	181	3	P	29	3	P	P	3	7	11	3
81	P	13	3	P	17	3	71	P	3	7	151	3	97	P	3	P	7	3	P	P
83	3	7	P	3	P	17	3	P	7	3	37	P	3	151	11	3	19	157	3	P
87	19	P	3	31	59	3	7	17	3	13	11	3	P	7	3	P	P	3	41	19
89	3	P	53	3	P	7	3	67	11	3	P	19	3	23	43	3	13	P	3	7
91	P	3	239	19	3	P	11	3	P	17	3	7	109	3	79	107	3	11	7	3
93	107	23	3	7	11	3	3	P	3	103	7	3	P	P	3	11	179	3	131	P
97	11	3	113	71	3	13	31	3	7	P	3	11	17	3	P	7	3	19	13	3
99	7	43	3	P	P	3	23	7	3	11	P	3	13	17	3	P	P	3	P	31

	660	661	662	663	664	665	666	667	668	669	670	671	672	673	674	675	676	677	678	679
01	13	7	3	P	23	3	P	P	3	149	11	3	17	13	3	7	P	3	P	P
03	3	P	239	3	P	73	3	7	11	3	P	P	3	17	7	3	67	79	3	11
07	149	P	3	61	11	3	43	41	3	23	37	3	7	P	3	11	P	3	P	7
09	3	P	11	3	7	P	3	19	P	3	113	7	3	11	P	3	17	P	3	59
11	11	3	73	7	3	227	59	3	71	13	3	11	P	3	P	P	3	7	19	3
13	251	17	3	13	P	3	29	P	3	7	19	3	P	83	3	181	7	3	17	113
17	7	3	23	17	3	11	P	3	109	61	3	41	P	3	7	107	3	13	73	3
19	107	37	3	11	17	3	7	137	3	P	29	3	P	7	3	251	P	3	P	23
21	3	11	P	3	127	7	3	P	P	3	P	P	3	23	P	3	19	241	3	7
23	103	3	47	29	3	P	17	3	19	P	3	7	13	3	191	P	3	P	7	3
27	3	89	7	3	181	71	3	53	17	3	97	19	3	13	P	3	7	11	3	P
29	P	3	103	19	3	P	P	3	7	17	3	P	23	3	P	7	3	89	P	3
31	7	13	3	113	3	P	3	23	7	3	P	17	3	P	11	3	P	3	29	P
33	3	41	107	3	31	P	3	P	13	3	P	11	3	7	P	3	47	P	3	P
37	P	P	3	P	7	3	37	11	3	13	43	3	71	17	3	P	239	3	7	41
39	3	19	19	3	29	11	3	P	89	3	7	P	3	P	17	3	11	7	3	P
41	P	3	7	11	3	P	103	3	P	7	3	P	19	3	11	17	3	P	179	3
43	211	7	3	P	13	3	P	31	3	P	P	3	11	P	3	7	17	3	P	P
47	P	3	31	P	3	13	7	3	11	P	3	83	P	3	P	P	3	37	13	3
49	257	29	3	43	P	3	11	P	3	P	P	3	7	P	3	31	61	3	19	7
51	3	83	97	3	7	61	3	P	P	2	19	7	3	47	37	3	P	P	3	13
53	13	3	11	7	3	P	P	3	P	23	3	P	109	3	P	43	3	7	P	3
57	3	7	59	3	P	19	3	241	7	3	P	P	3	193	13	3	29	P	3	P
59	7	3	173	P	3	101	191	3	13	P	3	239	103	3	7	P	3	P	11	3
61	31	P	3	P	41	7	7	191	3	29	P	3	P	7	3	13	11	3	79	P
63	3	109	23	3	P	3	3	P	3	199	47	3	31	11	3	71	P	3	7	P
67	P	127	3	7	P	163	179	3	167	7	3	137	23	3	P	157	3	P	P	P
69	3	P	7	3	13	3	3	23	11	3	47	P	3	P	19	3	7	13	3	11
71	P	3	P	31	3	P	11	3	7	193	3	13	P	3	109	7	3	11	67	3
73	7	P	3	P	11	3	61	7	3	P	P	3	P	89	3	11	31	3	13	101
77	11	3	191	P	3	7	13	3	P	P	3	11	7	3	P	P	3	P	103	3
79	13	P	3	41	7	2	131	43	3	11	P	3	19	13	3	P	P	3	7	P
81	3	17	79	3	19	139	3	11	47	3	7	P	3	43	P	3	53	7	3	157
83	P	3	7	P	3	11	P	3	P	7	3	23	61	3	13	19	3	P	P	3
87	3	11	13	3	17	P	3	7	211	3	73	P	3	79	7	3	113	53	3	P
89	P	3	151	197	3	17	7	3	P	13	3	P	P	3	P	P	3	P	29	3
91	29	P	3	13	P	3	17	P	3	31	23	3	7	P	3	257	13	3	P	7
93	3	37	P	3	7	P	3	17	151	3	13	7	3	19	P	3	139	11	3	P
97	257	53	3	67	29	3	P	P	3	7	229	3	173	11	3	23	7	3	43	97
99	3	7	167	3	P	13	3	67	7	3	17	11	3	P	P	3	P	151	3	53

	680	681	682	683	684	685	686	687	688	689	690	691	692	693	694	695	696	697	698	699
01	3	11	7	3	73	P	3	23	107	3	P	43	3	37	P	3	7	47	3	13
03	13	3	241	167	3	61	31	3	7	P	3	19	P	3	P	7	3	43	29	3
07	3	13	P	3	67	P	3	127	83	3	151	29	3	7	13	3	47	11	3	53
09	47	3	P	83	3	7	19	3	53	P	3	P	7	3	31	11	3	P	3	3
11	23	P	3	P	7	3	P	P	3	137	P	3	67	11	3	13	151	3	7	P
13	3	P	P	3	37	131	3	P	P	3	7	11	3	P	41	3	67	7	3	151
17	17	7	3	53	31	3	59	11	3	P	13	3	19	P	3	7	43	3	11	139
19	3	17	P	3	13	11	3	7	P	3	P	P	3	103	7	3	11	13	3	29
21	251	3	17	11	3	P	7	3	P	41	3	13	P	3	11	19	3	113	P	3
23	P	11	3	17	53	3	163	19	3	157	23	3	7	181	3	37	P	3	13	7
27	59	3	P	7	3	17	13	3	11	P	3	P	37	3	P	251	3	7	P	3
29	13	193	3	P	41	3	11	P	3	7	P	3	107	13	3	23	7	3	P	P
31	3	7	31	3	11	P	3	13	7	3	P	73	3	19	P	3	179	103	3	P
33	7	3	11	23	3	19	P	3	17	29	3	257	P	3	7	31	3	137	P	3
37	3	61	13	3	P	7	3	P	19	3	17	47	3	P	23	3	83	P	3	7
39	19	3	P	37	3	P	P	3	23	13	3	7	P	3	P	3	P	7	3	3
41	P	P	3	7	89	3	83	53	3	71	3	17	3	17	P	3	197	11	3	211
43	3	83	7	3	P	P	3	P	43	3	13	P	3	17	11	3	7	97	3	23
47	7	P	3	41	P	3	19	7	3	P	11	3	P	31	3	17	257	3	P	113
49	3	23	139	3	P	13	3	P	11	3	29	P	3	7	37	3	17	19	3	11
51	17	3	131	P	3	7	11	3	31	19	3	P	7	3	199	157	3	11	23	3
53	P	17	3	29	7	3	13	197	3	53	199	3	23	223	3	11	P	3	7	13
57	11	3	7	17	3	179	71	3	37	7	3	11	P	3	P	P	3	79	P	3
59	P	7	3	197	17	3	P	29	3	11	53	3	P	43	3	7	41	3	P	P
61	3	P	P	3	223	17	3	7	13	3	P	23	3	139	7	3	P	P	3	43
63	29	3	13	137	3	11	7	3	P	P	3	P	P	3	P	13	3	P	19	3
67	3	11	19	3	7	P	3	P	17	3	P	7	3	71	P	3	13	P	3	31
69	43	3	233	7	3	191	P	3	61	17	3	263	113	3	127	73	3	7	109	3
71	P	P	3	P	13	3	43	P	3	7	17	3	53	P	3	29	7	3	107	11
73	3	7	67	3	P	47	3	97	7	3	P	13	3	173	P	3	19	11	3	167
77	19	79	3	101	P	3	7	P	3	23	67	3	13	7	3	41	P	3	P	19
79	3	29	P	3	31	7	3	109	P	3	37	11	3	P	17	3	59	P	3	7
81	13	3	P	19	3	P	173	3	P	11	3	7	29	3	P	17	3	31	7	3
83	103	41	3	7	P	3	P	11	3	101	7	3	79	P	3	149	17	3	11	47
87	P	3	23	11	3	107	P	3	7	149	3	43	193	3	11	7	3	19	17	3
89	7	11	3	P	P	3	149	7	3	19	59	3	11	P	3	13	227	3	47	17
91	3	19	47	3	P	113	3	P	P	3	11	P	3	7	P	3	P	101	3	P
93	149	3	31	13	3	7	73	3	11	P	3	P	7	3	P	P	3	71	37	3
97	3	47	163	3	11	P	3	89	P	3	7	P	3	29	P	3	P	7	3	P
99	P	3	7	P	3	181	P	3	P	3	7	13	23	3	P	79	3	223	P	3

	700	701	702	703	704	705	706	707	708	709	710	711	712	713	714	715	716	717	718	719
01	P	3	7	3	P	17	3	101	P	3	97	13	3	11	127	3	7	19	4	3
03	P	11	3	229	23	3	13	17	3	7	19	3	11	113	3	P	P	3	59	13
07	7	3	P	167	3	P	P	3	11	17	3	211	31	3	7	23	3	P	P	3
09	P	13	3	P	181	3	7	P	3	23	17	3	P	7	3	43	101	3	P	P
11	3	P	61	3	11	7	3	31	13	3	P	47	3	29	P	3	19	P	3	7
13	53	3	11	167	3	107	241	3	19	P	3	3	17	3	P	13	3	P	7	3
17	3	P	7	3	67	151	3	P	23	3	47	19	3	P	17	3	7	29	3	P
19	P	3	23	19	3	97	P	3	7	P	3	P	229	3	P	7	3	P	11	3
21	7	P	3	P	13	3	P	7	3	P	29	3	67	73	3	37	11	3	P	23
23	3	P	P	3	P	109	3	197	P	3	P	13	3	7	11	3	67	17	3	71
27	239	23	3	P	7	3	P	107	3	19	11	3	13	P	3	P	41	3	7	17
29	3	19	P	3	P	P	3	P	11	3	7	P	3	P	P	3	83	7	3	11
31	13	3	7	53	3	251	11	3	193	7	3	83	19	3	61	233	3	11	109	3
33	59	7	3	61	11	3	23	13	3	89	251	3	P	P	3	7	P	3	29	P
37	11	3	P	37	3	P	7	3	13	P	3	11	P	3	P	P	3	23	P	3
39	P	P	3	31	P	3	P	127	3	11	P	3	7	P	3	13	71	3	19	7
41	3	P	P	3	7	23	3	11	P	3	19	7	3	P	199	3	31	P	3	P
43	89	3	19	7	3	11	41	3	P	61	3	P	191	3	P	29	3	7	P	3
47	3	7	199	3	13	19	3	203	7	3	23	P	3	P	37	3	P	13	3	P
49	7	3	P	103	3	P	31	3	P	P	3	13	P	3	7	P	3	157	P	3
51	P	29	3	P	P	3	7	139	3	P	227	3	43	7	3	P	137	3	13	11
53	3	31	163	3	47	7	3	P	3	41	P	3	P	3	P	3	79	11	3	7
57	13	P	3	7	P	3	P	173	3	P	7	3	P	11	3	163	131	3	181	47
59	3	17	7	3	P	37	3	13	59	3	P	11	3	P	19	3	7	83	3	227
61	P	3	17	71	3	41	19	3	7	11	3	P	P	3	13	7	3	P	P	3
63	7	P	3	17	31	3	P	7	3	29	179	3	P	P	3	P	3	11	P	
67	P	3	29	11	3	7	P	3	P	13	3	P	7	3	11	59	3	43	P	3
69	41	11	3	13	7	3	17	P	3	P	3	11	23	3	P	13	3	7	79	
71	3	47	P	3	19	P	3	17	131	3	7	P	3	149	P	3	P	7	3	P
73	79	3	7	P	3	P	29	3	11	7	3	103	263	3	P	19	3	13	41	3
77	3	P	31	3	11	13	3	7	P	3	17	109	3	137	7	3	229	P	3	167
79	P	3	11	P	3	163	7	3	P	P	3	17	13	3	P	31	3	179	P	3
81	11	P	3	P	P	3	13	37	3	167	P	3	7	41	3	47	43	3	P	7
83	3	P	67	3	7	P	3	P	73	3	31	7	3	13	P	3	97	23	3	167
87	109	13	3	59	3	P	71	3	7	67	3	P	P	3	17	7	3	P	P	
89	3	7	P	3	P	P	3	29	7	3	P	257	3	P	11	3	17	P	3	193
91	7	3	13	43	3	73	223	3	P	P	3	P	11	3	7	13	3	17	29	3
93	29	17	3	P	157	3	7	P	3	13	11	3	P	7	3	P	P	3	17	P
97	191	3	P	17	3	227	11	3	31	P	3	7	83	3	19	P	3	11	7	3
99	P	P	3	7	11	3	19	83	3	P	7	3	37	P	3	11	P	3	P	P

	720	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739
01	89	p	3	17	7	3	97	p	3	p	37	3	71	23	3	31	11	3	7	67
03	3	p	103	3	17	p	3	23	47	3	7	41	3	p	11	3	89	7	3	263
07	13	7	3	p	61	3	17	p	3	p	11	3	19	13	3	7	p	3	23	p
09	3	p	163	3	19	31	3	7	11	3	p	29	3	p	7	3	p	p	3	11
11	107	3	p	167	3	59	7	3	17	p	3	113	179	3	13	19	3	11	31	3
13	23	37	3	p	11	3	p	19	3	17	p	3	7	167	3	11	p	3	223	p
17	11	3	257	7	3	127	p	3	p	13	3	11	211	3	p	p	3	7	97	3
19	p	41	3	13	139	3	101	p	3	7	p	3	17	157	3	37	7	3	p	193
21	3	7	p	3	p	47	3	11	7	3	13	p	3	17	p	3	83	p	3	29
23	7	3	p	31	3	11	p	3	p	p	p	83	37	3	7	p	3	13	p	3
27	3	11	p	3	23	7	3	p	19	3	103	p	3	p	101	3	17	p	3	7
29	17	3	p	151	3	29	59	3	67	233	3	7	13	3	97	p	3	17	7	3
31	p	17	3	7	p	3	13	257	3	p	7	3	67	p	3	23	29	3	17	11
33	3	53	7	3	113	p	3	p	173	3	199	p	3	13	p	3	7	11	3	17
37	7	13	3	p	17	3	19	7	3	p	p	3	p	11	3	151	p	3	47	107
39	3	p	29	3	107	17	3	p	13	3	p	11	3	7	23	3	211	19	3	p
41	61	3	13	p	3	7	17	3	23	11	3	p	7	3	271	13	3	37	41	3
43	p	19	3	73	7	3	p	11	3	13	p	3	p	71	3	251	p	3	7	p
47	p	3	7	11	3	p	p	3	97	7	3	193	89	3	11	p	3	29	p	3
49	109	7	3	71	13	3	p	23	3	p	17	3	11	41	3	7	47	3	p	73
51	3	23	p	3	53	p	3	7	263	3	7	13	3	p	7	3	p	p	3	p
53	p	3	p	p	3	13	7	3	11	p	3	191	17	3	p	p	3	131	13	3
57	3	59	19	7	7	73	3	31	41	3	43	7	3	109	17	3	73	p	3	13
59	13	3	11	3	3	p	113	3	p	p	3	149	p	3	p	17	3	7	p	3
61	11	p	3	269	p	3	p	13	3	7	p	3	61	p	3	p	7	3	233	p
63	3	7	127	3	233	149	3	p	7	3	p	23	3	p	13	3	19	17	3	37
67	19	p	3	p	p	3	7	p	3	131	31	3	41	7	3	13	11	3	p	17
69	3	p	p	3	p	7	3	53	p	3	89	19	3	p	11	3	23	71	3	7
71	97	3	p	13	3	31	p	3	p	43	3	7	11	3	p	p	3	p	7	3
73	p	p	3	7	23	3	p	61	3	p	7	3	47	239	3	29	p	3	31	p
77	p	3	p	157	3	p	11	3	7	p	3	13	p	3	p	7	3	11	p	3
79	7	89	3	p	11	3	p	7	3	19	p	3	127	p	3	11	p	3	13	29
81	3	19	11	p	p	181	3	73	31	3	107	p	3	7	179	3	p	89	3	167
83	11	3	41	3	3	7	13	3	p	59	3	11	7	3	p	p	3	p	p	3
87	3	37	p	3	173	29	3	11	23	3	7	163	3	p	43	3	31	7	3	241
89	p	3	7	191	3	11	p	3	p	7	3	p	83	3	13	p	3	113	37	3
91	p	7	3	11	71	3	157	83	3	47	p	3	p	79	3	7	59	3	19	23
93	3	11	13	3	p	229	3	7	p	3	19	53	3	23	p	3	p	109	3	61
97	17	23	3	13	p	3	139	p	3	p	67	3	7	19	3	p	13	3	p	7
99	3	17	197	3	7	19	3	43	259	3	13	7	3	29	67	3	p	11	3	p

	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759
01	3	P	P	3	47	7	3	11	131	3	179	13	3	257	P	3	19	17	3	7
03	43	3	P	67	3	11	61	3	19	P	3	7	157	3	P	P	3	P	7	3
07	3	11	7	3	37	P	3	P	239	3	107	19	3	P	P	3	7	P	3	13
09	13	3	P	19	3	P	P	3	7	173	3	P	P	3	73	7	3	P	41	3
11	7	37	3	P	P	3	P	7	3	23	P	3	P	127	3	P	P	3	47	11
13	3	13	47	3	P	269	3	P	79	3	P	31	3	7	13	3	83	11	3	P
17	P	137	3	P	7	3	29	P	3	19	P	3	P	11	3	13	P	3	7	69
19	3	19	P	3	P	43	3	P	23	3	7	11	3	109	53	3	P	7	3	31
21	P	3	7	13	3	P	71	3	P	7	3	43	19	3	199	P	3	P	P	3
23	79	7	3	P	19	3	P	11	3	P	13	3	P	P	3	7	47	3	11	23
27	P	3	199	11	3	P	7	3	P	31	3	13	P	3	11	P	3	41	191	3
29	181	11	3	239	263	3	37	P	3	P	P	3	7	P	3	47	P	3	13	7
31	3	P	P	3	7	P	3	P	P	3	11	7	3	71	P	3	53	P	3	P
33	101	3	19	7	3	73	13	3	11	P	3	P	23	3	241	P	3	7	P	3
37	3	7	61	3	11	19	3	13	7	3	P	227	3	P	P	3	43	53	3	P
39	7	3	11	79	3	131	101	3	67	137	3	29	P	7	P	3	23	181	3	
41	11	151	3	17	P	3	7	31	3	P	P	3	61	7	3	P	P	3	149	P
43	3	P	13	3	17	7	3	41	P	3	101	163	3	59	37	3	67	P	3	7
47	P	53	3	7	109	3	17	P	3	149	7	3	47	P	3	31	11	3	73	173
49	3	P	7	3	P	127	3	17	29	3	13	P	3	151	11	3	7	211	3	53
51	P	3	41	149	3	P	19	3	7	241	3	223	11	3	197	7	3	13	101	3
53	7	29	3	P	P	3	P	7	3	17	11	3	P	P	3	P	3	P	3	151
57	103	3	P	P	3	7	11	3	P	23	3	17	7	3	61	P	3	11	31	3
59	31	P	3	23	7	3	13	P	3	P	47	3	17	179	3	11	P	3	7	13
61	3	P	11	3	19	P	3	P	P	3	7	P	3	11	59	3	29	7	3	37
63	11	3	7	P	3	173	197	3	43	7	3	11	73	3	17	19	3	239	107	3
67	3	P	23	3	113	P	3	7	13	3	271	P	3	P	7	3	17	P	3	P
69	17	3	13	31	3	11	7	3	P	61	3	P	P	3	163	13	3	17	P	3
71	P	17	3	11	P	3	89	P	3	13	41	3	7	23	3	P	31	3	17	7
73	3	11	17	3	7	P	3	23	P	3	37	7	3	19	71	3	13	P	3	17
77	P	P	3	P	13	3	53	37	3	7	193	3	P	P	3	P	7	3	23	11
79	3	7	P	3	71	17	3	P	7	3	P	13	3	43	P	3	P	11	3	P
81	7	3	59	P	3	13	17	3	103	97	3	P	83	3	7	11	3	P	13	3
83	23	31	3	P	211	3	7	17	3	167	P	3	13	7	3	P	P	3	P	P
87	13	3	P	73	3	P	P	3	P	11	3	7	79	3	19	131	P	P	7	3
89	43	P	3	7	P	3	19	11	3	31	7	3	P	P	3	269	P	3	11	P
91	3	13	7	3	163	11	3	29	P	3	61	17	3	P	13	3	7	19	3	P
93	P	3	P	11	3	97	113	3	7	19	3	P	17	3	11	7	3	P	29	3
97	3	P	P	3	23	P	3	P	P	3	11	29	3	7	17	3	59	P	3	P
99	P	3	191	13	3	7	P	3	11	37	3	139	7	3	103	17	3	229	71	3

	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779
01	P	3	181	41	3	113	7	3	P	11	3	P	P	3	17	19	3	13	P	3
03	P	P	3	P	P	3	P	11	3	53	P	3	7	23	3	17	71	3	11	7
07	17	3	3	7	3	P	P	3	89	P	3	83	13	3	11	179	3	7	29	3
09	29	11	3	137	109	3	13	74	3	7	53	3	11	97	3	P	7	3	17	13
11	3	7	17	3	43	P	3	41	7	3	11	29	3	13	199	3	P	P	3	17
13	7	3	P	17	3	19	23	3	11	P	3	59	P	3	7	P	3	P	P	3
17	3	103	199	3	11	7	3	P	13	3	P	67	3	P	P	3	P	23	3	7
19	19	3	11	167	3	P	17	3	P	P	3	7	37	3	P	13	3	P	7	3
21	11	163	3	7	P	3	193	17	3	13	7	3	31	167	3	P	P	3	59	67
23	3	P	7	3	P	59	3	73	17	3	P	233	3	P	139	3	7	P	3	29
27	7	269	3	127	13	3	19	7	3	43	17	3	29	53	3	P	11	3	223	149
29	3	P	31	3	23	103	3	277	P	3	P	13	3	7	11	3	149	19	3	P
31	P	3	P	37	3	7	P	3	P	19	3	137	7	3	P	31	3	P	13	3
33	139	19	3	P	7	3	197	P	3	107	11	3	13	17	3	23	29	3	7	P
37	13	3	7	23	3	P	11	3	P	7	3	P	P	3	211	17	3	11	277	3
39	P	7	3	97	11	3	173	13	3	47	41	3	P	P	3	7	17	3	P	59
41	3	13	11	3	P	7	3	7	43	3	P	P	3	11	7	3	P	17	3	41
43	11	3	P	P	3	P	7	3	13	P	3	11	P	3	43	P	3	P	17	3
47	3	P	19	3	7	41	3	11	P	3	P	7	3	P	P	3	P	P	3	23
49	113	3	P	7	3	11	P	3	31	P	3	179	P	3	41	P	3	7	P	3
51	59	271	3	11	89	3	P	23	3	7	13	3	67	P	3	P	7	3	127	P
53	3	7	P	3	13	37	3	P	7	3	29	P	3	103	73	3	19	13	3	137
57	19	P	3	29	101	3	7	P	3	41	251	3	23	7	3	P	79	3	13	11
59	3	P	P	3	157	7	3	59	151	3	263	19	3	P	29	3	P	11	3	7
61	23	3	P	19	3	P	13	3	101	P	3	7	P	3	71	11	3	P	7	3
63	13	P	3	7	P	3	31	29	3	P	7	3	P	11	3	P	37	3	P	53
67	29	3	53	P	3	23	P	3	7	11	3	P	P	3	13	7	3	19	P	3
69	7	59	3	P	47	3	43	7	3	19	P	3	P	P	3	P	101	3	11	P
71	3	19	13	3	P	11	3	P	P	3	37	P	3	7	P	3	11	83	3	103
73	127	3	89	11	3	7	P	3	P	13	3	229	7	3	11	P	3	P	43	3
77	3	17	83	3	31	73	3	P	59	3	7	71	3	P	P	3	173	7	3	P
79	P	3	7	P	3	P	P	3	11	7	3	113	P	3	P	23	3	13	47	3
81	P	7	3	17	P	3	11	P	3	23	P	3	109	223	3	7	P	3	19	29
83	3	29	P	3	11	13	3	7	P	3	19	79	3	P	7	3	131	P	3	P
87	11	47	3	P	P	3	13	31	3	167	157	3	7	19	3	P	P	3	71	7
89	3	61	P	3	7	19	3	17	23	3	127	7	3	13	P	3	P	107	3	167
91	P	3	23	7	3	191	53	3	17	P	3	P	P	3	P	P	3	7	11	3
93	47	13	3	79	P	3	271	41	3	7	P	3	37	193	3	31	7	3	P	23
97	7	3	13	241	3	P	3	131	3	37	3	17	11	3	7	13	3	P	61	3
99	P	23	3	19	227	3	7	61	3	13	11	3	17	7	3	73	P	3	P	P

	780	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799
01	3	P	3	P	P	3	83	7	3	P	13	3	P	P	3	107	P	3	P	P
03	7	83	P	3	13	29	3	211	P	3	199	P	3	7	271	3	23	13	3	P
07	P	37	3	P	7	3	P	P	3	19	41	3	103	71	3	43	11	3	7	P
09	3	19	197	3	89	P	3	31	P	3	7	239	3	P	11	3	P	7	3	41
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13	13	7	3	71	19	3	127	P	3	23	11	3	113	13	3	7	P	3	P	157
17	P	3	17	P	3	P	7	3	269	53	3	61	37	3	13	131	3	11	P	3
19	61	191	3	17	11	3	29	223	3	P	31	3	7	P	3	11	103	3	19	7
21	3	P	11	3	7	233	3	P	23	3	19	7	3	11	43	3	P	29	3	229
23	11	3	19	7	3	17	P	3	P	13	3	11	227	3	P	281	3	7	P	3
27	3	7	137	3	P	19	3	11	7	3	13	67	3	23	P	3	P	61	3	257
29	7	3	P	29	3	11	61	3	17	P	3	53	P	3	7	67	3	13	P	3
31	P	23	3	11	107	3	7	131	3	17	P	3	P	7	3	P	P	3	97	67
33	3	11	P	3	41	7	3	43	31	3	17	P	3	P	P	3	P	71	3	7
37	73	P	3	7	P	3	13	P	3	193	7	3	17	P	3	P	97	3	29	11
39	3	P	7	3	P	P	3	71	P	3	P	P	3	13	19	3	7	11	3	P
41	P	3	P	P	3	P	19	3	7	P	3	29	P	3	17	7	3	23	P	3
43	7	13	3	157	47	3	P	7	3	89	P	3	109	11	3	17	73	3	P	P
47	17	3	13	P	3	7	31	3	37	11	3	P	7	3	53	13	3	17	P	3
49	P	17	3	47	7	3	P	11	3	13	137	3	19	P	3	P	23	3	7	31
51	3	31	17	3	19	11	3	61	29	3	7	P	3	73	P	3	11	7	3	17
53	89	3	7	11	3	P	P	3	P	7	3	P	41	3	11	19	3	173	47	3
57	3	P	139	3	67	17	3	7	P	3	11	13	3	P	7	3	P	P	3	37
59	P	3	P	127	3	13	7	3	11	23	3	P	P	3	181	P	3	47	13	3
61	251	47	3	23	31	3	11	17	3	281	173	3	7	61	3	P	37	3	P	7
63	3	P	61	3	7	251	3	79	17	3	P	7	3	19	229	3	29	31	3	13
67	11	P	3	P	P	3	97	13	3	7	17	3	31	P	3	251	7	3	P	P
69	3	7	23	3	131	P	3	227	7	3	37	17	3	139	13	3	P	P	3	211
71	7	3	29	109	3	P	151	3	13	157	3	41	17	3	7	47	3	241	11	3
73	101	P	3	181	97	3	7	37	3	151	107	3	P	7	3	13	11	3	P	P
77	163	3	P	13	3	P	29	3	P	3	P	3	7	11	3	19	17	3	P	3
79	P	P	3	7	P	3	19	P	3	P	7	3	P	P	3	P	17	3	23	P
81	3	37	7	3	13	179	3	P	11	3	31	P	3	163	P	3	7	13	3	11
83	113	3	P	103	3	P	11	3	7	19	3	13	P	3	61	7	3	11	17	3
87	3	41	11	3	P	89	3	P	P	3	P	P	3	7	101	3	P	23	3	P
89	11	3	79	43	3	7	13	3	P	P	3	11	7	3	29	P	3	73	P	3
91	13	P	3	277	7	3	P	P	3	11	139	3	37	13	3	19	P	3	7	41
93	3	P	59	3	53	P	3	11	P	3	7	P	3	P	P	3	P	7	3	167
97	29	7	3	11	P	3	P	P	3	197	19	3	179	P	3	7	P	3	109	P
99	3	P	13	3	23	53	3	7	257	3	83	29	3	P	7	3	P	199	3	P

	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819
01	3	7	11	3	37	79	3	P	7	3	P	P	3	11	P	3	13	P	3	P
03	7	3	139	131	3	19	P	3	P	17	3	11	P	3	7	149	3	P	179	3
07	3	P	P	3	P	7	3	11	19	3	59	13	3	P	127	3	79	P	3	7
09	19	3	P	P	3	11	149	3	P	P	3	7	17	3	P	P	3	101	7	3
11	29	P	3	7	191	3	P	43	3	P	7	3	13	17	3	37	P	3	23	101
13	3	11	7	3	97	P	3	P	211	3	P	29	3	31	17	3	7	41	3	13
17	7	113	3	P	29	3	19	7	3	P	P	3	241	233	3	P	17	3	P	11
19	3	13	97	3	137	73	3	53	P	3	P	P	3	7	13	3	P	11	3	P
21	P	3	P	31	3	7	P	3	13	19	3	23	7	3	P	11	3	71	17	3
23	43	19	3	47	7	3	37	89	3	P	3	P	11	3	13	31	3	7	17	3
27	79	3	7	13	3	P	P	3	131	7	3	31	43	3	107	P	3	P	47	3
29	191	7	3	P	P	3	P	11	3	P	13	3	29	167	3	7	P	3	11	P
31	3	227	P	3	13	11	3	7	P	3	P	P	3	P	7	3	11	13	3	P
33	163	3	P	67	3	29	7	3	P	P	3	13	P	3	11	P	3	37	19	3
37	3	127	19	3	7	P	3	P	229	3	11	7	3	163	31	3	P	P	3	P
39	P	3	P	7	3	43	13	3	11	29	3	41	P	3	P	67	3	7	P	3
41	13	P	3	P	257	3	11	263	3	7	P	3	137	13	3	73	7	3	223	67
43	3	7	29	3	11	239	3	13	7	3	P	53	3	P	23	3	19	43	3	P
47	11	P	3	P	P	3	7	P	3	61	P	3	113	7	3	P	P	3	P	19
49	3	P	13	3	P	7	3	P	P	3	7	19	3	P	79	3	P	P	3	7
51	P	3	P	19	3	109	P	3	233	13	3	7	31	3	47	P	3	29	7	3
53	17	P	3	7	43	3	59	23	3	P	7	3	193	P	3	P	11	3	P	P
57	223	3	17	107	3	P	P	3	7	73	3	P	11	3	P	7	3	13	23	3
59	7	71	3	17	61	3	79	7	3	19	11	3	23	P	3	P	37	3	109	41
61	3	19	83	3	17	13	3	P	11	3	103	277	3	7	29	3	127	P	3	11
63	23	3	P	P	3	7	P	3	P	P	3	P	7	3	P	P	3	11	71	3
67	3	P	11	3	67	P	3	17	193	3	7	23	3	11	41	3	P	7	3	P
69	11	3	7	P	3	23	P	3	17	7	3	11	181	3	257	P	3	P	P	3
71	P	7	3	179	P	3	P	37	3	11	P	3	67	P	3	7	P	3	19	P
73	3	P	P	3	P	197	3	7	13	3	17	P	3	P	7	3	23	P	3	P
77	P	P	3	11	23	3	P	3	13	P	3	7	19	3	29	P	3	41	P	7
79	3	11	P	3	7	19	3	P	31	3	89	7	3	17	59	3	13	53	3	73
81	73	3	43	7	3	61	P	3	29	47	3	P	P	3	17	23	3	7	37	3
83	53	181	3	31	13	3	P	P	3	7	P	3	P	97	3	17	7	3	P	11
87	7	3	P	P	3	13	P	3	47	109	3	19	29	3	7	11	3	17	13	3
89	283	17	3	19	P	3	7	P	3	P	131	3	13	7	3	83	P	3	17	163
91	3	P	17	3	P	7	3	173	23	3	83	11	3	199	19	3	151	89	3	7
93	13	3	23	17	3	83	19	3	41	11	3	7	P	3	227	139	3	263	7	3
97	3	13	7	3	101	11	3	43	P	3	P	3	P	3	23	13	3	7	157	3
99	173	3	59	11	3	P	17	3	7	107	3	P	P	3	3	11	7	P	P	3

	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839
01	43	3	7	P	3	17	P	3	31	7	3	P	19	3	P	11	3	P	47	3
03	P	7	3	13	19	3	17	191	3	P	P	3	P	11	3	7	13	3	181	1
07	P	3	P	P	3	P	7	3	17	11	3	41	P	3	P	113	3	13	43	3
09	P	47	3	53	23	3	P	11	3	17	P	3	7	227	3	37	P	3	11	7
11	3	157	229	3	7	11	3	107	P	3	17	7	3	P	239	3	11	97	3	P
13	P	3	19	7	3	109	P	3	P	P	3	17	13	3	11	23	3	7	P	3
17	3	7	P	3	73	19	3	181	7	3	11	P	3	13	P	3	P	P	3	31
19	7	3	P	263	3	179	P	3	11	283	3	43	P	3	7	47	3	P	79	3
21	P	13	3	191	P	3	7	P	3	101	61	3	P	7	3	17	P	3	109	P
23	3	41	P	3	11	7	3	P	13	3	P	103	3	97	P	3	17	29	3	7
27	11	17	3	7	139	3	53	P	3	13	7	3	P	103	3	101	241	3	17	23
29	3	P	7	3	31	P	3	P	113	3	79	97	3	23	19	3	7	101	3	17
31	P	3	P	17	3	P	19	3	7	127	3	59	P	3	P	7	3	31	11	3
33	7	23	3	281	13	3	P	7	3	239	43	3	P	167	3	103	11	3	P	P
37	P	3	P	137	3	7	17	3	P	197	43	3	P	7	3	P	P	3	13	3
39	P	P	3	P	7	3	23	17	3	P	11	3	13	P	3	139	P	3	7	P
41	3	P	P	3	19	59	3	97	11	3	7	71	3	P	181	3	P	7	3	11
43	13	3	7	67	3	197	11	3	37	7	3	29	P	3	P	19	3	11	P	3
47	3	13	11	3	29	23	3	7	P	3	P	17	3	11	7	3	233	83	3	127
49	11	3	233	P	3	P	7	3	13	109	3	11	17	3	P	29	3	89	191	3
51	P	113	3	P	41	3	P	83	3	11	53	3	7	17	3	13	23	3	71	3
53	3	P	83	3	7	31	3	11	29	3	23	7	3	19	17	3	P	61	3	37
57	31	29	3	11	P	3	P	P	3	7	13	3	P	P	3	P	7	3	P	59
59	3	7	43	3	13	P	3	P	7	3	P	137	3	31	P	3	269	13	3	113
61	7	3	P	P	3	P	131	3	41	23	3	13	139	3	7	P	3	P	17	3
63	137	P	3	23	P	3	7	P	3	P	P	3	53	7	3	P	P	3	13	11
67	P	3	P	31	3	P	13	3	173	163	3	7	P	3	19	11	3	211	7	3
69	13	127	3	7	P	3	19	37	3	29	7	3	P	11	3	193	31	3	P	P
71	3	P	7	3	P	3	13	79	3	P	11	3	263	P	3	7	19	3	131	3
73	P	3	29	P	3	71	47	3	7	11	3	31	P	3	13	7	3	P	P	3
77	3	37	13	3	67	11	3	23	179	3	P	3	7	P	3	11	P	3	79	3
79	211	3	P	11	3	7	29	3	67	13	3	223	7	3	11	P	3	199	37	3
81	79	11	3	13	7	3	89	P	3	P	251	3	11	199	3	19	13	3	7	137
83	3	P	107	3	P	269	3	19	P	3	7	193	3	P	31	3	67	7	3	P
87	23	7	3	P	P	3	11	P	3	31	19	3	37	61	3	7	53	3	149	P
89	3	P	19	3	11	13	3	7	P	3	P	41	3	P	7	3	23	3	47	3
91	103	3	11	47	3	P	7	3	P	37	3	23	13	3	29	P	3	P	P	3
93	11	P	3	P	P	3	13	P	3	149	P	3	7	89	3	179	127	3	43	7
97	53	3	17	7	3	151	41	3	19	P	3	271	31	3	P	P	3	7	11	3
99	19	13	3	17	P	3	P	P	3	7	23	3	P	P	3	41	7	3	53	19

	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859
01	167	37	3	7	P	3	11	P	3	59	7	3	P	197	3	13	P	3	239	17
03	3	31	7	3	21	P	3	71	137	3	167	P	3	P	41	3	7	P	3	P
07	7	151	3	P	P	3	19	7	3	197	13	3	139	23	3	37	P	3	53	271
09	3	241	107	3	13	P	3	23	P	3	P	3	7	223	3	59	13	3	P	
11	P	3	P	57	3	7	211	3	P	19	3	13	7	3	P	233	3	P	11	3
13	29	19	3	P	7	3	191	P	3	P	151	3	P	P	3	P	11	3	7	53
17	P	3	7	P	3	223	13	3	89	7	3	47	11	3	229	P	3	P	P	3
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21	3	P	P	3	P	P	3	7	11	3	P	P	3	41	7	3	P	23	3	11
23	73	3	P	37	3	P	7	3	271	163	3	23	P	3	13	P	3	11	19	3
27	3	P	11	3	7	181	3	193	P	3	P	7	3	11	P	3	P	59	3	29
29	11	3	P	7	3	137	P	3	41	13	3	11	P	3	P	31	3	7	P	3
31	17	P	3	13	P	3	P	P	3	7	23	3	29	P	3	P	7	3	P	P
33	3	7	131	3	23	P	3	11	7	3	13	P	3	P	37	3	19	P	3	P
37	19	P	3	11	P	3	7	P	3	157	P	3	P	7	3	23	29	3	P	19
39	3	11	P	3	17	7	3	101	43	3	277	19	3	61	P	3	P	83	3	7
41	31	3	61	19	3	17	53	3	37	29	3	7	13	3	43	123	3	179	7	3
43	229	P	3	7	P	3	13	83	3	173	7	3	P	31	3	131	P	3	P	11
47	P	3	P	P	3	59	47	3	7	P	3	P	P	3	P	7	3	19	P	3
49	7	13	3	P	P	3	P	7	3	17	P	3	163	11	3	P	41	3	293	61
51	3	19	173	3	79	P	3	P	13	3	17	11	3	7	P	3	97	P	3	23
53	P	3	13	67	3	7	P	3	53	11	3	17	7	3	P	13	3	29	P	3
57	3	23	109	3	P	11	3	131	P	3	7	31	3	17	97	3	11	7	3	43
59	P	3	7	11	3	P	P	3	P	7	3	P	P	3	11	67	3	191	23	3
61	P	7	3	29	13	3	31	P	3	P	P	3	11	P	3	7	P	3	19	67
63	3	P	P	3	P	103	3	7	113	3	11	13	3	P	7	3	17	139	3	31
67	P	17	3	239	P	3	11	29	3	P	257	3	7	19	3	41	P	3	17	7
69	3	73	17	3	7	19	3	103	P	3	97	7	3	P	P	3	P	199	3	13
71	13	3	11	7	3	23	227	3	P	31	3	53	71	3	127	P	3	7	43	3
73	11	41	3	139	17	3	P	13	3	7	241	3	269	59	3	83	7	3	79	149
77	7	3	71	P	3	83	17	3	13	P	3	19	53	3	7	P	3	31	11	3
79	83	P	3	19	23	3	7	17	3	P	149	3	107	7	3	13	11	3	157	127
81	3	P	271	3	P	7	3	149	17	3	P	103	3	P	11	3	47	P	3	7
83	47	3	89	13	3	41	19	3	29	17	3	7	11	3	73	23	3	109	7	3
87	3	29	7	3	15	251	3	P	11	3	P	17	3	103	P	3	7	13	3	11
89	P	3	31	P	3	P	11	3	7	37	3	13	17	3	53	7	3	11	P	3
91	7	P	3	P	11	3	P	7	3	P	P	3	19	17	3	11	P	3	13	P
93	3	59	11	3	19	29	3	P	23	3	P	P	3	7	17	3	67	P	3	113
97	13	269	3	37	7	3	P	19	3	11	43	3	P	13	3	P	17	3	7	23
99	3	P	P	3	3	31	3	11	73	3	7	P	3	23	193	3	43	7	3	P

	860	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879
01	3	29	P	3	7	P	3	27	11	3	19	7	3	67	71	3	17	P	3	11
03	17	3	13	7	3	23	11	3	61	43	3	P	29	3	P	13	3	7	P	3
07	3	7	11	3	71	19	3	31	7	3	167	P	3	11	P	3	13	229	3	17
09	7	3	P	17	3	P	257	3	47	233	3	11	3	3	7	P	3	139	277	3
11	P	P	3	P	13	3	7	P	3	11	P	3	P	7	3	P	79	3	P	P
13	3	P	73	3	P	7	3	11	P	3	P	13	3	P	61	3	P	239	3	7
17	P	P	3	7	103	3	37	17	3	23	7	3	13	P	3	P	41	3	137	P
19	3	11	7	3	89	241	3	P	17	3	173	P	3	29	19	3	7	P	3	13
21	13	3	151	37	3	31	19	3	7	17	3	P	P	3	P	7	3	P	53	3
23	7	71	3	P	P	3	29	7	3	P	17	3	P	P	3	P	P	3	31	11
27	P	3	23	173	3	7	P	3	13	P	3	151	7	3	P	11	3	37	71	3
29	P	43	3	131	7	3	P	P	3	P	29	3	19	11	3	13	P	3	7	23
31	3	P	53	3	19	P	3	43	31	3	7	11	3	23	17	3	P	7	3	P
33	227	3	7	13	3	P	41	3	71	7	3	P	83	3	P	17	3	59	P	3
37	3	P	83	3	13	11	3	7	P	3	P	79	3	P	7	3	11	13	3	47
39	97	3	P	11	3	P	7	3	37	P	3	13	23	3	11	P	3	P	17	3
41	139	11	3	P	P	3	23	127	3	227	P	3	7	167	3	P	P	3	13	7
43	3	P	P	3	7	37	3	P	P	3	11	7	3	19	P	3	P	P	3	P
47	13	277	3	79	137	3	11	223	3	7	61	3	43	13	3	P	7	3	107	31
49	3	7	P	3	11	23	3	13	7	3	P	P	3	113	157	3	P	47	3	37
51	7	3	11	P	3	41	73	3	P	P	3	P	P	3	7	29	3	P	59	3
53	11	101	3	P	P	3	7	P	3	89	263	3	P	7	3	P	23	3	P	281
57	47	3	P	P	3	101	193	3	P	13	3	7	P	3	19	P	3	127	7	3
59	41	29	3	7	31	3	19	101	3	P	7	3	71	P	3	P	11	3	103	P
61	3	P	7	3	P	P	3	53	P	3	13	43	3	199	11	3	7	19	3	P
63	89	3	P	67	3	107	79	3	7	19	3	101	11	3	149	7	3	13	41	3
67	3	199	281	3	P	13	3	P	11	3	83	67	3	7	47	3	29	P	3	11
69	P	3	P	P	3	7	11	3	P	P	3	61	7	3	23	67	3	11	P	3
71	17	P	3	P	7	3	13	P	3	25	P	3	197	41	3	11	P	3	7	13
73	3	17	11	3	43	P	3	19	109	3	7	179	3	11	P	3	73	7	3	P
77	P	7	3	17	P	3	P	107	3	11	19	3	P	23	3	7	43	3	P	P
79	3	P	19	3	17	P	3	7	13	3	31	P	3	59	7	3	P	61	3	97
81	59	3	13	P	3	11	7	3	283	P	3	P	P	3	P	13	3	41	P	3
83	P	P	3	11	197	3	17	P	3	13	P	3	7	P	3	P	P	3	23	7
87	31	3	P	7	3	P	23	3	17	37	3	P	191	3	89	P	3	7	P	3
89	19	79	3	P	13	3	P	59	3	7	73	3	41	31	3	P	7	3	179	11
91	3	7	P	3	P	131	3	229	7	3	17	13	3	281	P	3	P	11	3	P
93	7	3	P	19	3	13	P	3	31	P	3	17	P	3	7	11	3	P	13	3
97	3	P	P	3	67	7	3	29	113	3	251	11	3	17	59	3	P	P	3	7
99	13	3	211	P	3	P	281	3	67	11	3	7	P	3	17	251	3	19	7	3

	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899
01	P	3	193	P	3	7	41	3	P	19	3	P	7	3	13	P	3	271	89	3
03	P	19	3	227	7	3	251	107	3	P	P	3	P	3	37	P	3	3	7	11
07	P	3	7	233	3	67	P	3	P	7	3	P	37	3	29	11	3	109	3	3
09	17	7	3	13	211	3	P	43	3	67	P	3	P	11	3	7	13	3	P	P
11	3	17	P	3	P	61	3	7	P	3	13	11	3	31	7	3	P	283	3	47
13	283	3	17	47	3	P	7	3	P	11	3	P	P	3	P	P	3	13	19	3
17	3	P	19	3	7	11	3	79	P	3	P	7	3	P	P	3	11	73	3	P
19	P	3	47	7	3	17	23	3	P	P	3	P	13	3	11	P	3	7	P	3
21	23	11	3	P	29	3	13	P	3	7	P	3	11	179	3	P	7	3	P	13
23	3	7	P	3	P	P	3	17	7	3	11	P	3	13	223	3	19	23	3	P
27	19	13	3	P	P	3	7	83	3	17	127	3	P	7	3	P	3	43	3	19
29	3	P	83	3	11	7	3	P	13	3	17	19	3	P	37	3	47	53	3	7
31	47	3	11	19	3	223	263	3	211	113	3	7	P	3	P	13	3	61	7	3
33	11	31	3	7	191	3	61	89	3	13	7	3	17	157	3	P	P	3	P	139
37	P	3	P	P	3	29	151	3	7	P	3	P	P	3	17	7	3	19	11	3
39	7	53	3	P	13	3	137	7	3	19	269	3	233	41	3	17	11	3	P	P
41	3	19	P	3	59	37	3	P	73	3	P	13	3	7	11	3	17	43	3	53
43	17	3	79	23	3	7	P	3	P	29	3	97	7	3	P	151	3	17	13	3
47	3	181	17	3	241	P	3	P	11	3	7	239	3	47	23	3	157	7	3	11
49	13	3	7	17	3	73	11	3	23	7	3	59	31	3	P	149	3	11	P	3
51	191	7	3	53	11	3	P	13	3	P	P	3	149	199	3	7	37	3	19	293
53	8	13	11	3	197	17	3	7	P	3	19	P	3	11	7	3	P	P	3	23
57	173	199	3	149	53	3	P	17	3	11	P	3	7	19	3	13	P	3	59	7
59	3	23	P	3	7	19	3	11	17	3	29	7	3	193	P	3	P	P	3	P
61	107	3	P	7	3	11	P	3	P	17	3	163	P	3	137	P	3	7	23	3
63	83	131	3	11	P	3	P	37	3	7	13	3	23	P	3	P	7	3	73	P
67	7	3	61	97	3	31	P	3	P	43	3	13	17	3	7	P	3	P	P	3
69	P	P	3	19	P	3	7	29	3	P	P	3	P	7	3	43	P	3	13	11
71	3	37	103	3	P	7	3	P	181	3	P	23	3	P	17	3	P	11	3	7
73	29	3	41	67	3	23	13	3	P	193	3	7	P	3	131	11	3	107	7	3
77	3	P	7	3	103	101	3	13	31	3	281	11	3	139	P	3	7	17	3	P
79	P	3	43	P	3	283	71	3	7	11	3	257	73	3	13	7	3	P	17	3
81	7	109	3	31	23	3	P	7	3	101	229	3	19	P	3	29	P	3	11	17
83	3	163	13	3	19	11	3	47	P	3	P	101	3	7	43	3	11	P	3	P
87	59	11	3	13	7	3	131	19	3	23	P	3	11	P	3	101	13	3	7	29
89	3	29	P	3	107	P	3	P	103	3	7	P	3	71	109	3	P	7	3	P
91	137	7	7	157	3	P	31	3	11	7	3	79	29	3	P	P	3	13	P	3
93	P	3	3	37	P	3	11	3	P	3	P	41	3	P	3	7	257	3	241	31
97	37	3	11	P	3	19	7	3	P	P	3	191	13	3	31	P	3	P	P	3
99	11	89	3	109	P	3	13	P	3	61	139	3	7	P	3	P	19	3	P	7

	900	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919
01	P	11	3	73	P	3	7	13	3	P	17	3	11	7	3	37	139	3	P	29
03	3	13	P	5	P	7	3	P	P	3	11	17	3	P	13	5	47	P	3	7
07	P	P	3	7	P	3	11	61	3	P	7	3	223	17	3	13	101	P	P	73
09	3	251	7	3	11	29	3	P	71	3	P	31	3	P	17	3	203	3	P	P
11	P	3	11	13	3	P	19	3	7	P	3	179	197	3	P	7	3	P	P	3
13	7	97	3	P	23	3	31	7	3	229	13	3	53	127	3	P	17	3	P	107
17	P	3	P	57	3	7	P	3	197	P	3	13	7	3	113	23	3	41	11	3
19	P	227	3	131	7	3	P	83	3	23	P	3	19	53	3	71	11	3	7	17
21	3	P	83	3	19	131	5	257	P	3	7	P	3	29	11	3	P	7	5	P
23	P	3	7	41	3	P	13	3	P	7	3	203	11	3	P	19	3	37	P	3
27	3	P	P	3	31	P	3	7	11	5	227	P	3	271	7	3	59	29	3	11
29	197	3	23	59	3	P	7	3	61	79	3	P	P	3	13	P	3	11	229	3
31	P	193	3	103	11	3	P	P	3	P	29	3	7	P	3	11	P	3	131	7
33	3	173	11	3	7	P	3	41	P	3	P	7	3	11	P	3	43	P	3	149
37	179	23	3	13	P	3	233	31	3	7	59	3	P	149	3	239	7	3	P	89
39	3	7	P	3	P	37	3	11	7	3	13	P	3	241	3	P	199	3	P	P
41	7	3	31	61	3	11	P	3	P	211	3	P	23	3	7	P	3	13	P	3
43	127	109	3	11	149	3	7	103	3	199	181	3	P	7	3	31	113	3	29	P
47	53	3	P	167	3	P	P	3	P	P	3	7	13	3	19	43	3	23	7	3
49	17	P	3	7	151	3	13	P	3	103	7	3	P	167	3	89	37	3	53	11
51	3	17	7	3	29	23	3	151	47	3	83	P	3	13	109	3	7	11	3	P
53	P	3	17	P	3	83	269	3	7	19	3	P	P	3	P	7	3	P	31	3
57	3	89	43	3	17	137	3	47	13	3	23	11	3	7	P	3	151	P	3	P
59	P	3	13	P	3	7	P	3	43	11	3	P	7	3	P	13	3	89	97	3
61	113	29	3	109	7	3	17	11	3	13	41	3	263	103	3	19	71	3	7	P
63	3	P	P	3	61	11	3	17	P	3	7	P	3	211	P	3	11	7	3	41
67	P	7	3	23	13	3	71	139	3	17	19	3	11	P	3	7	31	3	P	P
69	3	37	19	3	P	41	3	7	89	3	11	13	3	P	7	3	29	163	3	P
71	P	3	P	P	3	13	7	3	11	P	3	17	107	3	23	P	3	P	13	3
73	P	P	3	P	P	3	11	43	3	29	61	3	7	P	3	P	P	3	P	7
77	13	3	11	7	3	53	P	3	19	P	3	73	97	3	17	P	3	7	79	3
79	11	31	3	P	173	3	P	13	3	7	P	3	37	23	3	17	7	3	139	19
81	3	7	P	3	P	239	3	23	7	3	P	19	3	P	13	3	17	P	3	59
83	7	3	137	19	3	P	29	3	13	37	3	P	P	3	7	P	3	17	11	3
87	3	P	17	3	41	7	3	P	P	3	79	67	3	P	11	3	277	263	3	7
89	P	3	P	13	5	157	23	3	97	P	3	P	11	3	191	67	3	19	7	3
91	23	P	3	7	17	3	89	163	3	19	7	3	P	59	3	P	P	3	43	67
93	3	19	7	3	13	17	3	P	11	3	71	P	3	P	P	3	7	23	3	11
97	7	P	3	P	11	3	P	7	3	P	P	3	P	P	3	11	47	3	13	P
99	3	P	11	3	P	P	3	29	17	3	P	P	3	7	P	3	107	43	3	197

	920	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936	937	938	939
01	3	31	137	3	P	233	3	3	P	8	P	151	3	13	7	3	P	P	3	P
03	P	3	P	241	3	P	7	7	17	61	3	P	11	3	23	P	3	P	19	3
07	P	3	P	19	3	7	P	3	P	11	3	17	7	3	P	P	3	83	3	13
09	P	3	13	7	3	29	11	3	P	53	3	17	83	3	29	13	3	7	P	3
11	101	P	3	P	11	3	37	83	3	7	281	3	17	23	3	11	7	3	P	P
13	3	7	11	3	P	71	3	23	7	3	47	P	3	11	109	3	13	31	3	P
17	19	251	3	P	13	3	7	P	3	11	191	3	31	7	3	17	179	3	23	19
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21	17	3	P	19	3	11	23	3	P	P	3	7	73	3	103	41	3	17	7	3
23	23	17	3	7	29	8	P	P	3	43	7	3	13	P	3	P	251	3	17	P
27	13	3	P	17	3	67	P	3	7	P	3	23	53	3	P	7	3	19	P	3
29	7	181	3	127	17	3	211	7	3	19	41	3	P	P	3	P	P	3	101	11
31	3	13	149	3	P	17	3	47	P	3	31	P	3	7	13	3	109	11	3	29
33	P	3	P	P	3	7	17	3	13	199	3	P	7	3	233	11	3	67	103	3
37	3	199	P	3	23	37	3	P	17	3	7	11	3	P	223	3	P	7	3	P
39	31	3	7	13	3	29	P	3	263	7	3	P	P	3	41	89	3	107	P	3
41	P	7	3	107	97	3	P	11	3	P	13	3	P	31	3	7	29	3	11	P
43	3	P	P	3	13	11	3	7	227	3	19	17	3	269	7	3	11	13	3	37
47	83	11	3	P	191	3	P	163	3	41	P	3	7	17	3	139	37	3	13	7
49	3	41	29	3	7	19	3	137	P	3	11	7	3	277	17	3	71	241	3	P
51	P	3	P	7	3	P	13	3	11	P	3	P	P	3	113	17	3	7	P	3
53	13	P	3	P	59	3	11	P	3	7	P	3	P	13	3	P	7	3	127	47
57	7	3	11	P	3	P	P	3	P	P	3	19	P	3	7	P	3	29	17	3
59	11	157	3	19	P	3	7	23	3	P	P	3	179	7	3	P	73	3	47	17
61	3	23	13	3	P	7	3	P	P	3	29	52	3	89	19	3	229	P	3	7
63	43	3	257	P	3	151	19	3	P	13	3	7	P	3	P	P	3	P	7	3
67	3	37	7	3	P	P	3	P	P	3	13	151	3	73	11	3	7	41	3	P
69	23	3	P	P	3	P	P	3	7	31	3	P	11	3	151	7	3	13	37	3
71	7	61	3	71	89	3	P	7	3	239	11	3	19	P	3	137	47	3	P	P
73	P	P	53	3	19	13	3	163	11	3	163	23	3	7	211	3	283	79	3	11
77	P	P	3	P	7	3	13	19	3	109	P	3	37	P	3	11	113	3	7	13
79	3	P	11	3	P	43	3	P	131	3	7	P	3	11	P	3	23	7	3	P
81	11	3	7	P	3	P	P	3	293	7	3	11	P	3	P	P	3	191	269	3
83	P	7	3	P	23	3	P	31	3	11	P	3	P	P	3	7	P	3	223	P
87	71	3	13	P	3	11	7	3	29	P	3	P	P	3	P	13	3	P	P	3
89	17	P	3	11	P	3	59	P	3	13	P	3	7	47	3	51	19	3	P	7
91	3	11	41	3	7	53	3	P	19	3	127	7	3	61	P	3	13	71	3	191
93	19	3	17	3	P	3	P	P	3	41	29	3	P	173	3	P	3	43	11	3
97	3	7	P	3	17	29	P	71	7	3	P	13	3	59	P	3	3	11	3	P
99	7	3	23	P	3	13	P	3	P	113	3	P	79	3	7	11	3	97	13	3

	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959
01	23	3	P	181	3	11	13	3	7	43	3	P	31	3	P	7	3	P	P	3
03	7	139	3	11	67	3	P	7	3	P	P	3	P	13	3	43	P	3	P	29
07	P	P	P	P	3	7	89	3	113	P	3	P	7	3	13	P	3	P	149	3
09	P	P	3	P	7	3	37	P	3	107	P	3	19	191	3	149	67	3	7	11
11	3	P	13	3	19	29	3	53	P	3	7	P	3	P	73	3	23	7	3	P
13	41	3	7	37	3	P	P	3	59	7	3	227	P	3	P	11	3	P	P	3
17	3	P	71	3	263	47	3	7	53	3	13	11	3	P	7	3	P	P	3	P
19	149	3	P	257	3	31	7	3	P	11	3	73	P	3	P	23	3	13	P	3
21	167	P	3	P	P	3	P	11	3	23	P	3	7	199	3	59	P	3	11	7
23	3	61	59	3	7	11	3	P	P	3	167	7	3	19	37	3	11	P	3	P
27	17	11	3	P	3	3	13	P	3	7	P	3	11	P	3	P	7	3	79	13
29	3	7	P	3	89	P	3	43	7	3	11	251	3	13	P	3	P	29	3	P
31	7	3	17	P	3	P	173	3	11	59	3	P	P	3	7	P	3	P	61	3
33	P	13	3	17	P	3	7	61	3	P	29	3	P	7	3	83	P	3	47	23
37	271	3	11	29	3	17	101	3	P	139	3	7	131	3	19	13	3	P	7	3
39	11	23	3	7	P	3	17	211	3	13	7	3	P	P	3	P	59	3	239	197
41	3	47	7	3	P	P	3	17	P	3	101	89	3	67	P	3	7	19	3	37
43	157	3	73	P	3	P	31	3	7	19	3	P	23	3	P	7	3	67	11	3
47	3	31	79	3	P	P	3	P	P	3	17	13	3	7	11	3	101	P	3	P
49	P	3	307	P	3	7	P	3	P	P	3	17	7	3	31	P	3	23	13	3
51	163	P	3	P	7	3	P	41	3	P	11	3	13	97	3	19	P	3	7	229
53	3	P	P	3	29	23	3	19	11	3	7	P	3	17	53	3	41	7	3	11
57	P	7	3	157	11	3	103	13	3	269	19	3	P	167	3	7	23	3	P	P
59	3	13	11	3	59	P	3	7	29	3	23	43	3	11	7	3	17	31	3	P
61	11	3	P	127	3	P	7	3	13	P	3	11	P	3	P	P	3	17	257	3
63	P	17	3	197	P	3	181	193	3	11	P	3	7	47	3	13	271	3	17	7
67	109	3	107	7	3	11	137	3	19	23	3	59	P	3	P	227	3	7	37	3
69	19	P	3	11	17	3	41	97	3	7	13	3	47	P	3	P	7	3	P	19
71	3	7	31	3	13	17	3	P	7	3	P	19	3	281	P	3	29	13	3	P
73	7	3	P	19	3	P	17	3	P	73	3	13	P	3	7	31	3	P	P	3
77	3	41	23	3	P	7	3	P	17	3	31	P	3	127	307	3	241	11	3	7
79	P	3	29	P	3	271	13	3	79	17	3	7	P	3	P	11	3	19	7	3
81	13	53	3	7	107	3	73	P	3	19	7	3	151	11	3	P	163	3	P	41
83	3	19	7	3	P	P	3	13	239	3	P	11	3	P	P	3	7	P	3	53
87	7	97	3	37	19	3	P	7	3	43	P	3	P	17	3	61	103	3	11	P
89	3	131	13	3	61	11	3	P	P	3	P	P	3	7	17	3	11	P	3	P
91	37	3	P	11	3	7	23	3	31	13	3	P	7	3	11	17	3	P	P	3
93	23	11	3	13	7	3	P	P	3	P	P	3	11	P	3	109	13	3	7	59
97	73	3	7	P	3	P	281	3	11	7	3	23	233	3	29	P	3	P	17	3
99	P	7	3	P	53	3	11	47	3	P	61	3	157	19	3	7	83	3	41	17

	960	961	962	963	964	965	966	967	968	969	970	971	972	973	974	975	976	977	978	979
01	p	17	3	23	p	3	p	11	3	7	p	3	13	p	3	p	7	3	11	47
03	3	7	17	3	149	11	3	p	7	3	p	p	3	p	257	3	11	41	3	3
07	19	11	3	193	17	3	7	13	3	p	p	3	11	7	3	281	p	3	47	19
09	3	13	23	3	229	7	3	97	131	3	11	19	3	31	13	3	p	199	3	7
11	67	3	p	19	3	103	17	3	11	p	3	7	41	3	29	p	3	p	7	3
13	p	223	3	7	67	3	11	17	3	199	7	3	p	23	3	13	p	3	p	179
17	p	3	11	13	3	p	79	3	7	17	3	p	67	3	61	7	3	19	29	3
19	7	277	3	61	p	3	53	7	3	19	13	3	191	307	3	113	31	3	23	p
21	3	19	p	3	13	263	3	311	p	3	p	17	3	7	37	3	41	13	3	181
23	131	3	p	p	3	7	23	3	p	103	3	13	7	3	p	p	3	79	11	3
27	3	97	41	3	211	p	3	197	p	3	7	p	3	p	11	3	233	7	3	p
29	109	3	7	p	3	83	13	3	37	7	3	23	11	3	p	17	3	p	3	3
31	13	7	3	p	p	3	71	p	3	p	11	3	p	13	3	7	17	3	19	p
33	3	251	p	3	73	37	3	7	11	3	19	137	3	131	7	3	89	17	3	11
37	137	p	3	p	11	3	41	p	3	31	23	3	7	19	3	11	163	3	227	7
39	3	127	11	3	7	19	3	p	179	3	p	7	3	11	139	3	251	43	3	37
41	11	3	157	7	3	29	241	3	113	13	3	11	p	3	p	103	3	7	p	3
43	p	79	3	13	p	3	p	89	3	7	53	3	47	311	3	23	7	3	p	p
47	7	3	109	23	3	11	127	3	p	29	3	19	31	3	7	p	3	13	p	3
49	139	p	3	11	43	3	7	p	3	67	107	3	79	7	3	p	p	3	p	41
51	3	11	29	3	p	7	3	31	p	3	37	p	3	67	19	3	p	239	3	7
53	p	3	101	p	3	p	19	3	23	p	3	7	13	3	p	p	3	67	7	3
57	3	p	7	3	p	p	3	p	p	3	71	p	3	13	41	3	7	11	3	23
59	p	3	p	167	3	223	163	3	7	p	3	p	p	3	p	7	3	29	p	3
61	7	13	3	173	p	3	p	7	3	47	31	3	19	11	3	p	61	3	p	p
63	3	23	p	3	19	61	3	p	13	3	29	11	3	7	p	3	127	59	3	163
67	17	p	3	29	7	3	p	11	3	13	113	3	23	p	3	43	101	3	7	p
69	3	17	p	3	p	11	3	p	157	3	7	p	3	p	29	3	11	7	3	113
71	23	3	7	11	3	269	p	3	73	7	3	p	211	3	11	p	3	p	p	3
73	191	7	3	17	13	3	277	29	3	p	p	3	11	p	3	7	p	3	97	p
77	29	3	43	p	3	13	7	3	11	37	3	p	89	3	107	p	3	p	13	3
79	p	p	3	31	p	3	11	p	3	p	193	3	7	p	3	p	19	3	p	7
81	3	p	p	3	7	p	3	17	19	3	p	7	3	p	43	3	23	277	3	13
83	13	3	11	7	3	59	109	3	17	293	3	157	p	3	71	p	3	7	p	3
87	3	7	73	3	p	p	3	p	7	3	17	p	3	p	13	3	p	p	3	p
89	7	3	p	113	3	p	31	3	13	p	3	17	271	3	7	23	3	p	11	3
91	307	43	3	41	47	3	7	151	3	23	79	3	17	7	3	13	11	3	53	29
93	3	29	p	3	p	7	3	43	p	3	151	83	3	17	11	3	211	19	3	7
97	p	19	3	7	p	3	p	p	3	p	7	3	149	p	3	17	151	3	223	43
99	3	p	7	3	13	29	3	p	11	3	89	37	3	173	p	3	7	13	3	11

	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999
91	3	P	283	3	19	13	3	89	P	3	7	113	3	199	P	3	103	7	3	P
93	23	3	7	197	3	137	151	3	29	7	3	P	13	3	107	19	3	179	11	3
97	3	17	P	3	P	P	3	7	P	3	181	23	3	13	7	3	P	P	3	P
99	P	3	1	37	3	23	7	3	P	3	P	3	P	11	3	P	151	3	P	3
11	P	13	3	17	P	3	31	P	3	P	11	3	7	47	3	191	P	3	151	7
13	3	41	P	3	7	29	3	P	11	3	P	7	3	19	89	3	23	P	3	11
17	P	59	3	P	11	3	17	P	3	7	P	3	47	P	3	11	7	3	P	41
19	3	7	11	3	P	P	3	17	7	3	83	P	3	11	37	3	13	P	3	163
21	7	3	P	P	3	83	P	3	17	31	3	11	313	3	7	23	3	P	173	3
23	83	P	3	P	13	3	7	109	3	11	P	3	P	7	3	P	P	3	P	P
27	61	3	P	P	3	11	P	3	37	P	3	7	67	3	19	P	3	31	7	3
29	167	P	3	7	P	3	19	P	3	P	7	3	13	71	3	P	67	3	P	P
31	3	11	7	3	257	37	3	P	23	3	107	P	3	17	P	3	7	19	3	13
33	13	3	23	107	3	P	53	3	7	19	3	P	P	3	17	7	3	P	P	3
37	3	13	193	3	173	411	3	P	P	3	97	P	3	7	13	3	17	11	3	37
39	17	3	31	29	3	7	P	3	13	P	3	P	7	3	P	11	3	17	P	3
41	P	17	3	43	7	3	P	193	3	163	P	7	P	11	3	13	37	3	7	139
43	3	P	17	3	P	P	3	19	97	3	7	11	3	41	17	3	P	7	3	17
47	P	7	3	P	17	3	23	11	3	P	13	3	61	P	3	7	251	3	11	89
49	3	61	19	3	13	11	3	7	P	3	37	P	3	P	7	3	11	13	3	127
51	71	3	P	11	3	39	7	3	41	53	3	13	P	3	11	P	3	23	31	3
53	31	11	3	59	P	3	47	17	3	P	P	3	7	73	3	113	227	3	13	7
57	P	3	P	7	3	67	13	3	11	17	3	229	P	3	471	29	3	7	61	3
59	13	103	3	41	P	3	11	61	3	7	17	3	P	13	3	P	7	3	P	19
61	3	7	97	3	11	P	3	13	7	3	23	17	3	67	79	3	P	P	3	P
63	7	3	11	19	3	P	P	3	109	P	3	53	17	3	7	P	3	67	37	3
67	3	89	13	3	P	7	3	283	P	3	157	131	3	P	17	3	P	P	3	7
69	281	3	P	P	3	241	P	3	P	13	3	7	53	3	P	17	3	19	7	3
71	101	127	3	7	59	3	79	43	3	19	7	3	37	P	3	P	11	3	P	P
73	3	19	7	3	P	P	3	P	3	13	P	3	43	11	3	7	17	3	257	
77	7	31	3	P	19	3	101	7	3	29	11	3	P	P	3	P	263	3	P	17
79	3	P	23	3	P	13	3	P	11	3	P	41	3	7	31	3	P	113	3	11
81	P	3	29	131	3	7	11	3	61	P	3	P	7	3	53	P	3	11	P	3
83	43	47	3	37	7	3	13	173	3	31	P	3	101	23	3	11	83	3	7	13
87	11	3	7	P	3	11	29	3	P	7	3	11	43	3	P	53	3	P	59	3
89	47	7	3	P	149	3	P	223	3	11	P	3	P	19	3	7	P	3	23	P
91	3	149	227	3	P	19	3	7	13	3	197	P	3	P	7	3	131	73	3	P
93	233	3	13	61	3	11	7	3	P	P	3	281	31	3	37	13	3	P	191	3
97	3	11	P	3	7	P	3	31	P	3	41	7	3	P	P	3	13	23	3	19
99	263	3	P	7	3	43	229	3	P	P	3	19	109	3	29	137	3	7	283	3

*Of Rational Numbers that express the Sides of
Right-angled Triangles.*

A P R O B L E M.

Article 1. To find as many right-angled triangles as we please, of which the three sides shall be expressible in rational numbers.

S O L U T I O N.

Let the numbers that express the lengths of the two sides that contain the right angle, be denoted by the letters m and n . Then will the number that denotes the hypotenuse of the triangle, or the line that subtends the right angle, be greater than either of the two numbers m and n , and its excess above either of the said numbers will be a rational number: for, if it were not, the number itself which expresses the said hypotenuse would not be a rational number. Let the excess of this number, which expresses the hypotenuse, above the number m , which expresses one of the sides containing the right angle, be called e . Then will the number which expresses the hypotenuse be $m + e$, and its square will be $mm + 2me + ee$. But (by El. 1, 47,) the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two sides of it. Therefore

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$mm + 2me + ee$ will be equal to $mm + nn$; and, consequently (subtracting mm from both sides), $2me + ee$ will be equal to nn ; and (subtracting ee from both sides, which is evidently less than $2me + ee$, and consequently must be less also than nn , or the right-hand side of the equation $2me + ee = nn$), $2me$ will be equal to $nn - ee$; and (dividing both sides of the equation by $2e$), m will be equal to $\frac{nn - ee}{2e}$, and consequently $m + e$ will be equal to $\frac{nn - ee}{2e} + e$, or to $\frac{nn - ee}{2e} + \frac{2e \times e}{2e}$, or to $\frac{nn - ee}{2e} + \frac{2ee}{2e}$, or to $\frac{nn + ee}{2e}$. And consequently the three numbers m , n , and $m + e$, that will express the three sides of a right-angled triangle, will be equal to $\frac{nn - ee}{2e}$, n , and $\frac{nn + ee}{2e}$, or $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$; or, if we take any number whatsoever, and call it e , and take any other number whatsoever that is greater than e , and call it n , the three numbers $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, will be three rational numbers that will express the three sides of a right-angled triangle. Q. E. I.

Examples of this Method of finding such Rational Numbers.

Art. 2. Thus, for example, if e is $= 1$, and n is $= 2$, (which are the simplest numbers we can choose,) we shall have $ee = 1$, and $nn = 4$, and consequently $\frac{nn - ee}{2e}$ ($= \frac{4 - 1}{2 \times 1} = \frac{4 - 1}{2} = \frac{3}{2}$), and $\frac{2en}{2e}$ ($= \frac{2 \times 1 \times 2}{2 \times 1} = \frac{4}{2}$), and $\frac{nn + ee}{2e}$ ($= \frac{4 + 1}{2 \times 1} = \frac{4 + 1}{2} = \frac{5}{2}$); and consequently

$$\frac{3}{2}, \frac{4}{2}, \frac{5}{2}$$

$\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, will be three rational numbers that will express the lengths of the three sides of a right-angled triangle. And accordingly we shall find that the square of the last of these numbers, to wit, $\frac{5}{2}$, which expresses the hypotenuse of the triangle, is equal to the sum of the squares of the two former numbers, to wit, $\frac{3}{2}$ and $\frac{4}{2}$, which express the two sides that contain the right angle. For the square of $\frac{5}{2}$ is $\frac{25}{4}$, and the squares of $\frac{3}{2}$ and $\frac{4}{2}$ are $\frac{9}{4}$ and $\frac{16}{4}$; and $\frac{25}{4}$ is $= \frac{9}{4} + \frac{16}{4}$.

Secondly, let e be $= 2$, and n be $= 3$:

Then we shall have $ee = 4$, and $nn = 9$, and $2e = 4$, and consequently $\frac{nn-ee}{2e} (= \frac{9-4}{4}) = \frac{5}{4}$, and $\frac{2en}{2e} (= \frac{2 \times 2 \times 3}{2 \times 2}) = \frac{12}{4}$, and $\frac{nn+ee}{2e} (= \frac{9+4}{4}) = \frac{13}{4}$.

Therefore $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, will be three rational numbers that will express the three sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{13}{4}$ will be equal to the sum of the squares of $\frac{5}{4}$ and $\frac{12}{4}$. For the square of $\frac{13}{4}$ is $= \frac{169}{16}$, and the squares of $\frac{5}{4}$ and $\frac{12}{4}$ are $\frac{25}{16}$ and $\frac{144}{16}$; and $\frac{169}{16}$ is $= \frac{25}{16} + \frac{144}{16}$.

Thirdly, let e be $= 3$, and n be $= 5$.

Then we shall have $ee = 9$, and $nn = 25$, and $2e = 6$, and consequently $\frac{nn-ee}{2e} (= \frac{25-9}{6}) = \frac{16}{6}$, and $\frac{2en}{2e} (= \frac{2 \times 3 \times 5}{6}) = \frac{30}{6}$, and $\frac{nn+ee}{2e} (= \frac{25+9}{6}) = \frac{34}{6}$.

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Therefore

Therefore $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, will be three rational numbers that will express the three sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{34}{6}$ will be equal to the sum of the squares of $\frac{16}{6}$ and $\frac{30}{6}$. For the square of $\frac{34}{6}$ is $\frac{1156}{36}$, and the squares of $\frac{16}{6}$ and $\frac{30}{6}$ are $\frac{256}{36}$ and $\frac{900}{36}$; and $\frac{1156}{36}$ is $= \frac{256}{36} + \frac{900}{36}$.

These three numbers $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, might have been reduced to smaller numbers, by dividing both their numerators and denominators by 2. For they would then have been $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. Therefore these three numbers $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, will express the three sides of a right-angled triangle.

Fourthly, let e be $= 3$, and $n = 7$.

Then we shall have $ee = 9$, and $nn = 49$, and $2e = 6$, and consequently $\frac{nn - ee}{2e} (= \frac{49 - 9}{6}) = \frac{40}{6}$, and $\frac{2en}{2e} (= \frac{2 \times 3 \times 7}{6}) = \frac{42}{6}$, and $\frac{nn + ee}{2e} (= \frac{49 + 9}{6}) = \frac{58}{6}$.

Therefore $\frac{40}{6}$, $\frac{42}{6}$, and $\frac{58}{6}$, or $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, will be three rational numbers that will express the sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{29}{3}$ will be equal to the sum of the squares of $\frac{20}{3}$ and $\frac{21}{3}$. For the square of $\frac{29}{3}$ is $\frac{841}{9}$, and the squares of $\frac{20}{3}$ and $\frac{21}{3}$ are $\frac{400}{9}$ and $\frac{441}{9}$; and $\frac{841}{9}$ is $= \frac{400}{9} + \frac{441}{9}$.

Fifthly,

Fifthly, let e be $= 3$, and $n = 11$.

Then we shall have $ee = 9$, and $nn = 121$, and $2e = 6$, and consequently $\frac{nn-ee}{2e} (= \frac{121-9}{6}) = \frac{112}{6}$, and $\frac{2en}{2e} (= \frac{6 \times 11}{6}) = \frac{66}{6}$, and $\frac{nn+ee}{2e} (= \frac{121+9}{6}) = \frac{130}{6}$. Therefore $\frac{112}{6}$, $\frac{66}{6}$, and $\frac{130}{6}$, or $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, will be three rational numbers that will express the three sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{65}{3}$ will be equal to the sum of the squares of $\frac{56}{3}$ and $\frac{33}{3}$. For the square of $\frac{65}{3}$ is $\frac{4225}{9}$, and the squares of $\frac{56}{3}$ and $\frac{33}{3}$ are $\frac{3136}{9}$ and $\frac{1089}{9}$; and $\frac{4225}{9}$ is $= \frac{3136}{9} + \frac{1089}{9}$.

Sixthly, let e be $= 5$, and $n = 13$.

Then we shall have $ee = 25$, and $nn = 169$, and $2e = 10$, and consequently $\frac{nn-ee}{2e} (= \frac{169-25}{10}) = \frac{144}{10}$, and $\frac{2en}{2e} (= \frac{10 \times 13}{10}) = \frac{130}{10}$, and $\frac{nn+ee}{2e} (= \frac{169+25}{10}) = \frac{194}{10}$.

Therefore $\frac{144}{10}$, $\frac{130}{10}$, and $\frac{194}{10}$, or $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, will be three rational numbers that will express the three sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{97}{5}$ will be equal to the sum of the squares of $\frac{72}{5}$ and $\frac{65}{5}$. For the square of $\frac{97}{5}$ is $\frac{9409}{25}$, and the squares of $\frac{72}{5}$ and $\frac{65}{5}$ are $\frac{5184}{25}$ and $\frac{4225}{25}$; and $\frac{9409}{25}$ is $= \frac{5184}{25} + \frac{4225}{25}$.

Thus we have obtained six different sets of rational numbers, which express the lengths of the sides of as many different

ferent right-angled triangles; to wit, 1st, $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$; and, 2dly, $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$; and, 3dly, $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$; and, 4thly, $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$; and, 5thly, $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$; and, 6thly, $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$. And, by changing either both the numbers denoted by e and n , or only one of those numbers, and computing the values of the three fractions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, we may obtain as many more such sets of numbers as we please.

Art. 3. All these numbers are fractions, because they are derived from the general fractional expressions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$. But, if we multiply the three fractions of each of these six sets of fractions by their common denominator, the products will be whole numbers expressing the sides of greater right-angled triangles similar to the former triangles, of which the sides were expressed by the foregoing fractions. Thus, if we multiply the three fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, by their common denominator 2, we shall have the whole numbers 3, 4, and 5, for the sides of a greater right-angled triangle similar to the former triangle, of which the sides were expressed by the fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$. And, if we multiply the three fractions $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, by their common denominator 4, we shall have the whole numbers 5, 12, and 13, for the sides of a greater right-angled triangle similar to the former triangle, of which the sides were $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$. And, if we multiply the three fractions

$$\frac{8}{3},$$

$\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, by their common denominator 3, we shall have the whole numbers 8, 15, and 17, for the sides of a greater right-angled triangle similar to the former triangle, of which the sides were $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. And, in like manner, from the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, we may derive the whole numbers 20, 21, and 29; and from the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, we may derive the whole numbers 56, 33, and 65; and from the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, we may derive the whole numbers 72, 65, and 97; all which sets of whole numbers will express the sides of right-angled triangles similar to the three former triangles, of which the sides were expressed by the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, and the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, and the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$.

Art. 4. And these whole numbers might have been obtained at once by computing only the numerators of the three general fractions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, to wit, the expressions $nn - ee$, $2en$, and $nn + ee$, which are the products of the multiplication of the said three fractions into their common denominator $2e$. For then we should have found, in the first example, in which e is $= 1$ and n is $= 2$, that the said expressions $nn - ee$, $2en$, and $nn + ee$, would have been equal to $(4 - 1, 2 \times 1 \times 2, \text{ and } 4 + 1, \text{ or})$ 3, 4, and 5; and, in the second example, in which e was $= 2$, and $n = 3$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(9 - 4, 2 \times 2 \times 3, \text{ and } 9 + 4, \text{ or})$ 5, 12,

5, 12, and 13; and, in the 3d example, in which e was $= 3$, and $n = 5$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(25 - 9, 2 \times 3 \times 5, \text{ and } 25 + 9, \text{ or})$ 16, 30, and 34, which, when divided by 2, become 8, 15, and 17; and in the 4th example, in which e was $= 3$, and $n = 7$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(49 - 9, 2 \times 3 \times 7, \text{ and } 49 + 9, \text{ or})$ 40, 42, and 58, which, when divided by 2, become 20, 21, and 29; and, in the 5th example, in which e is $= 3$, and n is $= 11$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(121 - 9, 2 \times 3 \times 11, \text{ and } 121 + 9, \text{ or})$ 112, 66, and 130, which, when divided by 2, become 56, 33, and 65; and in the 6th and last example, in which e was $= 5$, and n was $= 13$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(169 - 25, 2 \times 5 \times 13, \text{ and } 169 + 25, \text{ or})$ 144, 130, and 194, which, when divided by 2, become 72, 65, and 97. And thus we should have obtained the six foregoing sets of whole numbers to express the sides of different right-angled triangles, to wit, 1st, the numbers 3, 4, and 5; 2dly, the numbers 5, 12, and 13; 3dly, the numbers 8, 15, and 17; 4thly, the numbers 20, 21, and 29; 5thly, the numbers 56, 33, and 65; and, 6thly, the numbers 72, 65, and 97.

A S C H O L I U M.

Art. 5. It may be observed, that in the four first of the foregoing six sets of numbers, which express the sides of right-angled triangles, to wit, in the numbers 3, 4, and 5, and in the numbers 5, 12, and 13, and in the numbers 8, 15, and 17, and in the numbers 20, 21, and 29, the first number of each set is less than the second; but in the fifth and sixth sets of those numbers, to wit, in the numbers 56, 33, and 65, and in the numbers 72, 65, and 97, the first

first number of each set is greater than the second. Now this depends upon the proportion in which the number n (which is always greater than e ,) exceeds the number e . For, if n were not a number, but a quantity incommensurable to 1, which bore the same proportion to the number e as 1 bears to $\sqrt{2} - 1$, or to the excess of $\sqrt{2}$ above 1, or as the side of a square bears to the excess of its diagonal above its side, the general expression $mn - ee$, from which the first terms of all these sets of numbers are derived, would be exactly equal to the general expression $2en$, from which the second terms of the said sets of numbers are derived: and, when the proportion of n to e is less than that

of 1 to $\sqrt{2} - 1$, or n is less than $e \times \frac{1}{\sqrt{2}-1}$, the general expression $mn - ee$ will be less than $2en$: and, when the proportion of n to e is greater than the said proportion of 1 to $\sqrt{2} - 1$, or n is greater than $e \times \frac{1}{\sqrt{2}-1}$, the general expression $mn - ee$ will be greater than $2en$. These things may be demonstrated in the manner following.

Art. 6. In the 1st place, if n is $= e \times \frac{1}{\sqrt{2}-1}$, we shall

have $mn = ee \times \frac{1}{\sqrt{2}-1} = \frac{ee}{2-2\sqrt{2}+1}$, and $mn - ee (=$

$$\frac{ee}{2-2\sqrt{2}+1} - ee = \frac{ee}{2-2\sqrt{2}+1} - \frac{(2-2\sqrt{2}+1) \times ee}{2-2\sqrt{2}+1} =$$

$$\frac{ee}{2-2\sqrt{2}+1} - \frac{2ee - 2\sqrt{2} \times ee + ee}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} -$$

$$\frac{3ee - 2\sqrt{2} \times ee}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} - \frac{3ee + 2\sqrt{2} \times ee}{2-2\sqrt{2}+1} =$$

$$+ \frac{2\sqrt{2} \times ee - 2ee}{2-2\sqrt{2}+1} = \frac{2ee \times \sqrt{2}-1}{2-2\sqrt{2}+1} = \frac{2ee}{\sqrt{2}-1}; \text{ and we}$$

shall also have $2en (= 2e \times e \times \frac{1}{\sqrt{2}-1}) = \frac{2ee}{\sqrt{2}-1}$.

And consequently $mn - ee$ will in this case be equal to $2en$.

Q. E. D.

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2dly,

2dly, If n is less than $e \times \frac{1}{\sqrt{2}-1}$, the compound quantity $nn - ee$ will be less than $2en$.

For, if we suppose n , from being equal to $e \times \frac{1}{\sqrt{2}-1}$, to become less than that quantity, but still to be greater than e , and the decrement of n , or its difference from its first value, to be denoted by the letter d , it is evident that while n is decreasing from n to $n - d$, the compound quantity $nn - ee$ will decrease from $nn - ee$ to $(n - d)^2 - ee$, that is, to $nn - 2nd + dd - ee$, or to $nn - ee - 2nd + dd$, or to $nn - ee - (2nd - dd)$, or will be less than it was before by the quantity $2nd - dd$; and in the same time the quantity $2en$ will decrease from its first value, $2en$, (which was equal to $nn - ee$) to $2e \times n - d$, or $2en - 2ed$, or will be less than it was before by the quantity $2ed$. Now, because $n - d$ is greater than e , it follows that n must be greater than $e + d$, and consequently that $n - e$ must be greater than d . Therefore $2d \times n - e$ will be greater than $2d \times d$, or $2nd - 2ed$ will be greater than $2dd$, and consequently $2nd$ will be greater than $2dd + 2ed$, and $2nd - dd$ will be greater than $dd + 2ed$. Therefore, *à fortiori*, $2nd - dd$ will be greater than $2ed$; that is, the decrement of the quantity $nn - ee$ while n decreases from n to $n - d$, will be greater than the decrement of the quantity $2en$ in the same time: and consequently the quantity $nn - ee - (2nd - dd)$, to which $nn - ee$ will have decreased, while n was decreasing from n to $n - d$, will be less than the quantity $2en - 2ed$, to which the quantity $2en$ (which was at first equal to $nn - ee$), will have decreased in the same time; or, if n is of any magnitude less than $e \times \frac{1}{\sqrt{2}-1}$, but greater than e , the quantity $nn - ee$ will be less than the quantity $2en$. Q. E. D.

And, 3dly, if n is greater than $e \times \frac{1}{\sqrt{2}-1}$, the compound quantity $nn - ee$ will be greater than $2en$.

For,

For, if we suppose n , from being equal to $e \times \frac{1}{\sqrt{2}-1}$, to become greater than that quantity, and the increment of n , or its difference from its former value, to be denoted by the letter d , it is evident that, while n is increasing from n to $n + d$, the compound quantity $nn - ee$ will increase from $nn - ee$ to $(n + d)^2 - ee$, that is, to $nn + 2nd + dd - ee$, or to $nn - ee + 2nd + dd$, or it will be greater than it was before by the quantity $2nd + dd$; and in the same time the quantity $2en$ will increase from its first value, $2en$, (which was equal to $nn - ee$), to $2e \times n + d$, or $2en + 2ed$; or will be greater than it was before by the quantity $2ed$. Now, because n is greater than e , it follows that $2nd$ must be greater than $2ed$; and consequently, *a fortiori*, $2nd + dd$ will be greater than $2ed$; that is, the increment of $nn - ee$ during the increase of n from n to $n + d$ will be greater than the contemporary increment of $2en$. Therefore the quantity $nn - ee + 2nd + dd$, to which $nn - ee$ will have increased while n increased from n to $n + d$, will be greater than the quantity $2en + 2ed$, to which the quantity $2en$ (which was at first equal to $nn - ee$), will have increased in the same time; or, if n is of any magnitude greater than $e \times \frac{1}{\sqrt{2}-1}$, the quantity $nn - ee$ will be greater than the quantity $2en$.

Q. E. D.

ART. 7. If we take two numbers for e and n that are nearly in the proportion of $\sqrt{2}-1$ and 1, we shall find that $nn - ee$ will be very nearly equal to $2en$. Now $\sqrt{2}$ is ≈ 1.414 &c. Therefore $\sqrt{2}-1$ is ≈ 1.414 &c $- 1 \approx 0.414$ &c, and $\sqrt{2}-1$ is to 1 pretty nearly in the proportion of 0.414 to 1, or of 414 to 1000, or of 207 to 500. Therefore, if we suppose e to be ≈ 207 , and n to be ≈ 500 , the value of $nn - ee$ ought to be nearly equal to that of $2en$. And so we shall find them to be. For, upon these suppositions, we shall have ee ($\approx 207^2$) $\approx 42,849$, and nn ($\approx 500^2$) $\approx 250,000$, and $nn - ee$ ($\approx 250,000 - 42,849$) $\approx 207,151$. And $2en$ will be ($\approx$

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2 x

$2 \times 207 \times 500 = 207 \times 1000 = 207,000$; which is very nearly equal to $207,151$, or $nn - ee$.

Art. 8. And, upon these suppositions, $nn + ee$ will be $(= 250,000 + 42,849) = 292,849$; which gives us a seventh set of numbers that express the sides of a right-angled triangle, to wit, the numbers $207,151$, $207,000$, and $292,849$. And accordingly we shall find that the square of the number $292,849$, which represents the hypotenuse of the triangle, will be equal to the sum of the squares of the other two numbers $207,151$ and $207,000$, which represent the sides that contain the right angle. For the square of $292,849$ is $85,760,536,801$, and the squares of $207,151$ and $207,000$, are $42,911,536,801$ and $42,849,000,000$; and $85,760,536,801$ is $= 42,911,536,801 + 42,849,000,000$.

Art. 9. If we suppose e to be $= 2$, and n to be $= 5$, we shall have $nn - ee (= 25 - 4) = 21$, and $2en (= 2 \times 2 \times 5) = 20$, and $nn + ee (= 25 + 4) = 29$. Therefore 21 , 20 , and 29 , will be three numbers that will express the lengths of the three sides of a right-angled triangle. And these numbers, we may observe, are the same with the three numbers 20 , 21 , and 29 , obtained above in art. 4, by supposing e to be equal to 3 , and n to be equal to 7 , excepting that the order of the two first numbers 20 and 21 is different in the two sets, 20 being the first number in the first set, 20 , 21 , and 29 , and being the second number in the second set, 21 , 20 , and 29 ; the reason of which is, that 20 is derived from the first general expression $nn - ee$ in the first set of numbers, 20 , 21 , and 29 , and it is derived from the second general quantity $2en$ in the second set of numbers, 21 , 20 , and 29 . This, however, has an odd appearance, that, when the original numbers n and e , from which the general expressions $nn - ee$, $2en$, and $nn + ee$ are derived, are made to bear different proportions to each other (for the proportion of 5 to 2 is greater than the proportion of 7 to 3 , being equal to that of 7 to 2.8 , or of 70 to 28 ,) the three numbers obtained by means of

of those general expressions should still be the same, though placed in a different order: and therefore it may not be amiss to inquire a little further into it.

Art. 10. In order therefore that the reason of this seeming irregularity may appear the more clearly, we will recur to the observation made above in art. 5, concerning the change in the proportion of the general expression $nn - ee$ to the general expression $2en$, when the proportion of n to e , from being at first a less proportion of majority than that of 1 to $\sqrt{2} - 1$, or of 1 to 0.414, &c, becomes equal to, and greater than, the said ratio; to wit, that, when the ratio of n to e is less than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is less than the quantity $2en$; and that, when the ratio of n to e is equal to the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is equal to the quantity $2en$; and that, when the ratio of n to e is greater than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is greater than the quantity $2en$. And to this observation we must add that, if the ratio of n to e , after having been equal to the ratio of 1 to 0.414, &c, is supposed to increase gradually *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increase gradually at the same time *ad infinitum*, or so as to become greater than any assigned ratio whatsoever. For the ratio of $nn - ee$ to $2en$ is equal to the ratio of $\frac{nn - ee}{2en}$ to $(\frac{2en}{2en}$, or) 1, or to the ratio of $\frac{nn}{2en} - \frac{ee}{2en}$ to 1, or to the ratio of $\frac{n}{2e} - \frac{e}{2n}$ to 1, which evidently increases *ad infinitum*, while the ratio of n to e increases *ad infinitum*. Therefore while the ratio of n to e increases, from being equal to the ratio of 1 to 0.414, &c, *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increase gradually from a ratio of equality *ad infinitum*, and consequently will become successively equal to all ratios of majority whatsoever. Therefore, if the ratio of n to e is at one time taken equal to the ratio of 7 to 3, (which is less than the ratio of 7 to 2.898, &c, or of 7 to 7×0.414 , &c, or of 1 to 0.414, &c,) and is afterwards supposed to increase

increase gradually till it becomes equal to the said ratio of 1 to 0.414, &c, and then to increase further *ad infinitum*, the ratio of the compound quantity $m - e$ to the quantity $2en$ (which is equal to the ratio of 20 to 21, when the ratio of n to e is equal to the ratio of 7 to 3,) will first become a ratio of equality, to wit, when the ratio of n to e becomes equal to the ratio of 1 to 0.414, &c, and afterwards will increase continually from being a ratio of equality, (which may be considered as an infinitely small ratio of majority, and is usually so considered by writers who treat of the magnitudes and measures of ratios,) till it becomes successively equal to every ratio of majority whatsoever. It therefore must at one point of time during its said increase become equal to the ratio of 21 to 20; or, there will be a certain ratio of majority, greater than that of 1 to 0.414, &c, to which when the ratio of n to e shall have become equal, the ratio of $m - e$ to $2en$ will be equal to the ratio of 21 to 20. And this ratio of majority is that of 5 to 2, as has been shewn in the foregoing, or 9th, article.

Art. 11. I will just add one more example of the foregoing method of finding three whole numbers that shall express the lengths of the sides of a right-angled triangle.

Let e be $= 5$, and n be $= 17$. And we shall have $ee = 25$, and $nn = 289$, and $nn - ee (= 289 - 25) = 264$, and $2en (= 2 \times 5 \times 17 = 10 \times 17) = 170$, and $nn + ee (= 289 + 25) = 314$. Therefore 264, 170, and 314, or (dividing all these numbers by 2,) 132, 85, and 157, will be three whole numbers that will express the three sides of a right-angled triangle.

And accordingly we shall find that the square of the number 157, which represents the hypotenuse, or line subtending the right angle, will be equal to the sum of the squares of the two numbers 132 and 85, which represent the sides that contain the right angle. For the square of 157 is 24,649, and the squares of 132 and 85 are 17,424 and 7225; and 24,649 is $= 17,424 + 7225$.

We

We have therefore now found the nine following sets of whole numbers for expressing the three sides of different right-angled triangles; to wit,

- 1st, The whole numbers 3, 4, and 5;
- 2dly, The whole numbers 5, 12, and 13;
- 3dly, The whole numbers 8, 15, and 17;
- 4thly, The whole numbers 20, 21, and 29;
- 5thly, The whole numbers 56, 33, and 65;
- 6thly, The whole numbers 72, 65, and 97;
- 7thly, The whole numbers 207, 151, 207,000, and 292,849;
- 8thly, The whole numbers 21, 20, and 29; and
- 9thly, The whole numbers 132, 85, and 157.

And we may easily find as many more sets of such numbers as we please, by substituting different numbers for e and n , or for either of them, in the three general expressions $nn - ee$, $2en$, and $nn + ee$.

Art. 12. The foregoing subject may also be treated in a somewhat different manner, by solving the following Problem.

A P R O B L E M.

To divide a given square number into two other square numbers, either whole numbers, or fractions, or mixt numbers.

SOLUTION.

SOLUTION.

Let the given square number that is to be so divided, be denoted by the letters aa , and let xx be one of the two square numbers that are sought, and yy be the other.

Then, since the two numbers sought are together to be equal to the given number, we shall, in the first place, have $xx + yy = aa$.

Now, since $xx + yy$ is $= aa$, it follows that yy will be $= aa - xx$. But $aa - xx$ is $= \overline{a + x} \times \overline{a - x}$. Therefore yy will be $= \overline{a + x} \times \overline{a - x}$. Therefore y will be a geometrical mean proportional between $a + x$ and $a - x$, and consequently will be less than $a + x$. Let the proportion of $a + x$ to y be that of the two numbers m and n , of which m is the greater. Then will y be $= \frac{n}{m} \times \overline{a + x}$,

and yy will be $= \frac{nn}{mm} \times \overline{a + x}^2 (= \frac{nn}{mm} \times \overline{aa + 2ax + xx})$
 $= \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$. Therefore xx

+ yy will be $= xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$
 $(= \frac{mm}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx)$
 $= \frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax$. But

$xx + yy$ is $= aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa$
 $+ \frac{nn}{mm} \times 2ax$ will also be $= aa$, and consequently $=$

$\frac{mm}{mm} \times aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times 2ax$ will
be $= \frac{mm}{mm} \times aa - \frac{nn}{mm} \times aa = \frac{mm - nn}{mm} \times aa$, and consequently

frequently (multiplying all the terms by mm ,) $\overline{mm + nn} \times xx + nn \times 2ax$ will be $= \overline{mm - nn} \times aa$, and (dividing all the terms by $mm + nn$,) $xx + \frac{nn}{mm + nn} \times 2ax$ will be $= \frac{mm - nn}{mm + nn} \times aa$. Therefore (adding $\frac{n^4}{(mm + nn)^2} \times aa$ to both sides of the equation,) we shall have $xx + \frac{nn}{mm + nn} \times 2ax + \frac{n^4}{(mm + nn)^2} \times aa = \frac{mm - nn}{mm + nn} \times aa + \frac{n^4}{(mm + nn)^2} \times aa = \frac{mm - nn \times mm + nn}{mm + nn \times mm + nn} \times aa + \frac{n^4}{(mm + nn)^2} \times aa = \frac{m^4 - n^4}{(mm + nn)^2} \times aa + \frac{n^4}{(mm + nn)^2} \times aa = \frac{m^4}{(mm + nn)^2} \times aa$. Therefore (extracting the square-roots of both sides,) we shall have $x + \frac{nn}{mm + nn} \times a = \frac{mm}{mm + nn} \times a$, and consequently $x = \frac{mm}{mm + nn} \times a - \frac{nn}{mm + nn} \times a = \frac{mm - nn}{mm + nn} \times a$. Therefore $a + x$ will be $= a + \frac{mm - nn}{mm + nn} \times a (= \frac{mm + nn}{mm + nn} \times a + \frac{mm - nn}{mm + nn} \times a) = \frac{2mm}{mm + nn} \times a$; and y , or $\frac{n}{m} \times a + x$, will be $= \frac{n}{m} \times \frac{2mm}{mm + nn} \times a = \frac{2mn}{mm + nn} \times a$, that is, x , or the root, or side, of the first of the two squares sought, to wit, xx and yy , will be equal to $\frac{mm - nn}{mm + nn} \times a$, and y , or the root, or side, of the latter of the said two squares, will be equal to $\frac{2mn}{mm + nn} \times a$; and consequently xx , or the first of the said two squares itself, will be equal to $\frac{(mm - nn)^2}{(mm + nn)^2} \times aa$, and yy , or the latter of the said two squares

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itself,

itself, will be equal to $\frac{2mn}{{mm + nn}} \times aa$, or $\frac{4m^2n^2}{{mm + nn}^2} \times aa$; or, if m and n be any two numbers whatsoever, of which m is the greater, $\frac{{mm - nn}^2}{{mm + nn}^2} \times aa$ and $\frac{4m^2n^2}{{mm + nn}^2} \times aa$ will be two square numbers that will together be equal to the original square number aa . Q. E. I.

Art. 13. That these two square numbers $\frac{{mm - nn}^2}{{mm + nn}^2} \times aa$ and $\frac{4m^2n^2}{{mm + nn}^2} \times aa$, will together be equal to the original square number aa , will be evident by adding them together. For $\frac{{mm - nn}^2}{{mm + nn}^2} \times aa + \frac{4m^2n^2}{{mm + nn}^2} \times aa = \frac{m^4 - 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa + \frac{4m^2n^2}{m^4 + 2m^2n^2 + n^4} \times aa = \frac{m^4 + 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa = aa$.

Q. E. D.

This solution of the foregoing Problem seems to me more easy and natural than that of Dr. Saunderson in his Algebra, Vol. 2, page 366, *et sequentibus*.

Examples of the foregoing Solution.

Art. 14. Let the given square number aa be 25, and let n be 1, and m be 2.

Then will nn be $= 1$, and mm be $= 4$, and $mm - nn$ will be $(= 4 - 1) = 3$, and $mm + nn$ will be $(= 4 + 1) = 5$, and consequently $\frac{{mm - nn}^2}{{mm + nn}^2}$ will be $= 9$, and $\frac{4m^2n^2}{{mm + nn}^2}$ will be $= 25$, and $4m^2n^2$ will be $(= 4 \times 4 \times 1) = 16$,

Let 16, and $\frac{(mm - nn)^2}{(mm + nn)^2} \times aa$ will be $= \frac{9}{25} \times 25 = 9$, and $\frac{4m^2n^2}{(mm + nn)^2} \times aa$ will be $= \frac{16}{25} \times 25 = 16$. Therefore 9 and 16 will be two square numbers that will together be equal to the given square number 25. And it is evident that 9 and 16 are equal to 25.

Secondly, let aa be $= 25$, and $n = 1$, and $m = 3$.

Then will nn be $= 1$, and mm be $= 9$, and consequently $mm - nn$ will be $(= 9 - 1) = 8$, and $mm + nn$ will be $= 9 + 1 = 10$, and $4m^2n^2$ will be $(= 4 \times 9 \times 1) = 36$. Therefore $\frac{(mm - nn)^2}{(mm + nn)^2}$ will be $= \frac{64}{100}$, and $\frac{(mm - nn)^2}{(mm + nn)^2} \times aa$ will be $= \frac{64}{100} \times aa = \frac{64}{100} \times 25 = \frac{16}{25} \times 25 = 16$, and $\frac{4m^2n^2}{(mm + nn)^2} \times aa$ will be $= \frac{36}{100} \times aa = \frac{9}{25} \times aa = \frac{9}{25} \times 25 = 9$. Therefore 16 and 9 will be two square numbers that will together be equal to the given square number 25. And it is evident that they are equal to 25.

These two square numbers 16 and 9 are the same with the two former square numbers 9 and 16, derived from the supposition that m was $= 3$, except in the order of their position.

Thirdly, let aa be $= 25$, and n be $= 1$, and m be $= 4$.

Then we shall have $nn = 1$, and $mm = 16$, and consequently $mm - nn (= 16 - 1) = 15$, and $mm + nn (= 16 + 1) = 17$, and $\frac{(mm - nn)^2}{(mm + nn)^2} (= \frac{225}{289}) = \frac{225}{289}$, and $\frac{(mm - nn)^2}{(mm + nn)^2} \times aa$ will be $(= \frac{225}{289} \times aa)$

$= \frac{225}{289} \times 25$, and $\frac{4m^2n^2}{mm + nn} \times aa$ will be $(= \frac{64}{289} \times aa)$
 $= \frac{64}{289} \times 25$. Therefore $\frac{225}{289} \times 25$ and $\frac{64}{289} \times 25$, will be
 two square numbers that will, together, be equal to the
 given square number 25. And it is evident that these two
 square numbers are equal to 25: for $\frac{225}{289} \times 25 + \frac{64}{289} \times 25$
 are $= \frac{225 + 64}{289} \times 25 = \frac{289}{289} \times 25 = 25$.

If we multiply these three numbers $\frac{225}{289} \times 25$, $\frac{64}{289} \times 25$,
 and 25, by 289, we shall thereby obtain the three follow-
 ing whole numbers, to wit, 225×25 , 64×25 , and
 289×25 , which will be, all of them, square numbers, and
 of which the two former will, together, be equal to the lat-
 ter. And, if we divide these three last numbers by 25, we
 shall obtain the three following lesser whole numbers, to wit,
 225, 64, and 289, which will be, all of them, square num-
 bers, and of which the two former will, together, be equal
 to the latter.

Fourthly, let aa be $= 25$, and n be $= 1$, and m be
 $= 5$.

Then we shall have $nn = 1$, and $mm = 25$, and conse-
 quently $mm - nn = 24$, and $mm + nn = 26$, and $\overline{mm - nn}^2$
 $(= 24^2 = 576)$, and $\overline{mm + nn}^2 (= 26^2) = 676$, and
 $4m^2n^2 (= 4 \times 25 \times 1) = 100$. Therefore $\frac{mm - nn}{mm + nn} \times aa$
 will be $(= \frac{576}{676} \times aa) = \frac{576}{676} \times 25$, and $\frac{4m^2n^2}{mm + nn} \times aa$
 will be $(= \frac{100}{676} \times aa) = \frac{100}{676} \times 25$. Therefore $\frac{576}{676} \times 25$
 and $\frac{100}{676} \times 25$ will be two square numbers that will, to-
 gether, be equal to the given square number 25. And
 accord-

accordingly it is evident that these two square numbers are equal to 25: for $\frac{576}{676} \times 25 + \frac{100}{676} \times 25$ are $= \frac{576 + 100}{676} \times 25 = \frac{676}{676} \times 25 = 25$.

If we multiply these three numbers, $\frac{576}{676} \times 25$, $\frac{100}{676} \times 25$, and 25, by 676, we shall thereby obtain the three following whole numbers, to wit, 576×25 , 100×25 , and 676×25 , which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three last numbers by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 576, 100, and 676, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these last numbers, 576, 100, and 676, by 4, we shall thereby obtain the three following still lesser whole numbers, to wit, 144, 25, and 169, which will likewise be, all of them, square numbers, and of which the two former will, together, be equal to the latter.

Art. 15. In the 5th place, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 3$.

Then we shall have $mn = 6$, and $mm = 9$, and consequently $mm - mn (= 9 - 6) = 3$, and $mm + mn (= 9 + 6) = 15$, and $\frac{mm - mn}{mm + mn} = \frac{3}{15} = \frac{1}{5}$, and $\frac{mm - mn}{mm + mn} = \frac{1}{5}$, and $\frac{mm - mn}{mm + mn} = \frac{1}{5}$, and $\frac{mm - mn}{mm + mn} = \frac{1}{5}$. Therefore

$\frac{mm - mn}{mm + mn} \times aa$ will be $(= \frac{1}{5} \times 25) = \frac{25}{5} \times 25$, and

$\frac{4m^2n^2}{mm + nn} \times aa$ will be $(= \frac{144}{169} \times aa) = \frac{144}{169} \times 25$. There-

fore $\frac{25}{169} \times 25$ and $\frac{144}{169} \times 25$ will be two square numbers that will, together, be equal to 25. And accordingly it is evident that these two square numbers $\frac{25}{169} \times 25$ and $\frac{144}{169} \times 25$

$\times 25$ are equal to 25. For they are equal to $\frac{25 + 144}{169} \times 25$
 $= \frac{169}{169} \times 25 = 25$.

If we multiply these three numbers, $\frac{25}{169} \times 25$, $\frac{144}{169} \times 25$, and 25, by 169, we shall thereby obtain the three following whole numbers, to wit, 25×25 , 144×25 , and 169×25 , which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three last numbers 25×25 , 144×25 , and 169×25 , by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 25, 144, and 169, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter.

Sixthly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 5$.

Then we shall have $nn = 4$, and $mm = 25$, and consequently $mm - nn (= 25 - 4) = 21$, and $mm + nn (= 25 + 4) = 29$, and $\overline{mm - nn}^2 (= 21^2) = 441$, and $\overline{mm + nn}^2 (= 29^2) = 841$, and $4m^2n^2 (= 4 \times 25 \times 4) = 400$. Therefore $\frac{mm - nn^2}{mm + nn^2} \times aa$ will be $(= \frac{441}{841} \times aa)$
 $= \frac{441}{841} \times 25$, and $\frac{4m^2n^2}{mm + nn^2} \times aa$ will be $= \frac{400}{841} \times 25$.

Therefore $\frac{441}{841} \times 25$ and $\frac{400}{841} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And so we shall find them to be: for $\frac{441}{841} \times 25 + \frac{400}{841} \times 25$ are $= \frac{441 + 400}{841} \times 25 = \frac{841}{841} \times 25 = 25$.

If

If we multiply these three numbers, $\frac{441}{841} \times 25$, $\frac{400}{841} \times 25$, and 25, by 841; we shall thereby obtain the three following whole numbers, to wit, 441×25 , 400×25 , and 841×25 , which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall obtain the three following lesser whole numbers, to wit, 441, 400, and 841, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter.

Seventhly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 7$.

Then we shall have $mn = 4$, and $mm = 49$, and consequently $mm - nn (= 49 - 4) = 45$, and $mm + nn (= 49 + 4) = 53$, and $\overline{mm - nn}^2 (= 45^2) = 2025$, and $\overline{mm + nn}^2 (= 53^2) = 2809$, and $4m^2n^2 (= 4 \times 49 \times 4)$

$= 784$. Therefore $\frac{mm - nn}{mm + nn} \times aa$ will be $(= \frac{2025}{2809} \times aa)$
 $= \frac{2025}{2809} \times 25$, and $\frac{4m^2n^2}{mm + nn} \times aa$ will be $= \frac{784}{2809} \times 25$.

Therefore $\frac{2025}{2809} \times 25$ and $\frac{784}{2809} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly it is evident that these two square numbers $\frac{2025}{2809} \times 25$, and $\frac{784}{2809} \times 25$, are equal to 25. For they are equal to $\frac{2025 + 784}{2809} \times 25 = \frac{2809}{2809} \times 25 = 25$.

If we multiply these three numbers, $\frac{2025}{2809} \times 25$, $\frac{784}{2809} \times 25$, and 25 by 2809, we shall thereby obtain the three following whole numbers, to wit, 2025×25 , 784×25 , and 2809×25 , which will be, all of them, square numbers,
and

and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 2025, 784, and 2809, which will, all of them, be square numbers, and of which the two former will, together, be equal to the latter.

Art. 16. In the 8th place, let aa be, as before, equal to 25, and let n be = 3, and m be = 5.

Then we shall have $nn = 9$, and $mm = 25$, and consequently $mm - nn (= 25 - 9) = 16$, and $mm + nn (= 25 + 9) = 34$, and $\overline{mm - nn}^2 (= 16^2) = 256$, and $\overline{mm + nn}^2 (= 34^2) = 1156$, and $4m^2n^2 (= 4 \times 25 \times 9 = 100 \times 9) = 900$. Therefore $\frac{mm - nn}{mm + nn} \times aa$ will be

$(= \frac{256}{1156} \times aa) = \frac{256}{1156} \times 25$, and $\frac{4m^2n^2}{mm + nn} \times aa$ will

be $(= \frac{900}{1156} \times aa) = \frac{900}{1156} \times 25$. Therefore $\frac{256}{1156} \times 25$,

and $\frac{900}{1156} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly it is evident that these two square numbers

$\frac{256}{1156} \times 25$, and $\frac{900}{1156} \times 25$, are equal to 25. For they are $= \frac{256 + 900}{1156} \times 25 = \frac{1156}{1156} \times 25 = 25$.

If we multiply these three square numbers, $\frac{256}{1156} \times 25$,

$\frac{900}{1156} \times 25$, and 25, by 1156, we shall thereby obtain the three following whole numbers, to wit, 256×25 , 900×25 , and 1156×25 , which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three following lesser,

leffer whole numbers, to wit, 256, 900, and 1156, which are, all of them, square numbers, and of which the two former are, together, equal to the latter.

Ninthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 7$.

Then we shall have $nn = 9$, and $mm = 49$, and consequently $mm - nn (= 49 - 9) = 40$, and $mm + nn (= 49 + 9) = 58$, and $\overline{mm - nn}^2 (= 40^2) = 1600$, and $\overline{mm + nn}^2 (= 58^2) = 3364$, and $4m^2n^2 (= 4 \times 49 \times 9) = 1764$. Therefore $\frac{\overline{mm - nn}^2}{\overline{mm + nn}^2} \times aa$ will be $(= \frac{1600}{3364} \times aa) = \frac{1600}{3364} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}^2} \times aa$ will be $(= \frac{1764}{3364} \times aa) = \frac{1764}{3364} \times 25$. Therefore $\frac{1600}{3364} \times 25$, and $\frac{1764}{3364} \times 25$, will be two square numbers that, together, will be equal to the given square number 25. And accordingly it is evident that these two numbers are equal to 25. For $\frac{1600}{3364} \times 25 + \frac{1764}{3364} \times 25$, are $= \frac{1600 + 1764}{3364} \times 25 = \frac{3364}{3364} \times 25 = 25$.

If we multiply these three numbers, $\frac{1600}{3364} \times 25$, $\frac{1764}{3364} \times 25$, and 25, by 3364, we shall thereby obtain the three whole numbers 1600×25 , 1764×25 , and 3364×25 , which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three following leffer whole numbers, to wit, 1600, 1764, and 3364, which are, all of them, square numbers, and of which the two former are equal to the latter.

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Tenthly;

Tenthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 11$.

Then we shall have $nn = 9$, and $mm = 121$, and consequently $mm - nn (= 121 - 9) = 112$, and $mm + nn (= 121 + 9) = 130$, and $\overline{mm - nn}^2 (= 112^2) = 12,544$, and $\overline{mm + nn}^2 (= 130^2) = 16,900$, and $4m^2n^2 (= 4 \times 121 \times 9 = 484 \times 9) = 4356$. Therefore

$\frac{mm - nn}{mm + nn} \times aa$ will be $(= \frac{12,544}{16,900} \times aa) = \frac{12,544}{16,900} \times 25$,

and $\frac{4m^2n^2}{mm + nn} \times aa$ will be $(= \frac{4356}{16,900} \times aa) = \frac{4356}{16,900} \times 25$.

Therefore $\frac{12,544}{16,900} \times 25$, and $\frac{4356}{16,900} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly it will easily appear that these two numbers are equal to 25. For $\frac{12,544}{16,900} \times 25 + \frac{4356}{16,900} \times 25$ are $= \frac{12,544 + 4356}{16,900} \times 25 = \frac{16,900}{16,900} \times 25 = 25$.

If we multiply these three numbers, $\frac{12,544}{16,900} \times 25$, $\frac{4356}{16,900} \times 25$, and 25, by 16,900, we shall thereby obtain the three following whole numbers, to wit, $12,544 \times 25$, 4356×25 , and $16,900 \times 25$, which are, all of them, square numbers, and of which the two former are, together, equal to the latter. And, if we divide these whole numbers by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 12,544, 4356, and 16,900, which are, all of them, square numbers, (to wit, the squares of the numbers 112, 66, and 130,) and of which the two former are, together, equal to the latter.

Art. 17. In the 11th place, let aa be equal, as before, to 25, and let n be $= 5$, and m be $= 7$.

Then

Then we shall have $nn = 25$, and $mm = 49$, and consequently $mm - nn (= 49 - 25) = 24$, and $mm + nn (= 49 + 25) = 74$, and $\overline{mm - nn}^2 (= 24^2) = 576$, and $\overline{mm + nn}^2 (= 74^2) = 5476$, and $4m^2n^2 (= 4 \times 49 \times 25 = 100 \times 49) = 4900$. Therefore $\frac{mm - nn}{mm + nn} \times aa$ will be $(= \frac{576}{5476} \times aa) = \frac{576}{5476} \times 25$, and $\frac{4m^2n^2}{mm + nn} \times aa$ will be $(= \frac{4900}{5476} \times aa) = \frac{4900}{5476} \times 25$. Therefore $\frac{576}{5476} \times 25$, and $\frac{4900}{5476} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly, if we add these numbers together, we shall find them to be $(= \frac{576 + 4900}{5476} \times 25 = \frac{5476}{5476} \times 25) = 25$.

If we multiply these three numbers, $\frac{576}{5476} \times 25$, $\frac{4900}{5476} \times 25$, and 25, by 5476, we shall thereby obtain the three following whole numbers, to wit, 576×25 , 4900×25 , and 5476×25 , which are, all of them, square numbers, and of which the two former are, together, equal to the latter. And, if we divide these whole numbers by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 576, 4900, and 5476, which are also, all of them, square numbers, and of which the two former are, together, equal to the latter.

Twelfthly, let aa be, as before, $= 25$, and let n be $= 5$, and m be $= 11$.

Then we shall have $nn = 25$, and $mm = 121$, and consequently $mm - nn (= 121 - 25) = 96$, and $mm + nn (= 121 + 25) = 146$, and $\overline{mm - nn}^2 (= 96^2) = 9216$, and

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and $\overline{mm + nn}^2 (= 146)^2 = 21,316$, and $4m^2n^2 (= 4 \times 121 \times 25 = 121 \times 100) = 12,100$. Therefore $\frac{mm - nn}{mm + nn}^2$

$\times aa$ will be $(= \frac{9216}{21,316} \times aa) = \frac{9216}{21,316} \times 25$, and

$\frac{4m^2n^2}{mm + nn}^2 \times aa$ will be $(= \frac{12,100}{21,316} \times aa) = \frac{12,100}{21,316} \times 25$.

Therefore $\frac{9216}{21,316} \times 25$, and $\frac{12,100}{21,316} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And so we shall easily find them to be.

For $\frac{9216}{21,316} \times 25 + \frac{12,100}{21,316} \times 25$, are equal to $\frac{9216 + 12,100}{21,316}$

$\times 25 = \frac{21,316}{21,316} \times 25 = 25$.

If we multiply these three numbers, $\frac{9216}{21,316} \times 25$, $\frac{12,100}{21,316} \times 25$, and 25, by 21,316, we shall thereby obtain the three following whole numbers, to wit, 9216×25 , $12,100 \times 25$, and $21,316 \times 25$, which are, all of them, square numbers, and of which the two former are, together, equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three lesser whole numbers 9216, 12,100, and 21,316, which are, all of them, square numbers, (to wit, the squares of the numbers 96, 110, and 146,) and of which the two former are, together, equal to the latter.

Art. 18. We have now obtained, by means of the three general expressions $\frac{mm - nn}{mm + nn}^2 \times aa$, $\frac{4m^2n^2}{mm + nn}^2$, and aa , the twelve following sets of three whole numbers each, that are, all of them, square numbers, and of which the two first numbers in every set are, together, equal to the third number; to-wit,

- 1st, The numbers 9, 16, and 25;
- 2dly, The numbers 16, 9, and 25, which differ from the three former numbers 9, 16, and 25, only in the order in which the two first numbers 9 and 16 are placed ;
- 3dly, The numbers 225, 64, and 289 ;
- 4thly, The numbers 144, 25, and 169 ;
- 5thly, The numbers 25, 144, and 169, which differ from the three foregoing numbers only in the order in which the two first numbers 25 and 144 are placed ;
- 6thly, The numbers 441, 400, and 841 ;
- 7thly, The numbers 2025, 784, and 2809 ;
- 8thly, The numbers 256, 900, and 1156 ;
- 9thly, The numbers 1600, 1764, and 3364 ;
- 10thly, The numbers 12,544, 4356, and 16,900 ;
- 11thly, The numbers 576, 4900, and 5476 ; and,
- 12thly, The numbers 9216, 12,100 and 21,316.

Art. 19. The square-roots of the foregoing twelve sets of numbers are as follows ; to wit,

- 1st, The numbers 3, 4, and 5 ;
- 2dly, The numbers 4, 3, and 5 ;
- 3dly, The numbers 15, 8, and 17 ;
- 4thly, The numbers 12, 5, and 13 ;
- 5thly, The numbers 5, 12, and 13 ;
- 6thly, The numbers 21, 20, and 29 ;
- 7thly, The numbers 45, 28, and 53 ;
- 8thly, The numbers 16, 30, and 34 ;
- 9thly, The numbers 40, 42, and 58 ;
- 10thly, The numbers 112, 66, and 130 ;
- 11thly, The numbers 24, 70, and 74 ; and
- 12thly, The numbers 96, 110, and 146.

Art. 20. If we divide the numbers of some of the foregoing

going sets of numbers by 2, (which will not alter the proportion of such numbers to each other,) the said twelve sets of numbers will be as follows; to wit,

- 1st, 3, 4, and 5;
- 2dly, 4, 3, and 5;
- 3dly, 15, 8, and 17;
- 4thly, 12, 5, and 13;
- 5thly, 5, 12, and 13;
- 6thly, 21, 20, and 29;
- 7thly, 45, 28, and 53;
- 8thly, 8, 15, and 17;
- 9thly, 20, 21, and 29;
- 10thly, 56, 33, and 65;
- 11thly, 12, 35, and 37; and
- 12thly, 48, 55, and 73.

And all these twelve sets of numbers will express the lengths of the sides of different right-angled triangles.

Art. 21. In the three foregoing general expressions

$$\frac{(mm - nn)^2}{mm + nn^2} \times aa, \frac{4m^2n^2}{mm + nn^2} \times aa, \text{ and } aa, \text{ or } \frac{mm + nn^2}{mm + nn^2} \times aa,$$

obtained in art. 12, the letter n answers to the letter e in the three former general expressions $nn - ee$, $2ne$, and $nn + ee$, obtained in art. 4; and the letter m in the three expressions obtained in art. 12, answers to the letter n in the three former expressions obtained in art. 4. And accordingly we find that, where the same two numbers have been substituted instead of the letters m and n in the general expressions

$$\frac{(mm - nn)^2}{mm + nn^2} \times aa, \frac{4m^2n^2}{mm + nn^2} \times aa, \text{ and } \frac{mm + nn^2}{mm + nn^2} \times aa, \text{ as}$$

were substituted in art. 4, instead of the letters n and e respectively, in the general expressions $nn - ee$, $2ne$, and $nn + ee$, they have produced the same three numbers to represent the lengths of the three sides of a right-angled triangle.

triangle. The only difference between these two sets of general expressions is, that the three expressions $\frac{mm - nn^2}{mm + nn^2} \times aa$, $\frac{4m^2n^2}{mm + nn^2} \times aa$, and $\frac{mm + nn^2}{mm + nn^2} \times aa$, give us the squares of such numbers as will express the sides of right-angled triangles, and the three expressions $mm - ee$, $2ne$, and $nn + ee$, give us the said numbers themselves.

Art. 22. The whole numbers that express the lengths of the sides of a right-angled triangle, cannot, when they are reduced to the lowest numbers possible by dividing them by their common divisors, be, all of them, even numbers.

For, if they were all even numbers, they might all be divided by 2, either once, or more than once, till at last some of the quotients would be odd numbers. Thus, for example, the three even numbers 16, 30, and 34, which have been found above to express the three sides of a right-angled triangle, are all divisible by 2, and are by such division reduced to the three lesser numbers 8, 15, and 17, of which the two latter are odd numbers.

Art. 23. And further, the said numbers that express the lengths of the sides of a right-angled triangle, cannot be, all of them, odd numbers.

For, if the two numbers expressing the lengths of the two sides of the triangle that contain the right angle, were, both of them, odd numbers, their squares would also be odd numbers; because the square of every odd number is an odd number taken an odd number of times, and consequently must be an odd number: and consequently the sum of the said two squares must be an even number, because two odd numbers added together always make an even number. Therefore the square of the number representing the hypotenuse of the triangle, being equal to the said sum of the two other squares, must be an even number. And

And consequently the square-root of the said number, that is, the number representing the hypotenuse of the triangle, must be an even number likewise, which is contrary to the supposition. Therefore it is impossible that all the three numbers which represent the lengths of the sides of a right-angled triangle, should be odd numbers.

Art. 24. There is also another way of finding several whole numbers that shall represent the lengths of the sides of different right-angled triangles; which consists in forming a list, or table, of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, set down in their proper order; and a list of the differences of the said squares, (which, it is well known, are the several odd numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, &c, taken in their natural order,) and adding together the said differences that follow any given square number in the list of squares till their sum amounts to another square number. Such a table, carried as far as the square of 100, will be as follows.

A TABLE

A TABLE of the Squares of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 100, and of their several Differences from each other, and likewise of the Differences of those Differences, or of the Second Differences of the said Squares.

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the said Squares.	Their 2d Differences, or the Differences of their Differences.
1	1		
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2
7	49	13	2
8	64	15	2
9	81	17	2
10	100	19	2
11	121	21	2
12	144	23	2
13	169	25	2
14	196	27	2
15	225	29	2
16	256	31	2
17	289	33	2
18	324	35	2
19	361	37	2
20	400	39	2
21	441	41	2
22	484	43	2
23	529	45	2
24	576	47	2
25	625	49	2
26	676	51	2
27	729	53	2
28	784	55	2
		57	2

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The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the said Squares.	Their 2d Differences, or the Differences of their Differences.
29	841	59	2
30	900	61	2
31	961	63	2
32	1024	65	2
33	1089	67	2
34	1156	69	2
35	1225	71	2
36	1296	73	2
37	1369	75	2
38	1444	77	2
39	1521	79	2
40	1600	81	2
41	1681	83	2
42	1764	85	2
43	1849	87	2
44	1936	89	2
45	2025	91	2
46	2116	93	2
47	2209	95	2
48	2304	97	2
49	2401	99	2
50	2500	101	2
51	2601	103	2
52	2704	105	2
53	2809	107	2
54	2916	109	2
55	3025	111	2
56	3136	113	2
57	3249	115	2
58	3364	117	2
59	3481	119	2
60	3600	121	2
61	3721	123	2
62	3844	125	2
63	3969	127	2
64	4096	129	2

The

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the said Squares.	Their 2d Differences, or the Differences of their Differences.
65	4225	131	2
66	4356	133	2
67	4489	135	2
68	4624	137	2
69	4761	139	2
70	4900	141	2
71	5041	143	2
72	5184	145	2
73	5329	147	2
74	5476	149	2
75	5625	151	2
76	5776	153	2
77	5929	155	2
78	6084	157	2
79	6241	159	2
80	6400	161	2
81	6561	163	2
82	6724	165	2
83	6889	167	2
84	7056	169	2
85	7225	171	2
86	7396	173	2
87	7569	175	2
88	7744	177	2
89	7921	179	2
90	8100	181	2
91	8281	183	2
92	8464	185	2
93	8649	187	2
94	8836	189	2
95	9025	191	2
96	9216	193	2
97	9409	195	2
98	9604	197	2
99	9801	199	2
100	10,000		

Art. 25. In the foregoing table the first column contains all the natural numbers 1, 2, 3, 4, 5, &c, as far as 100; and the second column contains the squares of the said numbers set down even with the said numbers themselves, or so that every square number shall be in the same horizontal line with the natural number of which it is the square; and the third column contains the differences of the square numbers in the second column, set down in lines between the lines in which the squares themselves, of which they are the differences, are set down; and the fourth column contains the differences of the foregoing differences that are set down in the third column, or the second differences of the square numbers that are set down in the second column; and each of these second differences is set down in a line that lies between the two lines in which the two first differences, of which it is the difference, are set down. And we may observe, that the differences set down in the third column are the odd numbers 3, 5, 7, 9, 11, 13, 15, &c, taken in their natural order; and the second differences set down in the fourth column, being the differences of the said odd numbers, are all equal to each other, and to the number 2.

Art. 26. From the construction of this table, it is obvious that, if we take any number whatsoever in the first column, and look out its square in the second column, and then add together the several differences in the third column that follow the said square number in the second column, till the sum of the said differences shall amount to a square number, the square-root of the said square number, or the number in the first column that is placed even with it, and the number first taken, will express the lengths of the two sides of a right-angled triangle that contain the right angle, and the number in the first column that immediately follows the last of the said differences in the third column, so added together, will express the length of the hypotenuse of the same triangle. Thus, for example, if we take the number 3 in the first column, and find its square, to wit, 9, in the second column, and then add up
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the differences 7, 9, 11, 13, &c, in the third column, which come after the square number 9, till their sum becomes equal to a square number, (for which purpose we need only add together the two differences 7 and 9, because their sum is 16, which is a square number,) the square root of the said square number 16, or the number in the first column which is placed even with it, to wit, the number 4, and the number 3, which was taken at first, will express the lengths of the two sides of a right-angled triangle that contain the right angle, and the number 5 in the first column, which immediately follows the last of the said two differences, so added together, to wit, 9, will express the length of the hypotenuse of the same triangle. So that we shall hereby obtain the whole numbers 3, 4, and 5, to express the lengths of the three sides of a right-angled triangle: which numbers we had before obtained by both the former methods of investigation.

Art. 27. If we look in the 3d column, or column of differences, for those differences that are themselves square numbers, without being added to any of the foregoing, or following, differences to make them so, (which differences are but few in number, being only the following six numbers, to wit, 9, 25, 49, 81, 121, and 169, in the whole table,) we may at once obtain from each of these differences a set of numbers that will express the lengths of the three sides of a right-angled triangle.

For, since 9 is a square number, and is likewise the difference between the two contiguous square numbers 16 and 25, and consequently $9 + 16 = 25$, it follows that the square-roots of these three numbers 9, 16, and 25, that is, the three numbers 3, 4, and 5, will express the three sides of a right-angled triangle.

And, in like manner, since 25 is a square number, and is likewise the difference between the two contiguous square numbers 144 and 169, and consequently $25 + 144 = 169$, it follows that the square-roots of the three numbers

25,

25, 144, and 169, that is, the three numbers 5, 12, and 13, will express the three sides of a right-angled triangle.

And, since 49 is a square number, and is likewise the difference of the two contiguous square numbers 576 and 625, and consequently $49 + 576$ are equal to 625, it follows that the square-roots of the three numbers 49, 576, and 625, that is, the three numbers 7, 24, and 25, will express the three sides of a right-angled triangle.

And, since 81 is a square number, and is likewise the difference of the two contiguous square numbers 1600 and 1681, and consequently $81 + 1600$ are $= 1681$, it follows that the square-roots of the three numbers 81, 1600, and 1681, that is, the three numbers 9, 40, and 41, will express the three sides of a right-angled triangle.

And, since 121 is a square number, and is likewise the difference of the two contiguous square numbers 3600 and 3721, and consequently $121 + 3600$ are $= 3721$, it follows that the square-roots of the three numbers 121, 3600, and 3721, that is, the three numbers 11, 60, and 61, will express the three sides of a right-angled triangle.

And, lastly, since 169 is a square number, and likewise is the difference of the two contiguous square numbers 7056 and 7225, and consequently $169 + 7056$ are $= 7225$, it follows that the square-roots of the three numbers 169, 7056, and 7225, that is, the three numbers 13, 84, and 85, will express the three sides of a right-angled triangle.

Art. 28. In this way of obtaining three numbers that shall express the three sides of a right-angled triangle, namely, by chusing such numbers in the 3d column, or column of differences, as are themselves square numbers, it is evident that the number expressing the hypotenuse of the triangle will always exceed the greater of the other two numbers, that express its sides, by an unit. But, when we take several successive differences, of which the sum is equal to a square number, the number that expresses the hypotenuse of the triangle, will exceed the number taken at first, and
between

between the square of which, and the square of the number representing the hypotenuse, the several differences that are added together lie, by as many units as there are differences that have been so added together in order to make a square number. Of this it will not be amiss to give a few examples.

Art. 29. Let us take 8 for the first number. Then, since the square of 8 is 64, we must begin with the difference 17, which comes immediately after 64, and we must add together this difference 17, and the following differences 19, 21, 23, 25, 27, &c, till their sum amounts to a square number. For this purpose we need add together only two of these differences, to wit, 17 and 19. For $17 + 19 = 36$, which is a square number, to wit, the square of 6. Therefore the first number 8, and the number 6, (or the square-root of the sum of these two differences), and the number 10, (or the square-root of the square number 100, which comes immediately after the last difference 19), will be three numbers that will express the three sides of a right-angled triangle. For $8^2 + 6^2$ will be $= 10^2$, or $64 + 36$ will be $= 100$. And the number 10, (which represents the hypotenuse of the triangle,) exceeds the first number 8, (which represents the greater of its two sides,) by 2, or two units, or the same number of units as there were differences added together, in order to produce the square number 36.

And, if, instead of taking only two of the differences 17, 19, 21, 23, 25, 27, 29, &c, we take nine of them, we shall find their sum to be equal to another square number, to wit, 225, which is the square of 15. For $17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 = 225$. Therefore the first number 8, and the number 15, (or the square-root of the sum of these nine differences,) and the number 17, (or the square-root of the square number 289, which comes immediately after the last difference 33,) will be three numbers that will express the three sides of a right-angled triangle. For $8^2 + 15^2$ will be $= 17^2$, or $64 +$
225

225 will be $= 289$. And the number 17, (which represents the hypotenuse of the triangle,) exceeds the first number 8, (which represents one of the sides that inclose the right angle,) by 9, or nine units, or the same number of units as there were differences added together, in order to produce the square number 225.

Art. 30. Now let the first number be 20. Then, since the square of 20 is 400, we must begin with the difference 41, which comes immediately after 400, and must add together this difference 41, and the following differences 43, 45, 47, 49, 51, &c, till their sum amounts to a square number. And for this purpose it will be necessary to add together nine of these differences. For $41 + 43 + 45 + 47 + 49 + 51 + 53 + 55 + 57$ are $= 441$, which is the square of 21. Therefore the first number 20, and the number 21, (or the square-root of the sum of these nine differences,) and the number 29, (or the square-root of the square number 841, which comes immediately after the last difference 57,) will be three numbers that will express the three sides of a right-angled triangle. For $20^2 + 21^2$ will be $= 29^2$, or $400 + 441$ will be $= 841$. And the number 29, (which represents the hypotenuse of the triangle,) exceeds the first number 20, (which represents one of the sides that inclose the right-angle,) by 9, or nine units, or the same number of units as there were differences added together in order to produce the square number 441.

Art. 31. Now let the first number be 28. Then, since the square of 28 is 784, we must begin with the difference 57, which comes immediately after 784, and we must add together this difference 57, and the following differences 59, 61, 63, 65, &c, till their sum amounts to a square number. And for this purpose it will be necessary to add together seven of these differences. For $57 + 59 + 61 + 63 + 65 + 67 + 69$ are $= 441$, which is the square of 21. Therefore the first number 28, and the number 21,

(or

(or the square-root of the sum of these seven differences,) and the number 35, (or the square-root of the square number 1225, which comes immediately after the last difference 69,) will be three numbers that will express the three sides of a right-angled triangle. For $28^2 + 21^2$ will be $= 35^2$, or $784 + 441$ will be $= 1225$. And the number 35, (which represents the hypotenuse of the triangle,) exceeds the first number 28, (which represents one of the sides that inclose the right angle,) by 7, or seven units, or the same number of units as there were differences added together, in order to produce the square number 441.

These examples, I apprehend, are sufficient to explain this method of obtaining different sets of whole numbers that shall express the lengths of the sides of different right-angled triangles. And with them I shall conclude this little tract,

End of the Discourse concerning the Methods of finding Rational Numbers that express the Sides of Right-angled Triangles.

OF THE
DIFFERENCES

OF THE
CUBES

OF THE
NATURAL NUMBERS 1, 2, 3, 4, 5, 6, 7, &c.

Article 1. We have seen in the table of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, set down in the foregoing Tract, that the first differences of the squares of those numbers are the odd numbers 3, 5, 7, 9, 11, 13, 15, 17, &c, in their natural order, and their second differences, or the differences of their first differences, are all equal to each other, and to the number 2. We will now set down a table of the cubes of the several natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and of their differences, and the differences of those first differences, and the differences of those second differences; by which it will appear that the cubes of the said numbers have three orders of differences, and that their differences of the third order are all equal to each other and to the number 6. This Table will be as follows :

3 N 2

A TABLE

A TABLE of the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, as far as 100; together with their First, Second, and Third, Differences.

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the said Cubes.
1	1	7	12	6
2	8	19	28	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	6
9	729	271	60	6
10	1000	331	66	6
11	1331	397	72	6
12	1728	469	78	6
13	2197	547	84	6
14	2744	631	90	6
15	3375	721	96	6
16	4096	817	102	6
17	4913	919	108	6
18	5832	1027	114	6
19	6859	1141	120	6
20	8000	1261	126	6
21	9261	1387	132	6
22	10648	1519	138	6
23	12167	1657	144	6
24	13824	1801	150	6
25	15625	1951	156	6
26	17576	2107	162	6
27	19683	2269	168	6
28	21952	2437	174	6
29	24389	2611	180	6

The

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Diffs. of the said Cubes.
30	27,000	2791	186	
31	29,791	2977	192	6
32	32,768	3169	198	6
33	35,937	3367	204	6
34	39,304	3571	210	6
35	42,875	3781	216	6
36	46,656	3997	222	6
37	50,653	4219	228	6
38	54,872	4447	234	6
39	59,319	4681	240	6
40	64,000	4921	246	6
41	68,921	5167	252	6
42	74,088	5419	258	6
43	79,507	5677	264	6
44	85,184	5941	270	6
45	91,125	6211	276	6
46	97,336	6487	282	6
47	103,823	6769	288	6
48	110,592	7057	294	6
49	117,649	7351	300	6
50	125,000	7651	306	6
51	132,651	7957	312	6
52	140,608	8269	318	6
53	148,877	8587	324	6
54	157,464	8911	330	6
55	166,375	9241	336	6
56	175,616	9577	342	6
57	185,193	9919	348	6
58	195,112	10267	354	6
59	205,379	10621	360	6
60	216,000	10981	366	6
61	226,981	11347	372	6
62	238,328	11719	378	6
63	250,047	12097	384	6
64	262,144	12481	390	6

The

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Diffs. or the Diffs. of the said Cubes.
65	274,625			
66	287,496	12871	396	6
67	300,763	13267	402	6
68	314,432	13669	408	6
69	328,509	14077	414	6
70	343,000	14491	420	6
71	357,911	14911	426	6
72	373,248	15337	432	6
73	389,017	15769	438	6
74	405,224	16207	444	6
75	421,875	16651	450	6
76	438,976	17101	456	6
77	456,533	17557	462	6
78	474,552	18019	468	6
79	493,039	18487	474	6
80	512,000	18961	480	6
81	531,441	19441	486	6
82	551,368	19927	492	6
83	571,787	20419	498	6
84	592,704	20917	504	6
85	614,125	21421	510	6
86	636,056	21931	516	6
87	658,503	22447	522	9
88	681,472	22969	528	6
89	704,969	23497	534	6
90	729,000	24031	540	6
91	753,571	24571	546	6
92	778,688	25117	552	6
93	804,357	25669	558	6
94	830,584	26227	564	6
95	857,375	26791	570	6
96	884,736	27361	576	6
97	912,673	27937	582	6
98	941,192	28519	588	6
99	970,299	29107	594	6
100	1,000,000	29701		

Art. 2.

Art. 2. And in like manner it will be found that the fourth powers of the natural numbers 1, 2, 3, 4, 5, &c, will have four orders of differences, and that their fifth powers will have five orders of differences; and, in general, that their n th powers, n being any whole number whatsoever, will have n orders of differences. This is a curious property of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, and has been long known to Mathematicians. The celebrated Mr. Leibnitz, of Hanover, had taken notice of it before the month of February, 1673; and it had been observed before him by a French Mathematician, named *Mouton*, (who was a Canon of the Church of Lyons,) in a book on the apparent diameters of the Sun and Moon; but which Mr. Leibnitz declared he had not seen at the time he made the same discovery. Mr. Leibnitz's manner of considering the subject is explained pretty fully in a large extract from a letter of his to Mr. Henry Oldenburgh, the Secretary of the Royal Society of London, dated at London on the 3d of February, 1673, which has been published in the *Commercium Epistolicum* of Mr. John Collins and other Mathematicians of that time. This extract, as it contains several interesting particulars relating to these numbers, I shall here insert at length from the said *Commercium Epistolicum*, pages 108, 109, 110, - - - 114. It is as follows.

Art. 3. *Excerpta ex Epistola Domini Gothofredi Gulielmi Leibnitzii ad Dominum Oldenburgh, Londini, Anno 1673, 3^{tio} die Februarii, data.*

Hujus Autographon in scriniis Regiæ Societatis extat, et exemplar ejus in Libro Epistolarum dictæ Societatis, N^o. 6, pag. 35, descriptum legitur.

CUM heri apud illustrissimum *Boylium* incidissem in clarissimum *Pellium* Mathematicum insignem, ac de Numeris incidisset

incidisset mentio, commemoravi ego, ductus occasione Sermonum, esse mihi methodum ex quodam differentiarum genere, quas voco *generatrices*, colligendi terminos Seriei cujuscunque continuè crescentis vel decrescantis. Differentias autem generatrices voco, si datæ Seriei inveniantur differentiæ, & differentiæ differentiarum, & ipsarum ex differentiis differentiarum differentiæ, &c. & series constituatur ex termino primo & primâ differentiâ, & primâ differentiâ differentiarum, & primâ differentiâ ex differentiis differentiarum, &c. ea Series erit differentiarum generatricium, ut si Series continuè crescens vel decrescens fuerit a, b, c, d .

Positâ igitur differentiæ Notâ,] differentiæ generatrices erunt:

$$1a . 2a \cup b . 3a \cup b \cup b \cup c . 4a \cup b \cup b \cup b \cup c \cup b \cup c \cup c \cup d .$$

$$\begin{array}{ccccccc} & & & & 4 & \overline{a \cup b \cup b \cup c} & \cup \overline{b \cup c \cup c \cup d} \\ & & & & 3 & a \cup b \cup b \cup c & \cup b \cup c \cup c \cup d \\ & & & 2 & a \cup b & b \cup c & c \cup d \\ & 1 & a & & b & & c & & d \end{array}$$

Aut in Numeris; si Series sit Numerorum cubicorum deinceps ab unitate crescentium, differentiæ generatrices erunt numeri 0, 1, 6, 6. Voco autem *generatrices*, quia ex iis certo modo multiplicatis producentur termini Seriei; cujus usus tum maximè apparet, cum differentiæ generatrices sunt finitæ, termini autem Seriei infiniti; ut in proposito exemplo Numerorum Cubicorum.

		0	0	0		
		6	6	6	6	
	6	12	18	24	30	
1	7	19	37	61	91	
0	1	8	27	64	125	216

Hoc cum audisset clarissimus *Pellius*, respondit, id jam fuisse in literas relatum à D. *Mouton*, Canonico *Lugdunensi*, ex obser-

observatione nobilissimi viri *Francisci Regnaldi Lugdunensis*, dudum in literario Orbe celebris, in libro laudati D. *Mouton* de diametris apparentibus Solis & Lunæ. Ego qui ex Epistolâ quâdam à *Regnaldo* ad *Monconisium* scriptâ, & Diario itinerum *Monconisiano* insertâ, nomen D. *Moutoni* & designata ejus duo didiceram; Diametros Luminarium apparentes, & consilium de mensuris rerum ad posteros transmittendis; ignorabam tamen librum ipsum prodiisse: quare apud D. *Oldenburgium Societatis Regalis* Secretarium, sumtum tumultuariè percurri, & inveni verissimè dixisse *Pellium*. Sed & mihi tamen dandam operam credidi, ne qua in animis relinqueretur suspicio, quasi, tacito * inventoris nomine, alienis meditationibus honorem mihi quærere voluissim; & spero apparitum esse, non adedè egenum me meditationum propriarum ut cogar alienas emendicare. Duobus autem argumentis ingenuitatem meam vindicabo. Primò, si ipsas Schedas meas confusas, in quibus non tantum inventio mea sed & inveniendi modus occasioque apparet, monstrem: deinde, si quædam momenti maximi *Regnaldo Mouton*que indicta addam, quæ ab hesterno vespere confinxisse me non sit verisimile, quæque non possunt facile expectari à Transcriptore.

Ex Schedis meis occasio inventi hæc apparet: quærebam modum inveniendi differentias omnis generis potestatum, quemadmodum constat differentias Quadratorum esse numeros impares; inveneramque regulam generalem ejusmodi.

Datâ potentiâ gradûs dati præcedente, invenire sequentem (vel contrâ) distantiam datæ vel radicum datarum; seu invenire potentiarum gradûs dati utcunque distantium differentias. Multiplicetur potentia gradûs, proximè præcedentis radice majoris per differentiam radicum; & differentia potentiarum gradûs proximè præcedentis multiplicetur per radicem minorem: productorum summa erit quæsitâ differentia potentiarum, quarum radices sunt datæ. Eandem regulam ita inflexeram, ut sufficeret, præter radices, cujuslibet gradûs, etiamsi non proximè præcedentis, potentias datarum radicum dari, ad differentias potentiarum alterius cu-

* Id est, celata.

juscunque, licet altioris, gradûs inveniendas. Et ostendi quod in Quadratis observatur, numeros impares esse eorum differentias, id non nisi regulæ propositæ subsumptionem esse.

His meditationibus defixus, quemadmodum in Quadratis differentiæ sunt numeri impares, ita quoque quæsiui quales essent differentiæ Cuborum, quæ cum irregulares videntur, quæsiui differentias differentiarum, donec inveni differentias tertias esse numeros senarios. Hæc observatio mihi aliam peperit: videbam enim ex differentiis præcedentibus generari terminos differentiâsque sequentes, ac proinde, ex primis, quas idèd voco *generatrices*, (ut hoc loco 0 . 1 . 6 . 6,) sequentes omnes. Hoc concluso, restabat invenire, quo additionis, multiplicationisve, aut horum complicationis, genere, termini sequentes ex differentiis generatricibus producerentur. Atque ita resolvendo experiundoque deprehendi primum Terminum 0 componi ex primâ differentiâ generatrice 0 sumtâ semèl, seu vice unâ: Secundum 1 ex primâ 0 semèl & secundâ 1 semèl: Tertium 8 ex primâ 0 semèl, secundâ 1 bis & tertiâ 6 semèl: nam $0 \times 1 + 1 \times 2 + 6 \times 1 = 8$. Quartum 27, ex primâ 0 semèl, secundâ 1 tèr, tertiâ 6 tèr, quartâ 6 semèl: nam $0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27$, &c. Idque Analysis mihi universale esse comprobavit. Hæc fuit occasio observationis meæ, longè alia à *Moutonianâ*, qui cum in Tabulis condendis laboraret, in hoc calculandi compendium cum *Reginaldo* incidit: nec vel illi vel *Reginaldo* adimenda laus; quod & *Briggius* in Logarithmicis suis jam olim talia quædam, observante *Pillio*, ex parte advertit. Mihi hoc superest ut addam nonnulla illis indicta, ad amoliendum Transcriptoris nomen; neque enim interest Reipublicæ quis observaverit, interest quid observetur. Primum, ergò illud adjicio, quod apud *Moutonium* non extat, & caput tamen rei est: quinam sint illi numeri, quorum Tabulam ille exhibet in infinitum continuandam, quorum ductu in differentias generatrices, productis inter se junctis, termini Serierum generentur. Vides enim ex ipso modo quo tabula ab eo pag. 385, exhibetur, non fuisse id ei satis exploratum; aliqui enim verisimile est ita Tabulam fuisse dispositurum, ut ea numerorum connexio atque

atque harmonia appareret; nisi quis de industriâ texisse dicat: ita enim se habet pars Tabulæ.

1	1					
2	1	1				
3	1	2	1			
(4)	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Apparet ex hujus Tabulæ constructione solam haberi rationem correspondens numerorum generantium cum numero Termini generati; ut cum terminus est quartus (4) producitur ex primâ differentiâ semel, secundâ ter 3, tertiâ ter 3, quartâ semel 1; ideo in eâdem (4) Lineâ transversâ locantur 1. 3. 3. 1. Sed vel non observavit vel dissimulavit autor correspondens numerorum, si à summo deorsum eundo per columnas disponantur hoc modo,

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Ita enim statim vera genuinâque eorum natura ac generatio apparet; esse, scilicet, eos numeros quos *Combinatorios* appellare soleo, de quibus multa dixi in dissertatiunculâ de Arte Combinatoriâ; quosque alii appellant Ordines numeri-

3 O 2

cos;

cos; alii specie primam columnam Unitatum; secundam Numerorum naturalium, tertiam Triangularem, quartam Pyramidalium, quintam Triangulo-Triangularem, &c. de quibus integer extat Tractatus *Paschalii* sub titulo Trianguli Arithmetici; in quo tamen proprietatem numerorum ejusmodi tam illustrem tamque naturalem * non observatam sum miratus. Sed est profectò casus quidam in inveniendis, qui non semper maximis ingeniis maxima, sed sæpe etiam mediocribus nonnulla offert.

Hinc jam vera numerorum istorum natura, & Tabulæ constructio, sive à *Reginaldo* sive à *Moutonio* dissimulata, intelligitur: semper enim terminus datus columnæ datæ componitur ex termino præcedente columnæ tam præcedentis quam datæ: Atque illud quoque apparet, non opus esse molesto calculo ad Tabulam à *Moutonio* propositam continuandam, ut ipse postulat; cum hæ numerorum Series passim jam tradantur calculenturque.

Cæterùm *Moutonius* observatione istâ ad interponendas medias proportionales inter duos extremos numeros datos; ego ad inveniendos ipsos numeros extremos in infinitum cum eorum differentiis, utendum censebam. Hinc ille, non nisi cum differentiæ ultimæ evanescunt (aut penè evanescunt) usum regulæ invenit; ego detexi innumerabiles casus, regulâ quâdam inobservatâ comprehendendos; ubi possum ex datis numeris finitis certo modo multiplicatis producere numeros plurimarum Serierum in infinitum euntium, etsi differentiæ earum non evanescant.

Ex iisdem fundamentis possum efficere in progressionibus problemata plurima; aut in Numeris singularibus, aut in Rationibus vel Fractionibus: possum enim progressionem ad-

* Imò observata fuit. Vide *Paschalii Triangulum Arithmeticum*, Parisiis Anno 1665 editum, pag. 2. ubi definitionum antepenultima hæc est.

Le nombre de chaque cellule est égal à celui de la cellule qui la précède dans son rang perpendiculaire, plus à celui de la cellule qui la précède dans son rang parallèle. Ainsi la cellule F, c'est à dire le nombre de la cellule F, égale la cellule C plus la cellule E; & ainsi des autres.

ere subtrahereque, imò multiplicare quoque & dividere, id-
que compendiosè.

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$
$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$	$\frac{1}{16}$
$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{19}$	$\frac{1}{20}$
$\frac{1}{21}$	$\frac{1}{22}$	$\frac{1}{23}$	$\frac{1}{24}$
$\frac{1}{25}$	$\frac{1}{26}$	$\frac{1}{27}$	$\frac{1}{28}$
$\frac{1}{29}$	$\frac{1}{30}$	$\frac{1}{31}$	$\frac{1}{32}$
$\frac{1}{33}$	$\frac{1}{34}$	$\frac{1}{35}$	$\frac{1}{36}$
$\frac{1}{37}$	$\frac{1}{38}$	$\frac{1}{39}$	$\frac{1}{40}$
$\frac{1}{41}$	$\frac{1}{42}$	$\frac{1}{43}$	$\frac{1}{44}$
$\frac{1}{45}$	$\frac{1}{46}$	$\frac{1}{47}$	$\frac{1}{48}$
$\frac{1}{49}$	$\frac{1}{50}$	$\frac{1}{51}$	$\frac{1}{52}$
$\frac{1}{53}$	$\frac{1}{54}$	$\frac{1}{55}$	$\frac{1}{56}$
$\frac{1}{57}$	$\frac{1}{58}$	$\frac{1}{59}$	$\frac{1}{60}$
$\frac{1}{61}$	$\frac{1}{62}$	$\frac{1}{63}$	$\frac{1}{64}$
$\frac{1}{65}$	$\frac{1}{66}$	$\frac{1}{67}$	$\frac{1}{68}$
$\frac{1}{69}$	$\frac{1}{70}$	$\frac{1}{71}$	$\frac{1}{72}$
$\frac{1}{73}$	$\frac{1}{74}$	$\frac{1}{75}$	$\frac{1}{76}$
$\frac{1}{77}$	$\frac{1}{78}$	$\frac{1}{79}$	$\frac{1}{80}$
$\frac{1}{81}$	$\frac{1}{82}$	$\frac{1}{83}$	$\frac{1}{84}$
$\frac{1}{85}$	$\frac{1}{86}$	$\frac{1}{87}$	$\frac{1}{88}$
$\frac{1}{89}$	$\frac{1}{90}$	$\frac{1}{91}$	$\frac{1}{92}$
$\frac{1}{93}$	$\frac{1}{94}$	$\frac{1}{95}$	$\frac{1}{96}$
$\frac{1}{97}$	$\frac{1}{98}$	$\frac{1}{99}$	$\frac{1}{100}$

$\frac{1}{101}$ $\frac{1}{102}$ $\frac{1}{103}$ $\frac{1}{104}$

Multa alia circa hos numeros observata sunt à me, ex quibus illud eminet, quod modum habeo summam inveniendi Seriei Fractionum in infinitum decrescentium; quarum numerator Unitas, nominatores vero numeri isti Triangulares aut Pyramidales, aut Triangulo-Triangulares, &c.

End of the Extract from Mr. Leibnitz's Letter.

Art. 4. By the help of the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we may find the cube-root of any number exact to two places of figures, without the trouble of any calculation whatsoever, or by the mere inspection of the table. Thus, for example, if I wanted to find the cube-root of 2000, I need only look along the second column of the table, (which contains the cubes of the numbers set down in the first column,) till I found the two cube numbers which are nearest to the proposed number 2000, the one above it and the other below it. These numbers I should find to be 2197 and 1728; of which the former is the cube of 13, and the latter is the cube of 12. And hence I might conclude with certainty that, since the proposed number 2000 is greater than 1728, or the cube of 12, but less than 2197, or the cube of 13, its cube-root must be greater than 12, but less than 13, and consequently that the two first figures of it must be 12. And from the two first figures of the cube-root of any proposed number, we may derive the following figures of it to five, or six, or any greater number

number of figures that we may desire, by the method of approximation invented for this purpose by Monsieur De Lagney, which has since been approved and adopted by Dr. Halley and other Mathematicians, as the most convenient that can be taken. This method is as follows.

Monsieur De Lagney's Method of approximating to the Value of the Cube-root of any proposed Number, when the Two, or Three, first Figures of the said Cube-root are known.

Art. 5. If the number of which the cube-root is to be extracted be called c , and a number, consisting of two, or more figures, that is somewhat less than the true value of the said cube-root be called a , the remaining part of the said cube-root will be very nearly equal to the quantity $\frac{c - a^3}{c + 2a^3} \times a$, and consequently the whole of the said cube-root will be very nearly equal to $a + \frac{c - a^3}{c + 2a^3} \times a$; but it will always be a little greater than the said quantity. Also the said remaining part of the cube-root of c , which is to be added to its first value a , will be very nearly equal to the quantity $\sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}$, and consequently the whole of the said cube-root will be very nearly equal to $a + \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}$, or to $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; but it will always be a little less than that quantity. And this latter expression will be a little nearer to the true value of the cube-root of c than the former expression $a + \frac{c - a^3}{c + 2a^3} \times a$; but the difference will be so small as to be hardly worth considering.

Art. 6.

Art. 6. And, if a , or the first near value of the cube-root of the proposed number c , be a little greater than its true value, the quantity to be subtracted from a , in order to make it equal to the said true value, will be very nearly equal to the quantity $\frac{a^3 - c}{c + 2a^3} \times a$, and consequently the said cube-root will be very nearly equal to $a - \frac{a^3 - c}{c + 2a^3} \times a$; but it will be always a little greater than the said quantity.

Also the said difference between a and the true value of the cube-root of c , or quantity which is to be subtracted from a , in order to make it equal to the said cube-root, will be very nearly equal to the quantity $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$, and consequently the said cube-root will be very nearly equal to $a - \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, or to $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; but it will always be a little less than the said quantity. And this latter expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be a little nearer to the true value of the cube-root of c than the former expression $a - \frac{a^3 - c}{c + 2a^3} \times a$; but the difference will be so small as to be hardly worth considering.

Art. 7. The number of figures that will be exact in the second near value of the cube-root of the proposed number c that will be obtained by either of these four expressions $a + \frac{c - a^3}{c + 2a^3} \times a$, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, $a - \frac{a^3 - c}{c + 2a^3} \times a$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, (which last expression, we may observe, is the very same with the second expression,) is usually triple, or triple wanting one figure, and in the worst cases is triple wanting two figures, of the number of figures that are exact in the first near value, a .

An Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 8. Let it be required to extract the cube-root of the number 2; which answers to the solution of the Problem, so much celebrated amongst the Antients, of doubling the cube, or finding the length of the side of a cube that shall be double of a given cube.

Here I observe, in the first place, that, since the number 2000 is equal to 1000×2 , or to $10 \times 10 \times 10 \times 2$, the cube-root of 2000 must be equal to 10 times the cube-root of 2. But it appears from the foregoing table of cube numbers, that the cube-root of 2000 must be greater than 12, but less than 13, and consequently that the two first figures of it must be 12. Therefore the cube-root of 2 must be $(= \frac{12}{10} = \frac{10}{10} + \frac{2}{10}) = 1.2$, or the two first figures of the said cube-root must be 1.2. Here then we have $c = 2$, or 2.000,000, and $a = 1.2$, and consequently $a^3 = 1.728$, and $2a^3 = 3.456$, and $c + 2a^3 (= 2 + 3.456) = 5.456$, and $c - a^3 (= 2.000 - 1.728) = 0.272$, and $\overline{c - a^3} \times a (= 0.272 \times 1.2) = 3.264$, and $\frac{\overline{c - a^3} \times a}{c + 2a^3} (= \frac{3.264}{5.456}) = 0.5982$. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $= 1.2 + 0.05982$, or 1.259,82; or the second near value of the cube-root of 2, which is obtained by means of the first expression $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ given in art. 5, is 1.259,82.

The number of figures that are exact in this second near value, 1.259,82, of the cube-root of 2, is four, to wit, the figures 1.259, that is, three times as many figures, wanting two, as are contained in 1.2, or a , the first near value of the

the said cube-root, the more accurate value of which is 1.259,921,049, &c, which is greater than 1.259,82 by 0.000,1, &c.

The other expression given in art. 5, to wit, $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, may be computed as follows.

Since a is = 1.2, and c is = 2, we shall have $12a$ (= 12×1.2) = 14.4, and $\frac{a}{2}$ = 0.6, and $4c$ = 8.000, and $4c - a^3$ (= $8.000 - 1.728$) = 6.272, and $\frac{4c-a^3}{12a}$ (= $\frac{6.272}{14.4}$) = 0.435,555,555, &c, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{0.435,555,555, \text{ \&c.}}$) = 0.659,96, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ (= $0.6000,00 + 0.659,96$) = 1.259,96. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, or the second near value of the cube-root of 2, which is obtained by means of the second expression given in art. 5, is 1.259,96; which is exact in the first five figures 1.2599, and is greater than the true value of the said cube-root, to wit, 1.259,92, &c, by only 0.000,04, &c, which is less than the difference 0.000,1, by which 1.259,82, or the second value of the cube-root of 2, obtained by means of the former expression $a + \frac{c-a^3}{c+2a^3} \times a$, fell short of the true value of the said cube-root. But either of these second near values of the said cube-root, 1.259,82 and 1.259,96, is a great improvement upon its first near value, 1.2, though less than it would have been if the two first figures of the cube-root of the proposed number had been higher figures than 1 and 2.

Art. 9. And, if we repeat this process of approximation with either of the two expressions $a + \frac{c-a^3}{c+2a^3} \times a$, and $\frac{a}{2} +$

✓ $\sqrt[3]{\frac{4c-a^3}{12a}}$, taking the first four figures of the second values of the cube-root of 2, which have been already found, (and which are the same in both the foregoing calculations,) to wit, the four figures 1.259, for the basis of the next operation, we shall obtain the value of the said cube-root to a very great degree of exactness. This may be done in the manner following.

Let a be supposed = 1.259.

Then we shall have a^3 ($= 1.259^3$) = 1.995,616,979, and consequently $2a^3$ ($= 2 \times 1.995,616,979$) = 3.991,233,958, and $c + 2a^3$ ($= 2 + 3.991,233,958$) = 5.991,233,958, and $c - a^3$ ($= 2.000,000,000 - 1.995,616,979$) = 0.004,383,021, and $\frac{c-a^3}{c+2a^3} \times a$ ($= 0.004,383,021 \times 1.259$) = 0.005,518,223,439, and $\frac{\frac{c-a^3}{c+2a^3} \times a}{5.991,233,958}$ ($= \frac{0.005,518,223,439}{5.991,233,958}$) = 0.000,921,049,55, &c. Therefore $a + \frac{\frac{c-a^3}{c+2a^3} \times a}{5.991,233,958}$ will be ($= 1.259 + 0.000,921,049,55$, &c,) = 1.259,921,049,55, &c; which is exact in the first ten figures 1.259,921,049, the more accurate value of the cube-root of 2 being 1.259,921,049,89, which is greater than 1.259,921,049,55 by only the very small quantity 0.000,000,000,34.

Also we shall have $4c$ ($= 4 \times 2$) = 8.000,000,000, and $4c - a^3$ ($= 8.000,000,000 - 1.995,616,979$) = 6.004,383,021, and $12a$ ($= 12 \times 1.259$) = 15.108, and $\frac{4c-a^3}{12a}$ ($= \frac{6.004,383,021}{15.108}$) = 0.397,430,900,3, and $\sqrt[3]{\frac{4c-a^3}{12a}}$ ($= \sqrt[3]{0.397,430,900,3}$) = 0.630,421,050,01. Therefore $\frac{a}{2} + \sqrt[3]{\frac{4c-a^3}{12a}}$ will be ($= \frac{1.259}{2} + 0.630,421,050,01$) = 0.6295 + 0.630,421,050,01 = 1.259,921,050,01; which exceeds the more accurate value of the cube-root of 2, to wit,

wit, 1.259,921,049,89, by only the very small quantity 0.000,000,000,12, which is still less than the small quantity 0.000,000,000,34.

Another Example of the Extraction of the Cube-root of a Number, by the same Method of Approximation.

Art. 10. Let it be proposed to find, in inches and decimal parts of an inch, the side of a cube that is equal to the English measure called a gallon, which contains 231 cubick inches; or, in other words, let it be required to find the cube-root of the number 231.

Now, if we look along the column of cube numbers in the foregoing table, we shall find that 216 is the cube of 6, and that 343 is the cube of 7. Therefore we may conclude that the cube-root of the proposed number 231 must be greater than 6, but less than 7. We will therefore take 6 for the value of a , or for the first near value of $\sqrt[3]{231}$, with which we are to begin our approximation.

Now, since c is, in this example, equal to 231, and a is $= 6$, we shall have $a^3 (= 6^3) = 216$, and $2a^3 = 432$, and $c + 2a^3 (= 231 + 432) = 663$, and $c - a^3 (= 231 - 216) = 15$, and $c - a^3 \times a (= 15 \times 6) = 90$, and $\frac{c - a^3 \times a}{c + 2a^3} (= \frac{90}{663} = 0.13$. Therefore $a + \frac{c - a^3 \times a}{c + 2a^3}$ will be $(= 6 + 0.13) = 6.13$; which is therefore the second near value of the cube-root of 231.

Now let a be taken $= 6.13$, in order to obtain a third near value of the cube-root of 231.

Then we shall have $a^3 (= 6.13^3) = 230.346,397$, and $2a^3 (= 2 \times 230.346,397) = 460.692,794$, and $c + 2a^3$
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(=

($= 231 + 460.692,794$) $= 691.692,794$, and $c - a^3 (= 231.000,000 - 230.346,397) = 0.653,603$, and $\frac{c - a^3}{c + 2a^2} \times a$
 $(= 0.653,603 \times 6.13) = 4.006,586,39$, and $\frac{c - a^3}{c + 2a^2} \times a$
 $(= \frac{4.006,586,39}{691.692,794}) = 0.005,792,436$. Therefore $a + \frac{c - a^3}{c + 2a^2} \times a$
 will be ($= 6.13 + 0.005,792,436$) $= 6.135,792,436$;
 which is therefore the third near value of the cube-root of
 231, or of the length, in inches and decimal parts of an inch,
 of the side of a cube that contains an English gallon. Q. E. I.

This number 6.135,792,436, is exact in the first nine
 figures, 6.135,792,43, the more accurate value of the cube-
 root of 231, being, according to Dr. Halley, (from whose
 tract upon this subject in the Philosophical Transactions
 this example is taken,) 6.135,792,439,661,958, &c. There-
 fore the number of figures obtained exactly in this instance
 by means of the expression $a + \frac{c - a^3}{c + 2a^2} \times a$ is just triple of
 the number of figures that are contained in a , or 6.13,
 agreeably to what is observed above in art. 7.

Art. 11. The other expression of the third near value of
 the cube-root of 231, to wit, the expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$;
 may be computed as follows.

Since c is $= 231$, and a is $= 6.13$, we shall have $4c (= 4 \times 231) = 924$, and $\frac{a}{2} (= \frac{6.13}{2}) = 3.065$, and $12a (= 12 \times 6.13) = 73.56$, and $a^3 (= 6.13^3) = 230.346,397$,
 and $4c - a^3 (= 924.000,000 - 230.346,397) = 693.653,603$,
 and $\frac{4c - a^3}{12a} (= \frac{693.653,603}{73.56}) = 9.429,766,218,053,289,8$,
 and $\sqrt{\frac{4c - a^3}{12a}} (= \sqrt{9.429,766,218,053,289,8}) =$
 $3.070,792,441$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be ($= 3.065$
 +

+ 3.070,792,441) = 6,135,792,441; or the third near value of the cube-root of 231, obtained by means of the expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 6.135,792,441.

Q. E. I.

Art. 12. This number 6.135,792,441, obtained by means of the expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, exceeds the more accurate value of the cube-root of 231, to wit, 6.135,792,439, &c, by only the small fraction 0.000,000,002; which is somewhat less than the small-fraction 0.000,000,003, by which the former number 6.135,792,436, that was obtained by means of the expression $a + \frac{c-a^3}{c+2a^3} \times a$, falls short of the said more accurate value. But both these differences enter in the same place of decimal fractions, to wit, the ninth place, and therefore the small difference of exactness in these two expressions $a + \frac{c-a^3}{c+2a^3} \times a$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is not worth attending to. But Dr. Halley gives the preference to the latter expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ on another account, to wit, because he thinks the extraction of the square-root of the fraction $\frac{4c-a^3}{12a}$ a less laborious operation than the division of $c - a^3 \times a$ by the great divisor $c + 2a^3$. His words are as follows. "And this Formula [the irrational formula $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, though he uses a somewhat different notation,] is deservedly preferable to the rational [or $a + \frac{c-a^3}{c+2a^3} \times a$,] upon the account of the great divisor, which is not to be managed without a great deal of labour; whereas the extraction of the square-root proceeds much more easily, as manifold experience has taught me."

These

These two examples are taken from a very ingenious and useful tract on this subject, intitled, *A new, exact, and easy Method of finding the Roots of Equations Generally, and that without any previous Reduction*; written by the celebrated Dr. Edmund Halley, and published first in the Philosophical Transactions for the month of May 1694, Number 210, and afterwards in the year 1708, in the second volume of the Collection of Mathematical and Philosophical Tracts, intitled *Miscellanea Curiosa*, in three volumes octavo. See the 2d volume of the said *Miscellanea Curiosa*, pages 70, 71, 72, 73, 74, and 75.

A Third Example of the Extraction of the Cube-root of a Number, by the same Method of Approximation.

Art. 13. This example shall be that which is given by Mr. Raphson in his *Analysis Æquationum Universalis*, Problem 2d. It is to find the cube-root of the number 37,945.

Now, if we look along the column of cube-numbers in the foregoing table, we shall find that 35,937 is the cube of 33, and that 39,304 is the cube of 34. Therefore, since the proposed number 37,945 is greater than 35,937, but less than 39,304, it follows that the cube-root of 37,945 will be greater than 33, but less than 34, and consequently that the two first figures of it will be 33.

Here then we have $c = 37,945$, and $a = 33$. Therefore a^3 will be $= 35,937$, and $2a^3$ will be $(= 2 \times 35,937) = 71,874$, and $c + 2a^3$ will be $(= 37,945 + 71,874) = 109,819$, and $c - a^3$ will be $(= 37,945 - 35,937) = 2008$, and $\overline{c - a^3} \times a$ will be $(= 2008 \times 33) = 66,264$, and consequently $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $(= \frac{66,264}{109,819}) = 0.6033$.

Therefore

Therefore $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$, or the second near value of the cube-root of the proposed number 37,945, will be ($= 33 + 0.6033$, or) 33.6033; of which number the five first figures 33.603 are exact, the more accurate value of the said cube-root being 33.603,526,179,43, &c.

Now let us suppose a to be $= 33.6033$, or the second near value of the cube-root of 37,945 that has been already found; and let us, in order to obtain a third near value of it, repeat the foregoing process.

Then we shall have $a^3 = 37,944.233,801,747,937$, and $2a^3 (= 2 \times 37,944.233,801,747,937) = 75,888.467,603,495,874$, and $c+2a^3 (= 37,945.000,000,000,000 + 75,888.467,603,495,874) = 113,833.467,603,495,874$, and $c-a^3 (= 37,945.000,000,000,000 - 37,944.233,801,747,937) = 0.766,198,252,063$, and $\overline{c-a^3} \times a (= 0.766,198,252,063 \times 33.6033) = 25.746,789,723,548,607.9$, and consequently $\frac{\overline{c-a^3} \times a}{c+2a^3} (= \frac{25.746,789,723,548,607.9}{113,833.467,603,495,874}) = 0.000,226,$

179,437,95. Therefore $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$ will be ($= 33.6033 + 0.000,226,179,437,95$.) $= 33.603,526,179,437,95$; that is, the third near value of the cube-root of c , or 37,945, that is obtained by means of the rational expression $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$, will be 33.603,526,179,437,95; which I believe to be exact in the first fifteen figures 33.603,526,179,437,9, if not in the last, or sixteenth, figure 5 likewise.

Art. 14. The other expression of the third near value of the cube-root of 37,945, to wit, the irrational expression

$\frac{a}{2} + \sqrt[3]{\frac{4c-a^3}{12a}}$, may be computed as follows.

Since c is $= 37,945$, and a is $= 33.6033$, we shall have $\frac{a}{2} (= \frac{33.6033}{2}) = 16.801,65$, and $12a (= 12 \times 33.6033)$
 $=$

$= 403.2396$, and $4c (= 4 \times 37,945) = 151,780$, and
 $a^3 (= 33.6033^3) = 37,944.233,801,747,937$, and $4c - a^3$
 $(= 151,780.000,000,000,000 - 37,944,233,801,747,937)$
 $= 113,835.766,198,252,063$, and $\frac{4c - a^3}{12a} (=$
 $\frac{113,835.766,198,252,063}{403.2396}) = 282.303,043,149,165,069$, and
 $\sqrt{\frac{4c - a^3}{12a}} (= \sqrt{282.303,043,149,165,069}) = 16.801,$
 $876,179,437,96$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be $(=$
 $16.801,65 + 16.801,876,179,437,96) = 33.603,526,179,$
 $437,96$; that is, the third near value of the cube-root of c , or
 $37,945$, that is obtained by means of the irrational ex-
 pression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be $33.603,526,179,437,96$.

Art. 15. This number $33.603,526,179,437,96$ must (if
 there has been no mistake made in the calculation,) be some-
 what greater than the true value of the cube-root of c , or
 $37,945$; and the former number $33.603,526,179,437,95$,
 obtained by means of the rational expression $a + \frac{c - a^3 \times a}{c + 2a^3}$,
 must be something less than the said true value. These
 two numbers differ only by an unit in the last, or sixteenth,
 figure. And hence it follows that the first sixteen figures
 of the said true value must be the same with those of the
 lesser of those two numbers, or $33.603,526,179,437,95$.
 Mr. Raphson, however, computes it to be $33.603,526,179,$
 $438,08$. But I suspect that his three last figures are not
 exact. But, whether they are exact or not, we may, at
 least, conclude that, so far as these different calculations
 agree with each other, they must be exact, and consequently
 that the first thirteen figures of the cube-root of the number
 $37,945$ are $33.603,526,179,43$.

A fourth

A Fourth Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 16. This example shall be one that is given by Monsieur de Lagny himself, in his book intituled, *Nouveaux Elements d'Arithmétique et d'Algèbre*, which was published at Paris in Duodecimo, in the year 1697. It is to find the cube-root of the number 696,536,483,318,640,035,073,641,037, which consists of twenty seven figures, and may be expressed in the words following, to wit, 696 quadrillions, or fourth powers of a million, 536,483 trillions, or third powers of a million, 318,640 billions, or second powers of a million, 035,073 millions, and 641,037 units.

This number is so great that it will be convenient to divide it into these two parts, 696,536,483,000,000,000,000,000,000 and 318,640,035,073,641,037, and to begin by seeking the cube-root of the first part, 696,536,483,000,000,000,000,000,000.

Now this number 696,536,483,000,000,000,000,000,000 is $= 696,536,483 \times 1,000,000,000,000,000,000$, or $696,536,483 \times$ the cube of 1,000,000. Therefore its cube-root will be equal to 1,000,000 $\times$ the cube root of 696,536,483. Therefore, if we can find the cube-root of 696,536,483, we need only multiply it by 1,000,000, in order to obtain the cube root of 696,536,483,000,000,000,000,000,000. We will therefore endeavour to find the cube-root of 696,536,483.

Now this number 696,536,483 is greater than 696,536,000, or than $696,536 \times 1000$, or than $696,536 \times$ the cube of 10. Therefore the cube-root of 696,536,483 will be greater than the product of the multiplication of the cube root of 696,536 into 10. Therefore, if we can find the cube-root of 696,536, we need only multiply it by 10, in order to obtain the cube-root of 696,536,000, which will be something

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less than the cube-root of 696,536,483, and may serve as a basis from which to begin our approximation to the true value of the said cube-root. We will therefore now endeavour to find, to a small degree of exactness, the cube-root of 696,536.

Art. 17. Now, if we look into the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we shall find that 681,472 is the cube of 88, and that 704,969 is the cube of 89. It follows therefore that the cube-root of 696,536, (which is greater than 681,472, but less than 704,969,) must be greater than 88, but less than 89, and consequently that the two first figures of it must be 88. Therefore the cube-root of 696,536,000 must be greater than 88×10 , or 880, but less than 89×10 , or 890, and consequently the two first figures of it will be 88. Therefore 880, being less than the cube-root of the number 696,536,000, will be less also than the cube-root of the number 696,536,483, which is greater than 696,536,000; but it will approach sufficiently near to it to enable us to begin a further approximation to it by means of the foregoing expressions of Monsieur de Lagney.

Art. 18. Let us therefore suppose a to be $= 880$; and, in order to find a second near value of the cube-root of c , or 696,536,483, let us compute the expression $a + \frac{c-a^3}{c+2a^3} \times a$.

Here then we shall have $a^3 (= 880^3) = 681,472,000$, and $2a^3 = 1,362,944,000$, and $c + 2a^3 (= 696,536,483 + 1,362,944,000) = 2,059,480,483$, and $c - a^3 (= 696,536,483 - 681,472,000) = 15,064,483$, and $c - a^3 \times a (= 15,064,483 \times 880) = 13,256,745,040$, and consequently $\frac{c-a^3}{c+2a^3} \times a (= \frac{13,256,745,040}{2,059,480,483}) = 6.436,9$. Therefore $a + \frac{c-a^3}{c+2a^3} \times a$ will be $(= 880 + 6.436,9) = 886.4369$; and consequently 886.4369 will be a second near value of the cube-root of the number c , or 696,536,483.

Art. 19.

Art. 19. Therefore (by what is shewn in art. 16,) $886,4369 \times 1000,000$, or $886,436,900$ will be nearly equal to, but somewhat less than, the cube-root of the number $696,536,483,000,000,000,000,000,000$, and, *a fortiori*, will be less than the cube-root of the proposed number $696,536,483,318,640,035,073,641,037$. And, as $886,436,900$ is not much less than $886,437,000$, it seems probable that $886,437,000$ will likewise be somewhat less than the cube-root of the said number $696,536,483,318,640,035,073,641,037$. And so upon trial we shall find it to be. For the cube of $886,437,000$ is $696,535,206,998,055,453,000,000,000$, which is less than the said proposed number.

Art. 20. Now let a be supposed to be $= 886,437,000$, and let us endeavour to find a nearer value of the cube-root of the proposed number $696,536,483,318,640,035,073,641,037$, by computing the expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$.

Then, since c is $= 696,536,483,318,640,035,073,641,037$, and a is $= 886,437,000$, and consequently a^3 is $= 696,535,206,998,055,453,000,000,000$, we shall have $\frac{a}{2}$ ($= \frac{886,437,000}{2}$) $= 443,218,500$, and

$4c$ ($= 4 \times 696,536,483,318,640,035,073,641,037$)
 $= 2,786,145,933,274,560,140,294,564,148$
 and $4c - a^3$ ($= 2,786,145,933,274,560,140,294,564,148$
 $- 696,535,206,998,055,453,000,000,000$)
 $= 2,089,610,726,276,504,687,294,564,148$, and $12a$ ($=$
 $12 \times 886,437,000$) $= 10,637,244,000$, and $\frac{4c-a^3}{12a}$ ($=$
 $\frac{2,089,610,726,276,504,687,294,564,148}{10,637,244,000}$) $= 196,442,868,686,332,$
 983 , and $\sqrt{\frac{4c-a^3}{12a}}$ ($= \sqrt{196,442,868,686,332,983}$) $=$
 $= 443,218,759$. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be ($=$

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$443,218,500 + 443,218,759 = 886,437,259$; and consequently the cube-root of the proposed long number $696,536,483,318,640,035,073,641,037$, will be very nearly equal to $886,437,259$. Q. E. I.

Monsieur de Lagney determines this cube-root to be only $886,437,166$.[†] But this is owing, as I apprehend, to his intirely neglecting the last eighteen figures, $318,640,035,073,641,037$, of the proposed number, and consequently giving us only the cube-root of the number $696,536,483,000,000,000,000,000,000$, which is less than the proposed number: whereas in the last operations of the foregoing process we took notice of all the figures of the proposed number, when we found the value of $4c$, and extracted the square-root of the fraction $\frac{4c-a^2}{12a}$.

Mr. de Lagney adds, as a proof of the great usefulness of this method of extracting cube-roots, that the most skilful Arithmetician would not be able to find the cube root of this long number, $696,536483,318640,035073,641037$, to the same degree of exactness, or to nine places of figures, by the common method of extracting the cube-root, in the space of a whole month. See Monsieur de Lagney's *Nouveaux Eléments d'Arithmétique et d'Algèbre*, page 307.

A S C H O L I U M.

Art. 21. This very useful method of approximating to the cube-roots, and other roots, of numbers was first published by Mr. de Lagney, at Paris, in the *Journal des Sçavants* for the 14th of May 1691, and afterwards was published again at greater length, and with a demonstration, in a separate tract in quarto, in the month of May of the following year 1692. But Mr. Joseph Raphson had published his *Analysis Aequationum Universalis*, (which contains

[†]The number is $886,437,165.343,281,625,318,310,870,474$.
 Extracted after Young's method. See his *Arithmetica* p. 151. See this Example worked at length in my present Vol. I.st

tains a general method of finding the roots of all sorts of equations by approximation,) in the year 1690: and his method of approximation is not very different from this of Mr. de Lagney; and the ground, or principle, of it is exactly the same. So that, if Mr. de Lagney had seen Mr. Raphson's *Analysis Aëquationum* before he had discovered his own method of approximation, it would have been easy for him to have deduced his own method from Mr. Raphson's; and in that case it would have been candid in him to acknowledge that he had seen Mr. Raphson's book, and had been led by it to the discovery of his own method. This, however, he has not done; at least, not in his *Nouveaux Eléments d'Arithmétique et d'Algèbre*, which is the only book of his that I have seen. And therefore I suppose he invented his method of approximation by his own efforts, and without having met with Mr. Raphson's book.

Art. 22. It may further be observed, that Mr. Raphson himself was not the first inventor of the method of resolving equations by approximations of the kind he has made use of, that is, by approximations performed by transforming the original equation into another equation that involves in it the powers of the unknown difference between the first near value (already obtained, by conjecture or otherwise,) of the root of the first equation and its true value, and by resolving the said transformed equation in the manner of a simple equation, or by dropping all the terms that involve in them any higher powers of the said unknown difference, or root of the second equation, than its simple power. But this excellent method of discovering the roots of high equations had been found out by the great Sir Isaac Newton more than twenty-four years before the publication of Mr. Raphson's *Analysis Aëquationum Universalis*, to wit, in the year 1666, when he wrote his learned little tract, intitled *De Analysis per æquationes numero terminorum infinitas*, which is printed in the *Commercium Epistolicum* of Mr. John Collins and other Mathematicians, pages 67, 68, 69, &c. - - 93, of the 2d edition. This tract was first printed in the year 1712, in the first edition of the said *Commercium Epistolicum*.

licum. But it had been shewn in manuscript to Dr. Isaac Barrow, and by him sent to the said Mr. John Collins, with Mr. Newton's leave, in the month of July 1669, and shewn to the Lord Viscount Brouncker, (a great Mathematician of that time,) and, probably, to many other learned Mathematicians that were Fellows of the Royal Society, to which Mr. Collins was then a Secretary. And afterwards a part of it, containing a short specimen of Mr. Newton's method of resolving equations by approximation, was published by Dr. Wallis in the 94th chapter of his Algebra in the year 1685, which was five years before the publication of Mr. Raphson's *Analysis Æquationum Universalis*. Whether this specimen suggested to Mr. Raphson the discovery of his method of approximating to the roots of equations, (which differs but little from that of Sir Isaac Newton,) it is difficult to determine. He has not mentioned Newton's method in his treatise on this subject, though he was a great admirer of his genius, and ever ready to commend him : and therefore I am inclined to think that the above-mentioned specimen of Newton's method of approximation was not the circumstance that led him to the discovery of his own. But, whether it was or not, it is certain that the honour of priority with respect to this very useful invention is due to Sir Isaac Newton.

Of the Ground, or Principle, of the investigation of the foregoing Expressions, invented by Monsieur de Lagney, for Approximating to the Value of the Cube-root of a given Number.

Art. 23. The investigation of all the foregoing expressions invented by Monsieur de Lagney for approximating to the cube-root of a given number, when a first near value of the said cube-root that is exact to one, or two, or more, places

places of decimal figures, is already known, is not difficult. It results from the contemplation of the compound quantities that are equal to the cubes of a binomial quantity, (such as $a + b$,) and a residual quantity, (such as $a - b$,) according as a , or the first near value of $\sqrt[3]{c}$ which is already known, is less, or greater, than $\sqrt[3]{c}$; and therefore it ought properly to be divided into two parts, the one relating to the case in which a , or the first near value of the cube-root of the given number c that is already known, is less than the cube-root of c , and the other relating to the case in which a , or the said first near value of $\sqrt[3]{c}$, is greater than the said cube-root. The first of these investigations, (by which we shall also obtain Mr. Raphson's approximation to the value of the said cube-root, in the same case, or when a is less than $\sqrt[3]{c}$,) is as follows.

An Investigation of the two Expressions, $a + \frac{c - a^3}{c + 2a^3} \times a$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, given by Monsieur de Lagney, for a Second near Value of the Cube-root of a given Number c , when a , or a First near Value of it that is already known, is less than its true Value.

Art. 24. Let z be put for the unknown difference by which a , or the first near value of the cube-root of the given number c , falls short of its true value; so that $a + z$ shall be $= \sqrt[3]{c}$.

Then will $(a + z)^3$ be $= c$. But $(a + z)^3$ is $= a^3 + 3a^2z + 3aaz + z^3$. Therefore $a^3 + 3a^2z + 3aaz + z^3$ will also be $= c$. This cubick equation is the foundation both
of

of Mr. Raphson's and of Monsieur de Lagney's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 25. Mr. Raphson's approximation is obtained as follows. Since z is less, and usually much less, than a , to wit, about a 10th or a 100th part of it, or, perhaps, still less,) it follows that both $3azz$ and z^3 will be less, and usually much less, than $3aaz$, and consequently that $a^3 + 3aaz$ will be nearly equal to $a^3 + 3aaz + 3azz + z^3$, and therefore to c . Let them therefore be supposed to be equal to c . Then, since $a^3 + 3aaz$ are $= c$, we shall have $3aaz = c - a^3$, and consequently $z = \frac{c - a^3}{3aa}$, which fraction consists intirely of known quantities. Therefore $a + z$ will be $= a + \frac{c - a^3}{3aa}$, and $a + \frac{c - a^3}{3aa}$ will be a second near value of $\sqrt[3]{c}$, or the cube-root of the given number c . Q. E. I.

This is Mr. Raphson's approximation to the cube-root of c , when a is less than the said cube-root; and it is the simplest and easiest approximation that can well be imagined, and approaches very considerably beyond a to the true value of $\sqrt[3]{c}$. For it usually gives us twice as many figures exact as we had before in a , or the first near value of $\sqrt[3]{c}$. And it is less operose, or difficult to compute, than Mr. de

Lagney's first, or rational, expression $a + \frac{c - a^3}{c + 2a^3} \times a$; because it is easier to divide $c - a^3$ by $3aa$, or three times the square of a , than, first, to multiply $c - a^3$ by a , and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for these reasons Mr. Raphson, in the Appendix to the second edition of his *Analysis Aequationum Universalis*, (which was published in the year 1697, several years after the publication of Mr. de Lagney's method of approximation,) declares that he continued to prefer his own approximation to those of Mr. de Lagney, notwithstanding their greater exactness.

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We will now proceed to investigate Mr. de Lagney's first, or rational, expression above-mentioned, in obtaining which Mr. Raphson's approximation is made use of as a necessary step.

Art. 26. Mr. de Lagney's first, or rational, expression, $a + \frac{\sqrt{c-a^3} \times a}{c+2a^3}$, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, or by supposing $a^3 + 3aaz + 3azz$ to be equal to c , and resolving the quadratic equation $a^3 + 3aaz + 3azz = c$, resulting from that supposition, in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 + 3aaz + 3azz$ is $= c$, we shall have $3aaz + 3azz = c - a^3$, and (dividing both sides by $3aa + 3az$) $z = \frac{c-a^3}{3aa+3az}$. Let us now substitute, instead of z , in the

denominator of this fraction $\frac{c-a^3}{3aa+3az}$, the near value of z already obtained by the resolution of the simple equation $a^3 + 3aaz = c$, to wit, the fraction $\frac{c-a^3}{3aa}$; and we shall thereby obtain the equation $z = \frac{c-a^3}{3aa+3a \times \frac{c-a^3}{3aa}}$; which

last quantity is equal to $\frac{c-a^3}{3aa+\frac{c-a^3}{a}}$ ($= \frac{c-a^3}{\frac{3a^2+c-a^3}{a}} = \frac{c-a^3}{c+2a^3} = \frac{\sqrt{c-a^3} \times a}{c+2a^3}$). Therefore z

will be $= \frac{\sqrt{c-a^3} \times a}{c+2a^3}$, and consequently $a + z$ will be $= a + \frac{\sqrt{c-a^3} \times a}{c+2a^3}$. Therefore the true value of $a + z$, or the cube-root of the given number c , will be nearly $= a + \frac{\sqrt{c-a^3} \times a}{c+2a^3}$. Q. E. D.

Art. 27. This expression $a + \frac{\sqrt[3]{c-a^3} \times a}{c+2a^3}$ of the second near value of $\sqrt[3]{c}$, will always be less than its true value; as may be demonstrated in the manner following.

Since $c - a^3$ is $= 3aaz + 3azz + z^3$, and $3aaz + 3azz + z^3$ is $= 3aaz + azz + 2azz + z^3$, it follows that $c - a^3$ will be $= 3aaz + azz + 2azz + z^3$, and consequently will be greater than $3aaz + azz$. Therefore $\frac{c-a^3}{a}$ will be

greater than $\frac{3aaz+azz}{a}$, or than $3az + zz$. But $\frac{c-a^3}{a}$ is $= 3a \times \frac{\sqrt[3]{c-a^3}}{3aa}$. Therefore $3a \times \frac{\sqrt[3]{c-a^3}}{3aa}$ will be greater than $3az + zz$; and consequently (adding $3aa$ to both sides,) $3aa + 3a \times \frac{\sqrt[3]{c-a^3}}{3aa}$ will be greater than $3aa + 3az + zz$.

Therefore $\frac{c-a^3}{3aa + 3a \times \frac{\sqrt[3]{c-a^3}}{3aa}}$ will be less than $\frac{c-a^3}{3aa + 3az + zz}$.

But $\frac{c-a^3}{3aa + 3a \times \frac{\sqrt[3]{c-a^3}}{3aa}}$ is $= \frac{\sqrt[3]{c-a^3} \times a}{c+2a^3}$, and $\frac{c-a^3}{3aa + 3az + zz}$

is $=$ the true value of z . Therefore $\frac{\sqrt[3]{c-a^3} \times a}{c+2a^3}$ will be less than the true value of z . Therefore $a + \frac{\sqrt[3]{c-a^3} \times a}{c+2a^3}$ will be less than the true value of $a + z$, or than $\sqrt[3]{c}$.

Q. E. D.

This expression, $a + \frac{\sqrt[3]{c-a^3} \times a}{c+2a^3}$, gives usually three times as many figures of the value of $\sqrt[3]{c}$ exact as were given exactly by a , or the first near value of the said cube-root. But in some cases the figures which it gives exactly are only three times as many wanting one, and in some unfavourable cases only three times as many wanting two, as were exact

exact in a ; as we have seen in some of the foregoing examples.

Art. 28. Mr. de Lagney's second, or irrational, expression, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, for the second near value of the cube-root of c , when a is less than the said cube-root, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, and resolving the equation thence resulting, to wit, the quadratick equation $a^3 + 3aaz + 3azz = c$, in an accurate manner. This may be done as follows.

If we suppose $a^3 + 3aaz + 3azz$ to be equal to c , we shall have $3aaz + 3azz = c - a^3$, and (dividing all the terms by $3a$) $az + zz = \frac{c - a^3}{3a}$. Therefore (adding $\frac{aa}{4}$ to both sides,) we shall have $\frac{aa}{4} + az + zz = \frac{c - a^3}{3a} + \frac{aa}{4}$ ($= \frac{4c - 4a^3}{12a} + \frac{3a^2}{12a} = \frac{4c - 4a^3 + 3a^3}{12a}$) $= \frac{4c - a^3}{12a}$, and (extracting the square-roots of both sides,) $\frac{a}{2} + z = \sqrt{\frac{4c - a^3}{12a}}$. Therefore z will be $= \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}$, and $a + z$ will be $(= a + \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}) = \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore $\sqrt[3]{c}$ will be nearly equal to the same quantity $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Q. E. I.

Art. 29. This expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ must always be greater than the true value of $\sqrt[3]{c}$, or than the true value of $a + z$ in the original cubick equation $a^3 + 3aaz + 3azz + z^3 = c$.

For it is derived from a supposition that $a^3 + 3aaz + 3azz$

$3axz$ is equal to c , or to $a^3 + 3aaz + 3axz + z^3$, or is greater than it really is: from which it will necessarily follow that the value of z deduced from that supposition must be greater than its true value, and consequently that the value of $a + z$ deduced from that supposition, that is, the expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be greater than the true value of $a + z$, or than $\sqrt[3]{c}$. Q. E. D.

Art. 30. The irrational expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will approach a little nearer than the rational expression $a + \frac{c - a^3}{c + 2a^3} \times a$ to the true value of $\sqrt[3]{c}$, because it is obtained by resolving the quadratic equation $a^3 + 3aaz + 3axz = c$ accurately, whereas the rational expression $a + \frac{c - a^3}{c + 2a^3} \times a$ is obtained by resolving the same quadratic equation inaccurately, by substituting $\frac{c - a^3}{3a^2}$ instead of z in the quantity $3axz$ in the denominator of the fraction $\frac{c - a^3}{3a^2 + 3ax}$ in art. 26. But the difference of the two expressions in point of exactness is not considerable; and the principal reason for preferring the irrational expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ to the rational expression $a + \frac{c - a^3}{c + 2a^3} \times a$, is, that there is much less labour in extracting the square-root of the fraction $\frac{4c - a^3}{12a}$, than in dividing the numerator $(c - a^3) \times a$ by the denominator $c + 2a^3$, when that denominator is a very long number.

An Investigation of the Two Expressions, $a - \frac{\sqrt{a^3 - c} \times a}{c + 2a^2}$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, given by Monsieur de Lagny for a Second near Value of the Cube-root of a given Number c , when a , or a First near Value of it that is already known, is greater than its true Value.

Art. 31. Let z be put for the unknown difference by which a , or the first near value of the cube-root of the given number c , exceeds its true value; so that $a - z$ shall be $= \sqrt[3]{c}$.

Then will $a - z$ be $= \sqrt[3]{c}$. But $a - z$ is $= a^3 - 3aaz + 3azz - z^3$. Therefore $a^3 - 3aaz + 3azz - z^3$ will also be $= c$. This cubick equation is the foundation both of Mr. Raphson's and of Monsieur de Lagny's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 32. Mr. Raphson's approximation is obtained as follows.

Since z is less, and usually much less, than a , (to wit, about a 10th, or a 100th, part of it, or, perhaps, still less,) it follows that both $3azz$ and z^3 will be less, and usually much less, than $3aaz$, and consequently that $a^3 - 3aaz$ will be nearly equal to $a^3 - 3aaz + 3azz - z^3$, and therefore to c . Let them therefore be supposed to be equal to c . Then, since $a^3 - 3aaz$ are $= c$, we shall have $a^3 = c + 3aaz$, and $3aaz = a^3 - c$, and consequently $z = \frac{a^3 - c}{3aa}$.

Therefore $a - z$ will be $= a - \frac{a^3 - c}{3aa}$, and consequently

the expression $a - \frac{a^3 - c}{3aa}$ will be a second near value of $\sqrt[3]{c}$, or the cube-root of the given number c . Q. E. I.
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This is Mr. Raphson's approximation to the cube-root of c , when a is greater than the said cube-root; and it is the simplest and easiest approximation that can well be imagined, and approaches much nearer than a to the true value of the said cube-root. For it usually gives us the value of the said cube-root exact to twice as many decimal places of figures as were exact in a , or the first near value of the said cube-root. And it is less operose, or difficult to compute, than Mr. de Lagney's first, or rational, expression, $a - \frac{(a^3 - c) \times a}{c + 2a^3}$; because it is easier to divide $c - a^3$ by $3aa$, or three times the square of a , than, first, to multiply $a^3 - c$ by a , and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for these reasons Mr. Raphson always preferred it to Mr. de Lagney's approximations, notwithstanding their greater exactness.

We will now proceed to investigate Mr. de Lagney's first expression above-mentioned, to wit, the rational expression $a - \frac{(a^3 - c) \times a}{c + 2a^3}$, in obtaining which Mr. Raphson's approximation is made use of as a necessary step.

Art. 33. Mr. de Lagney's first, or rational, expression, $a - \frac{(a^3 - c) \times a}{c + 2a^3}$, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or by supposing $a^3 - 3aaz + 3azz$ to be equal to c , and resolving the quadratic equation $a^3 - 3aaz + 3azz = c$, resulting from that supposition, in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 - 3aaz + 3azz$ is supposed to be equal to c , we shall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$, and (dividing both sides of the equation by $3aa - 3az$), $z = \frac{a^3 - c}{3aa - 3az}$. Mr. de Lagney then substitutes,

tutes, instead of z , in the denominator of the fraction $\frac{a^3 - c}{3aa - 3az}$, the near value of z already obtained by the resolution of the simple equation $a^3 - 3aaz = c$, to wit, the fraction $\frac{a^3 - c}{3aa}$, and thereby obtains the equation $z =$

$$\frac{a^3 - c}{3aa - 3a \times \frac{a^3 - c}{3aa}}; \text{ which last quantity is equal to } \frac{a^3 - c}{3aa - \sqrt{\frac{a^3 - c}{a}}}$$

$$(\frac{a^3 - c}{3a^3 - \sqrt{a^3 - c}} = \frac{a^3 - c}{3a^3 - a^3 + c} = \frac{a^3 - c}{2a^3 + c} = \sqrt{a^3 - c} \times$$

$$\frac{a}{c + 2a^3}) = \frac{a^3 - c \times a}{c + 2a^3}. \text{ Therefore } z \text{ will be } = \frac{a^3 - c \times a}{c + 2a^3},$$

and consequently $a - z$ will be $= a - \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$. There-

fore the true value of $a - z$, or of the cube-root of the given number c , will be nearly $= a - \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$.

Q. E. I.

Art. 34. This expression, $a - \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$, of the second near value of $\sqrt[3]{c}$, will always be greater than its true value; as may be demonstrated in the manner following.

It has been shewn above in art. 31, that $a^3 - 3aaz + 3azz - z^3$ is $= c$. Therefore $a^3 + 3azz$ will be $= c + 3aaz + z^3$, and a^3 will be $= c + 3aaz - 3azz + z^3$, and $a^3 - c$ will be $= 3aaz - 3azz + z^3$. But $3aaz - 3azz + z^3$ is $= 3aaz - azz - 2azz + z^3$. Therefore $a^3 - c$ will be $= 3aaz - azz - 2azz + z^3$. But, because z is less than a , z^3 will be less than azz , and, *a fortiori*, less than $2azz$. Therefore $3aaz - azz - 2azz + z^3$ will be less than $3aaz - azz$; and consequently $a^3 - c$ (which is $= 3aaz - azz - 2azz + z^3$), will be less than $3aaz - azz$.

Therefore $\frac{a^3 - c}{a}$ will be less than $\frac{3aaz - azz}{a}$, or than $3az -$
 $zz.$

zx . But $\frac{a^3 - c}{a}$ is $= 3a \times \sqrt[3]{\frac{a^3 - c}{3aa}}$. Therefore $3a \times \sqrt[3]{\frac{a^3 - c}{3aa}}$ will be less than $3az - zx$. Therefore, if both these quantities be subtracted from $3aa$, the remainder $3aa - 3a \times \sqrt[3]{\frac{a^3 - c}{3aa}}$ will be greater than the remainder $3aa - (3az - zx)$, or than $3aa - 3az + zx$. Therefore $\frac{a^3 - c}{3aa - 3a \times \sqrt[3]{\frac{a^3 - c}{3aa}}}$ will be less than $\frac{a^3 - c}{3aa - 3az + zx}$. But $\frac{a^3 - c}{3aa - 3a \times \sqrt[3]{\frac{a^3 - c}{3aa}}}$ is $= \frac{\sqrt[3]{a^3 - c} \times a}{c + 2a^3}$. Therefore $\frac{\sqrt[3]{a^3 - c} \times a}{c + 2a^3}$ will be less than $\frac{a^3 - c}{3aa - 3az + zx}$. But $\frac{a^3 - c}{3aa - 3az + zx}$ is equal to the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or to its true value. Therefore $\frac{\sqrt[3]{a^3 - c} \times a}{c + 2a^3}$ will be less than the true value of z . And consequently $a - \frac{\sqrt[3]{a^3 - c} \times a}{c + 2a^3}$ will be greater than the true value of $a - z$, or than the cube-root of c . Q. E. D.

Art. 35. The other, or irrational, expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, given by Mr. de Lagney for the second near value of the cube-root of c , when a is greater than the said cube-root, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, obtained in art. 31, or by supposing the trinomial quantity $a^3 - 3aaz + 3azz$ to be equal to c , and resolving the quadratrick equation $a^3 - 3aaz + 3azz = c$ in an accurate manner. This may be done as follows.

If we suppose $a^3 - 3aaz + 3azz$ to be $= c$, we shall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$. Therefore

Therefore (dividing both sides of the equation by $3a$,) we shall have $az - zz = \frac{a^3 - c}{3a}$.

The compound quantity $az - zz$, which forms the left-hand side of this equation, is $= \sqrt{a - z} \times z$, and consequently (by Euclid's Elements, Book 2d, Prop. 5,) must be less than the square of $\frac{a}{2}$, or than $\frac{aa}{4}$. And consequently the other side of the equation, or the quantity $\frac{a^3 - c}{3a}$, will also be less than $\frac{aa}{4}$. They may therefore both be subtracted from $\frac{aa}{4}$. Let them be so subtracted. And we shall then have

$$\frac{aa}{4} - \sqrt{az - zz} = \frac{aa}{4} - \sqrt{\frac{a^3 - c}{3a}}, \text{ or } \frac{aa}{4} - az + zz = \frac{aa}{4} - \sqrt{\frac{a^3 - c}{3a}}$$

$$\left(= \frac{3a^3}{12a} - \sqrt{\frac{4a^3 - 4c}{12a}} = \frac{3a^3 - (4a^3 - 4c)}{12a} = \frac{3a^3 - 4a^3 + 4c}{12a} \right) = \frac{4c - a^3}{12a}.$$

Therefore the square-root of the trinomial quantity $\frac{aa}{4} - az + zz$, will be equal to the square-root of the fraction $\frac{4c - a^3}{12a}$. Now, if z could be of two different values, the one greater than $\frac{a}{2}$, and the other less than $\frac{a}{2}$, the trinomial quantity $\frac{aa}{4} - az + zz$ might have two square-roots, to wit, $\frac{a}{2} - z$, and $z - \frac{a}{2}$. But, as z in the present problem is supposed to be much less than $\frac{a}{2}$, the latter of these square-roots, to wit, $z - \frac{a}{2}$, cannot exist, and the other square-root, $\frac{a}{2} - z$, will be the only one consistent with the conditions of the Problem. Therefore we shall

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have

have $\frac{a}{2} - z = \sqrt{\frac{4c - a^3}{12a}}$, and consequently (adding z to both sides,) $\frac{a}{2} = \sqrt{\frac{4c - a^3}{12a}} + z$, and (subtracting $\sqrt{\frac{4c - a^3}{12a}}$ from both sides,) $z = \frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$. Therefore $a - z$ will be $= a - \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ ($= a - \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$) $= \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore the second near value of $\sqrt[3]{c}$ will be $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Q. E. I.

Art. 36. This expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will always be less than the true value of the cube-root of the given number c .

For, if we suppose z to increase continually from 0 till it becomes equal to $\frac{a}{2}$, the compound quantity $az - zz$, or $a - z \times z$, will increase continually at the same time till it becomes equal to $\left(\frac{a}{2}\right)^2$, or $\frac{aa}{4}$; and consequently the compound quantity $3a \times \sqrt{az - zz}$, or the compound quantity $3aaz - 3azz$, will increase continually till it becomes equal to $3a \times \frac{aa}{4}$, or $\frac{3a^3}{4}$. Therefore, when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c$, the quantity z will be greater than when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c - z^3$, which is less than $a^3 - c$; that is, the value of z in the quadratic equation $3aaz - 3azz = a^3 - c$ will be greater than the value of z in the cubick equation $3aaz - 3azz = a^3 - c - z^3$, or in the cubick equation $3aaz - 3azz + z^3 = a^3 - c$, or in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$. But the value of z in the quadratic equation $3aaz - 3azz =$
 $a^3 - c$

$a^3 - c$ is $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$; and the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$ is the true value of z , or of the excess of a above $\sqrt[3]{c}$. Therefore the expression $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$ will be greater than the true value of z . Therefore $a - \frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$, or $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be less than the true value of $a - z$, or than the cube-root of the given number c . Q. E. D.

Art. 37. The irrational Formula $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will approach a little nearer than the rational Formula $a - \frac{a^3 - c}{c + 2a^3} \times a$ to the true value of the cube-root of c ; because the irrational Formula is derived from the quadratick equation $a^3 - 3aaz + 3azz = c$ by an accurate resolution of it, and the rational Formula is derived from the same equation by an inaccurate resolution of it. But the difference of exactness between these two expressions is not great, and either of them will usually give us three times as many decimal figures of $\sqrt[3]{c}$ exact as were exact in a , or the first near value of it. But, when the given number c itself consists of nine, or ten, or more figures, and also when a consists of three, or four, figures, and consequently a^3 consists of nine, or ten, or more, figures, the irrational expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be found to be much easier to compute than the rational expression $a - \frac{a^3 - c}{c + 2a^3} \times a$, on account of the labour of dividing $a^3 - c \times a$ by the long number $c + 2a^3$; and therefore Dr. Halley thinks it ought to be preferred to the other.

Art. 38. I have now given very full investigations of the four expressions invented by Mr. de Lagney for the second near value of the cube-root of a given number c , of which a first near value, denoted by the letter a , is already known, to wit, the four expressions $a + \frac{c-a^3}{c+2a^3} \times a$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, and $a - \frac{a^3-c}{c+2a^3} \times a$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$; of which the two first relate to the case in which a , or the first near value of the said cube-root, is less than its true value; and the two last relate to the case in which a , or the first near value of the said cube-root, is greater than its true value. And I have given demonstrations of what is asserted concerning these four expressions in art. 5 and 6, to wit, that the first of them, or $a + \frac{c-a^3}{c+2a^3} \times a$, is always less than the true value of $\sqrt[3]{c}$, and that the second of them, or $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, is always greater than the said true value, and that the third expression $a - \frac{a^3-c}{c+2a^3} \times a$ is always greater than the said true value, and that the fourth expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is always less than the said true value. And the two first of these assertions have been confirmed upon trial in the examples given in art. 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, of the extraction of the cube-roots of the numbers 2, 231, 37,945, and the long number 696,536,483,318,640,035,073,641,037; the said cube-roots having been extracted by means of the two first expressions $a + \frac{c-a^3}{c+2a^3} \times a$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, which relate to the case in which a , or the first near value of $\sqrt[3]{c}$, is less than its true value. I will now therefore give an example, or two, of the extraction of the cube-roots of numbers by means of the two latter expressions

$a =$

$a = \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, which relate to the case in which a , or the first near value of $\sqrt[3]{c}$, is greater than its true value.

An Example of the Extraction of the Cube-root of a given Number, by means of Mr. de Lagney's Third and Fourth Expressions $a = \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, in which a , or the First near Value of $\sqrt[3]{c}$, is supposed to be greater than its true Value.

Art. 39. Let it be required to find the cube-root of 2, which was extracted above in art. 8 and 9; and let a , or the first near value of the said cube-root, be 1.26, which is somewhat greater than its true value, which is 1.259,921, 049, &c.

Here then we have $c = 2$, and $a = 1.26$. Therefore a^3 will be $= 2.000,376$, and $2a^3$ will be $= 4.000,752$, and $c + 2a^3$ will be $(= 2 + 4.000,752) = 6.000,752$, and $a^3 - c$ will be $(= 2.000,376 - 2) = 0.000,376$, and $\sqrt{a^3 - c} \times a$ will be $(= 0.000,376 \times 1.26) = 0.000,473,76$, and consequently $\frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$ will be $(= \frac{0.000,473,76}{6.000,752}) = 0.000,078,950$. Therefore $a = \frac{\sqrt{a^3 - c} \times a}{c + 2a^3}$, or the second near value of the cube-root of 2, will be $(= 1.260,000,000 - 0.000,078,950) = 1.259,921,050$; which is a little greater than its true value 1.259,921,049, &c, agreeably to what is asserted in art. 6, and demonstrated in art. 34.

And we shall have $4c (= 4 \times 2) = 8.000,000$, and $\frac{a}{2} (= \frac{1.26}{2}) = 0.63$, and $12a (= 12 \times 1.26) = 15.12$,
and

and $4c - a^3 (= 8.000,000 - 2.000,376) = 5.999,624$, and $\frac{4c - a^3}{12a} (= \frac{5.999,624}{15.12}) = 0.396,800,529,100,529,100,529, \&c$, and $\sqrt{\frac{4c - a^3}{12a}} (= \sqrt{0.396,800,529,100,529,100,529, \&c}) = 0.629,921,049,894,76$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}} (= 0.639,000,000,000,00 + 0.629,921,049,894,76) = 1.259,921,049,894,76$. Therefore the second near value of the cube-root of 2, obtained by the irrational expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be 1.259,921,049,894,76; which is a little less than the true value of the said cube-root, agreeably to what is asserted in art. 6, and demonstrated in art. 36, the more accurate value of the said cube-root being 1.259,921,049,894,873,164,76, &c.

Art. 40. The foregoing more accurate value of the cube-root of 2, to wit, 1.259,921,049,894,873,164,76, was obtained by taking 1.259,921,0 for a , or the first near value of the said cube-root, and computing the expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$.

For, if a is = 1.259,921,0, or 1.259,921, we shall have $\frac{a}{2} = 0.629,960,5$, and $a^3 (= 1.259,921^3) = 1.999,999,762,390,486,961$, and $4c (= 4 \times 2) = 8.000,000,000,000,000,000,000$, and $4c - a^3 = 8.000,000,000,000,000,000,000 - 1.999,999,762,390,486,961 = 6.000,000,237,609,513,039$, and $12a (= 12 \times 1.259,921) = 15.119,052$, and consequently $\frac{4c - a^3}{12a} (= \frac{6.000,000,237,609,513,039}{15.119,052}) = 0.396,850,294,423,850,982,125,069,746,436$, and $\sqrt{\frac{4c - a^3}{12a}} (= \sqrt{0.396850,294423,850982,125069,746436}) = 0.629,960,549,894,873,164,76, \&c$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$

$\sqrt[3]{\frac{4c - a^3}{12a}}$ will be $(= 0.629,960,5 + 0.629,960,549,894,873,164,76, \&c.) = 1.259,921,049,894,873,164,76, \&c.$, which therefore will be a very near value of the cube-root of 2.

All the twenty-one figures of this number 1.259,921,049,894,873,164,76, may be depended upon as exact, if no mistake has been made in computing the value of the expression $\sqrt[3]{\frac{4c - a^3}{12a}}$; because a , or the first value of the cube-root of 2, to wit, 1.259,921,0, consists of eight figures which are all exact, and the number of figures that are exact in $\frac{a}{2} + \sqrt[3]{\frac{4c - a^3}{12a}}$ is always triple, or triple wanting one or two figures, of the number of figures that are exact in a ; as was observed in art. 7.

Art. 41. And in like manner in the second example, relating to the extraction of the cube-root of 231, if we take a equal to 6.14 (which is somewhat greater than the true value of the said cube-root,) instead of taking it equal to 6.13, (which is somewhat less than the said true value,) and compute the two expressions $a - \frac{a^3 - c \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt[3]{\frac{4c - a^3}{12a}}$, we shall find the former of these expressions to be somewhat greater, and the latter of them to be somewhat less, than the true value of the said cube-root, agreeably to what is asserted in art. 6. These computations may be performed as follows.

If a is supposed to be $= 6.14$, we shall have $\frac{a}{2} (= \frac{6.14}{2}) = 3.07$, and $12a (= 12 \times 6.14) = 73.68$, and $a^3 (= 6.14^3) = 231.475,544$, and $a^3 - c (= 231.475,544 - 231) = 0.475,544$, and $a^3 - c \times a (= 0.475,544 \times 6.14) = 2.919,840,16$, and $2a^3 (= 2 \times 231.475,544) = 462.951,088$.

462.951,088, and $c + 2a^2 (= 231 + 462.951,088) = 693.951,088$, and consequently $\frac{a^3 - c}{c + 2a^2} (= \frac{2.919,840,16}{693.951,088}) = 0.004,207,559$. Therefore $a - \frac{(a^3 - c) \times a}{c + 2a^2}$ will be $(= 6.140,000,000 - 0.004,207,559) = 6.135,792,441$; which is greater than the true value of the cube-root of 231, agreeably to what is asserted in art. 6, and demonstrated in art. 34, the said true value being only 6.135,792,439,661,958, &c. See above, art. 10, page 476.

And we shall have $4c (= 4 \times 231) = 924$, and $4c - a^2 = 924.000,000 - 231.475,544 = 692.524,456$, and $\frac{4c - a^2}{12a} (= \frac{692.524,456}{73.68}) = 9.399,083,279,044,516,829$, and $\sqrt{\frac{4c - a^2}{12a}} (= \sqrt{9.399083,279044,516829}) = 3.065,792,439,004$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^2}{12a}}$ will be $(= 3.07 + 3.065,792,439,004) = 6.135,792,439,004$; which is less than the true value of the cube-root of 231, to wit, 6.135,792,439,661,958, &c, agreeably to what is asserted in art. 6, and demonstrated in art. 36.

Art. 42. These two examples are sufficient to illustrate and confirm what is asserted in art. 6, and demonstrated in art. 34 and 36, concerning the two expressions $a - \frac{(a^3 - c) \times a}{c + 2a^2}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^2}{12a}}$; which are given by Monsieur de Lagney for a second near value of the cube-root of a given number c , when a , or the first near value of it, is greater than its true value. And with them I shall conclude the present tract.

End of the Treatise on the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, &c, and on Mr. de Lagney's Method of Extracting the Cube-roots of Numbers by Approximation.

A GENE.

A
GENERAL METHOD
OF
EXTRACTING THE ROOTS OF NUMBERS
BY
APPROXIMATION;

INVENTED BY
MONSIEUR DE LAGNY,
A MEMBER OF THE ROYAL ACADEMY OF SCIENCES AT PARIS,
AND PUBLISHED IN THE YEAR 1697,

IN HIS
Nouveaux Eléments d'Arithmétique et d'Algèbre.

A

GENERAL METHOD

OF

EXTRACTING THE ROOTS OF NUMBERS

BY

APPROXIMATION.

Article 1. **I**N the foregoing Tract I have inserted Monsieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and have both given full investigations of it, and illustrated it by several examples that clearly prove its great utility. But Mr. de Lagny did not confine this method to the extraction of the cube-roots of numbers, but extended it to the extraction of their fifth roots, and their seventh roots, and all higher roots of them whatsoever. This he did by pursuing the same principle by which he had before been enabled to find his approximations to the cube-root of a given number, to wit, by considering the constitution of the compound quantity that is equal to any given power of a binomial quantity, (such as $a + b$,) or of a residual quantity, (such as $a - b$,) and substituting the sum, or difference, between a , the first near value of the root sought, (which is supposed to be already known,) and z , its unknown difference from the true value of the said root sought, instead of the said true value itself in the original equation derived from the conditions of the Problem, and then resolving the new equation, resulting

ing from such substitution, as if it were a quadratick equation, or neglecting all the terms of it which involve any higher powers of its root, or the difference z , than the square. This method I shall now endeavour to explain in the solutions of the two following Problems.

P R O B L E M I.

Art. 2. Let N be any proposed number whatsoever, and m any proposed whole number whatsoever; and let a be a known number that is nearly equal to, but less than, the m th root of the given number N . It is required to find a second near value of the said m th root of the given number N , that shall approach much nearer to it than a , or the former near value of it that is already known.

S O L U T I O N.

Let z be put for the unknown difference between a , the first near value of the m th root of the given number N , and the true value of the said number. Then, since a is supposed to be less than $\sqrt[m]{N}$, it follows that $a + z$ will be $= \sqrt[m]{N}$, and consequently that $\overline{a + z}^m$ will be $= N$.

But, by the binomial theorem in the first and simplest case of it, to wit, the case of integral powers, $\overline{a + z}^m$ will be $=$ the series $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2$
+

$+ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times$
 $\frac{m-3}{4} \times a^{m-4} z^4 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$
 $a^{m-5} z^5 + \&c,$ continued to $m + 1$ terms; or, if, for the
 sake of brevity, we substitute the capital letters A, B, C,
 D, E, F, &c, instead of the several numeral co-efficients
 1, m , $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times$
 $\frac{m-3}{4}$, and $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, respec-
 tively, $\overline{a + z}^m$ will be = the series $A a^m + B a^{m-1} z$
 $+ C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5$
 $+ \&c,$ continued to $m + 1$ terms. Therefore the said series
 $A a^m + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4$
 $+ F a^{m-5} z^5 + \&c,$ continued to $m + 1$ terms, will be
 = N. This is the original equation, by the solution of
 which we are to find a near value of z , and consequently
 a second near value of $a + z$, or $\sqrt[m]{N}$.

Art. 3. Now, since z is less than a , and usually much
 less, being about a 10th or 100th part of it, or some still
 lesser part of it, it is evident that all the terms in the afore-
 said series that involve zz , and z^3 , and z^4 , and the follow-
 ing powers of z , will be less, and usually much less, than
 the term $B a^{m-1} z$, which involves only the simple power
 of z . And therefore, if all the said terms of the series be
 neglected or omitted, the two first terms alone, to wit,
 $A a^m + B a^{m-1} z$, will be nearly equal to the given num-
 ber N; and consequently (if we subtract $A a^m$ from both
 sides of the equation,) we shall have $B a^{m-1} z$ nearly =
 $N - A a^m$, and (dividing both sides by $B a^{m-1}$) z nearly =

$= \frac{N - A a^m}{B a^{m-1}}$, or (because A is $= 1$, and B is $= m$) z

nearly $= \frac{N - a^m}{m a^{m-1}}$; which fraction may be derived from the

known quantities N and a by the operations of Multiplication, Subtraction, and Division. This therefore is an approximation to the true value of z , and consequently $a +$

$\frac{N - a^m}{m a^{m-1}}$ will be an approximation to the true value of $a + z$,

or of $\sqrt[m]{N}$, or will be a second near value of it that will approach nearer to it than a , or the first near value of it which was already known. But it will evidently be somewhat greater than the true value of $\sqrt[m]{N}$; because it arose

from a supposition that $A a^m + B a^{m-1} z$ were equal to the whole series of which they are only the two first terms, or that they were greater than they really were.

This quantity, $a + \frac{N - a^m}{m a^{m-1}}$, is the expression given by

Mr. Raphson for the second near value of the m th root of the given number N . And it is a very useful approximation: for it usually gives us twice as many figures of the true value of $\sqrt[m]{N}$ exact as were exact in a , or the first near value of the said m th root. And it is evidently the most simple and easy approximation to the value of the said m th root that can well be imagined.

Art. 4. But Mr. de Lagny, being desirous of finding z once a still nearer value of the m th root of the number N , retains the third term $C a^{m-2} z^2$, as well as the two first terms $A a^m$ and $B a^{m-1} z$, of the series $A a^m + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5 + \&c$, (which is equal to N), and thereby converts the original equation

equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 + Da^{m-3}z^3 +$
 $Ea^{m-4}z^4 + Fa^{m-5}z^5 + \&c, = N$ into a quadratick
equation, to wit, the equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2$
 $= N$, instead of converting it (as Mr. Raphson does,) into
the simple equation $Aa^m + Ba^{m-1}z = N$. And this
quadratick equation he resolves first imperfectly, or inac-
curately, by substituting in one of its terms, instead of z , the
inaccurate value of z already found by the resolution of the
simple equation $Aa^m + Ba^{m-1}z = N$, to wit, the fraction
 $\frac{N - a^m}{ma^{m-1}}$, (by which substitution the quadratick equation is
reduced to a simple equation,) and then resolving the said
simple equation thereby obtained ; which produces a second
value of z that is nearer than the former value $\frac{N - a^m}{ma^{m-1}}$ to
its true value. And this gives him a rational expression for
the value of $a + z$, or the second near value of $\sqrt[m]{N}$.
And then he resolves the same quadratick equation, Aa^m
 $+ Ba^{m-1}z + Ca^{m-2}z^2$, accurately, by the common
methods of resolving quadratick equations ; which produces
a furd, or irrational, expression for the value of z , and
consequently another furd, or irrational, expression for the
value of $a + z$, or for the second near value of $\sqrt[m]{N}$.
These resolutions of the said quadratick equation $Aa^m +$
 $Ba^{m-1}z + Ca^{m-2}z^2 = N$, may be performed in the
following manner,

Art. 5. Since $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 = N$,
we shall have $Ba^{m-1}z + Ca^{m-2}z^2 = N - Aa^m$; that
is,

is, $z \times B a^{m-1} + z \times C \times a^{m-2} z$ will be $= N - A a^m$, or $z \times \sqrt{B a^{m-1} + C \times a^{m-2} z}$ will be $= N - A a^m$. Therefore (dividing both sides of the equation by the compound quantity $B a^{m-1} + C \times a^{m-2} z$) we shall have $z = \frac{N - A a^m}{B a^{m-1} + C a^{m-2} z}$.

Now let $\frac{N - A a^m}{B a^{m-1}}$, or $\frac{N - a^m}{m a^{m-1}}$, (which has already been shewn to be nearly equal to z), be substituted instead of z in the second term $C a^{m-2} z$ of the denominator of the fraction last obtained, to wit, the fraction $\frac{N - A a^m}{B a^{m-1} + C a^{m-2} z}$.

And we shall have $z = \frac{N - A a^m}{B a^{m-1} + C a^{m-2} \times \sqrt{\frac{N - a^m}{m a^{m-1}}}}$, or

(because A is $= 1$, and B is $= m$),

$$\begin{aligned}
 z &= \frac{N - a^m}{m a^{m-1} + C a^{m-2} \times \sqrt{\frac{N - a^m}{m a^{m-1}}}}; \text{ which is} \\
 &= \frac{N - a^m}{m^2 a^{2m-2} + C a^{m-2} \times N - a^m} \\
 &= \frac{N - a^m}{m^2 a^{2m-2} + C N a^{m-2} - C a^{2m-2}} \\
 &= \sqrt{N - a^m} \times \frac{a^{m-1}}{m^2 a^{2m-2} + C N a^{m-2} - C a^{2m-2}}
 \end{aligned}$$

$$= \frac{\overline{N - a^m} \times m a^{m-1}}{m^2 a^{2m-2} + C N a^{m-2} - C a^{2m-2}}$$

$$= \frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + 2C N a^{m-2} - 2C a^{2m-2}} = \text{(because C is}$$

$$= m \times \frac{m-1}{2}, \text{ and consequently } 2C \text{ is } = m \times \overline{m-1},$$

which therefore may be substituted for it,)

$$\frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + m \times \overline{m-1} \times N a^{m-2} - m \times \overline{m-1} \times a^{2m-2}}$$

= (if we divide both the numerator and the denominator

by m ,)

$$\frac{\overline{N - a^m} \times 2a^{m-1}}{2m a^{2m-2} + \overline{m-1} \times N \times a^{m-2} - \overline{m-1} \times a^{2m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{2m - \overline{m-1} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{2m - m + 1 \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{m + 1 \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}} = \text{(if we mul-}$$

tiple both the numerator and denominator into a ,)

$$\frac{\overline{N - a^m} \times 2a^m}{m + 1 \times a^{2m-1} + \overline{m-1} \times N \times a^{m-1}} = \text{(by dividing}$$

both the numerator and the denominator of this fraction by

a^{m-1} ,)

$$\frac{\overline{N - a^m} \times 2a}{m + 1 \times a^m + \overline{m-1} \times N}, \text{ or } \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + m + 1 + a^m}.$$

Therefore z will be $= \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + m + 1 + a^m}$, and conse-

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quently

quently $a + z$ will be $= a + \frac{2a \times \sqrt[m]{N - a^m}}{(m-1) \times \sqrt[m]{N} + \sqrt[m]{m+1} \times a^{\frac{m}{m-1}}}$,
 or $a + \frac{2a \times \sqrt[m]{N - a^m}}{(m-1) \times \sqrt[m]{N} + \sqrt[m]{m+1} \times a^{\frac{m}{m-1}}}$ will be a second near value
 of $a + z$, or of the m th root of the proposed number N .

Q. E. I.

Art. 6. The quadratick equation mentioned above in
 art. 4, to wit, the equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 = N$, or (because A is $= 1$, and B is $= m$, and C is $= m \times \frac{m-1}{2}$), the equation $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2 = N$, may be accurately resolved in the manner
 following.

By doubling both sides of this equation we shall have
 $2a^m + 2ma^{m-1}z + m \times \overline{m-1} \times a^{m-2}zz = 2N$;
 and, by multiplying both sides of this equation into aa , we
 shall have $2a^{m+2} + 2ma^{m+1}z + m \times \overline{m-1} \times a^m \times zz = 2a^2 N$; and, by subtracting $2a^{m+2}$ from both
 sides, we shall have $2ma^{m+1}z + m \times \overline{m-1} \times a^m \times zz = 2aa N - 2a^{m+2}$; and, by dividing both sides of this
 last equation by $m \times \overline{m-1} \times a^m$, we shall have

$$\frac{2ma^{m+1}z}{m \times \overline{m-1} \times a^m} + zz = \frac{2aa N - 2a^{m+2}}{m \times \overline{m-1} \times a^m}, \text{ or } \frac{2a^{m+1}z}{\overline{m-1} \times a^m} +$$

$$zz = \frac{2aa N - 2a^{m+2}}{m \times \overline{m-1} \times a^m}, \text{ or (because } \frac{a^{m+1}}{a^m} \text{ is } = a_1) \frac{2az}{m-1} +$$

$$zz =$$

$$zx = \frac{2aaN - 2a^{m+2}}{m \times m-1 \times a^m}, \text{ or } \frac{2a}{m-1} \times z + zz = \frac{2N - 2a^m}{m \times m-1 \times a^{m-2}},$$

$$\text{or } \frac{2a}{m-1} \times z + zz = \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}.$$

Now let the square of $\frac{a}{m-1}$ (which is half the co-efficient of z in the term $\frac{2az}{m-1}$) be added to both sides of this equation. And we shall have $\left[\frac{a}{m-1}\right]^2 + \frac{2a}{m-1} \times z + zz = \left[\frac{a}{m-1}\right]^2 + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}} = \frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}$; or, if, for the sake of brevity, we put P for the quantity $\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}$, we shall have $\left[\frac{a}{m-1}\right]^2 + \frac{2a}{m-1} \times z + zz = P$. Therefore, (extracting the square-roots of both sides,) we shall have $\frac{a}{m-1} + z = \sqrt{P}$, and $z = \sqrt{P} - \frac{a}{m-1}$. Therefore $a + z$, or $\sqrt{m} \sqrt{N}$, will be $= a + \sqrt{P} - \frac{a}{m-1}$, or $a + \sqrt{\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}}$ $= \frac{a}{m-1}$. Q. E. I.

Art. 7. This irrational expression

$$a + \sqrt{\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}} - \frac{a}{m-1}, \text{ or } a + \sqrt{P} - \frac{a}{m-1},$$

will approach somewhat nearer to the true value of $\sqrt{m} \sqrt{N}$ than the former, or rational, expression,

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$a + \frac{2a \times N - a^m}{(m-1) \times N + (m+1) \times a^m}$; because it proceeds from the accurate resolution of the quadratick equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}zz = N$, whereas the rational expression $a + \frac{2a \times N - a^m}{(m-1) \times N + (m+1) \times a^m}$ was derived from an inaccurate resolution of the same quadratick equation. But the difference of the two expressions, in point of exactness, is not great; and either of them will usually give us three times as many decimal figures of the true value of $\sqrt[m]{N}$ exact as were exact in a , or the preceding near value of it.

Examples of the Extraction of the Roots of Numbers by Means of the foregoing Expressions.

E X A M P L E I.

Art. 8. Let it be required to find the cube-root of the number 2, having 1.259 for a , or the first near value of the said cube-root.

Here N is $= 2$, and m is $= 3$, and consequently $m - 1$ is $(= 3 - 1) = 2$, and $m + 1$ is $(= 3 + 1) = 4$, and

the expression $a + \frac{2a \times N - a^m}{(m-1) \times N + (m+1) \times a^m}$ becomes $= a + \frac{2a \times N - a^3}{2N + 4a^3}$, or $a + \frac{a \times N - a^3}{N + 2a^3}$, or $a + \frac{N - a^3}{N + 2a^3} \times a$, or, (if we substitute the small letter c instead of the capital letter N ,)

N,) $a + \frac{c - a^3}{c + 2a^3}$; which is the first, or rational, expression, given in the preceeding tract for the cube-root of the number c . And as a is supposed to be $= 1.259$, this expression $a + \frac{c - a^3}{c + 2a^3}$ will be $= 1.259 + \frac{2 - 1.259^3}{2 + 2 \times 1.259^3}$
 $= 1.259 + \frac{2 - 1.995,616,979}{2 + 2 \times 1.995,616,979} = 1.259 +$
 $\frac{0.004,383,021 \times 1.259}{2 + 3.991,233,958} = 1.259 + \frac{0.005,518,223,439}{5.991,233,958} = 1.259$
 $+ 0.000,921,049,55, \&c = 1.259,921,049,55, \&c$. Therefore $1.259,921,049,55, \&c$, will be a near value of the cube-root of 2. Q. E. I.

See the preceeding Tract, page 474.

Art. 9. And if we compute this cube-root by means of the irrational expression $a + \sqrt[m]{P - \frac{a}{m-1}}$, or $a - \frac{a}{m-1} + \sqrt[m]{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{m-1} + \frac{2 \times n - a^m}{m \times m-1 \times a^{m-2}}}$, we

shall find that this expression will, upon making the proper substitutions, co-incide with the irrational expression given for the cube-root of a given number c in the foregoing tract, to wit, the expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, and consequently give the same value of the cube-root of 2, as was obtained in the foregoing tract by means of that expression.

For, since m is $= 3$, we shall have $m - 1 = 2$, and $m - 1)^2 = 4$, and $m \times m - 1 (= 3 \times 2) = 6$, and $m - 2 (= 3 - 2) = 1$, and $a^{m-2} = a$, and consequently

$$\frac{aa}{m-1)^2} = \frac{aa}{4}, \text{ and } \frac{2 \times n - a^m}{m \times m-1 \times a^{m-2}} = \frac{2 \times n - a^3}{6 \times a} = \frac{n - a^3}{3a},$$

and

and P, or $\frac{aa}{(m-1)^2} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}} (= \frac{aa}{4} + \frac{N-a^3}{3a})$
 $= \frac{3a^2}{12a} + \frac{4N-4a^3}{12a} = \frac{3a^2+4N-4a^3}{12a} = \frac{4N-a^3}{12a}$, or, (if we
 substitute c instead of N , in order to adopt the notation
 used in the foregoing tract,) $P = \frac{4c-a^3}{12a}$, and $a = \frac{a}{m-1}$
 $+ \sqrt{P} (= a - \frac{a}{2} + \sqrt{P} = \frac{a}{2} + \sqrt{P}) = \frac{a}{2} +$
 $\sqrt{\frac{4c-a^3}{12a}}$; which is the irrational expression given in the
 foregoing tract for the second near value of the cube-root of
 the number c . So that the general, irrational, expression $a =$

$\frac{a}{m-1} + \sqrt{P}$, or $a = \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}}$,
 which is obtained by the general investigation of the m th
 root of the given number N in the present tract, agrees
 perfectly with the particular, irrational, expression $\frac{a}{2} +$
 $\sqrt{\frac{4c-a^3}{12a}}$, which was obtained in the foregoing tract, by
 the particular investigation of the cube-root of the given
 number c .

And, since c is $= 2$, and a is $= 1.259$, we shall have
 $a^3 (= 1.259^3) = 1.995,616,979$, and $4c - a^3 (= 8.000,$
 $000,000 - 1.995,616,979) = 6.004,383,021$, and $12a$
 $(= 12 \times 1.259) = 15.108$, and $\frac{4c-a^3}{12a} (= \frac{6.004,383,021}{15.108}) =$
 $0.397,430,700,3$, and $\sqrt{\frac{4c-a^3}{12a}} (= \sqrt{0.397430,7003}) =$
 $0.630,421,050,01$, and $\frac{a}{2} (= \frac{1.2590}{2}) = 0.6295$, and $\frac{a}{2}$
 $+ \sqrt{\frac{4c-a^3}{12a}} (= 0.6295 + 0.630,421,050,01) = 1.259,$

921,050,01. Therefore 1.259,921,050,01 will be a second near value of the cube-root of 2. Q. E. I.

The first eight figures 1.259,921,0, of this number 1.259,921,050,01, are exact, the more accurate value of the cube-root of 2 being 1.259,921,049,894,873,164,76, &c, of which I believe all the figures to be exact. See above, pages 502 and 503.

EXAMPLE II.

Art. 10. Let it be required to find the fifth root of the number 2, which the celebrated *Vieta*, or *Monsieur Viète*, has found to be 1.148,697.

Here m is = 5, and consequently $m - 1$ is = 4, and $m + 1$ is = 6, and the first general expression $a +$

$\frac{2a \times \overline{N - a^m}}{(m-1) \times N + m+1) \times a^m}$ will be $(= a + \frac{2a \times \overline{N - a^5}}{4N + 6a^5}) = a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, or $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$; and the other general expression

$a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} + \frac{2 \times \overline{N - a^m}}{m \times (m-1) \times a^{m-2}}}$

will be $(= a - \frac{a}{4} + \sqrt{\frac{aa}{16} + \frac{2 \times \overline{N - a^5}}{5 \times 4 \times a^3}} = \frac{3a}{4} +$

$\sqrt{\frac{aa}{16} + \frac{N - a^5}{10a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} + \frac{16N - 16a^5}{160a^3}} = \frac{3a}{4} +$

$\sqrt{\frac{10a^5 + 16N - 16a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{16N - 6a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

We must therefore compute one of the two expressions

$a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, in order to obtain a

second

second near value of the fifth root of N , or 2, when we shall have first found a , or a first near value of the said fifth root that is less than its true value, to a small degree of exactness.

Art. 11. The value of a , or the first approximation to the fifth root of 2, may be found in the following manner.

The fifth root of 2 is the second of six quantities in continued geometrical proportion, of which 1 is the first, or least, and 2 is the sixth, or greatest. Now the excess of the second of these proportionals above the first, is necessarily less than the fifth part of the excess of the greatest above the least, because the excesses increase in the same proportion as the terms themselves. The excess of the greatest of these six terms, to wit, 2, above the least of them, or 1, is 1; of which the fifth part is $\frac{1}{5}$, or $\frac{2}{10}$, or 0.2. Therefore the excess of the second term above the first will be less than $\frac{2}{10}$, or 0.2; and consequently the second term itself will be less than $1 + \frac{2}{10}$, or than 1.2. Let us therefore suppose this second term, or the fifth root of 2, which we are seeking, to be equal to 1.1; and let us raise this number to the fifth power, in order to try how nearly it approaches to the truth.

Now, if we raise 1.1 to the fifth power, or multiply it four times into itself, we shall find that 1.1^5 is $= 1.61051$; which is considerably less than 2. Therefore 1.1 must be considerably less than the true value of the 5th root of 2. But we have seen that the said true value must be less than 1.2. Let us therefore suppose it to be equally distant from 1.1 and 1.2, or to be $= 1.15$, and try whether this will not be pretty near the truth.

Now 1.15^5 is $= 2.011,357,187,5$; which is a little bigger than 2. Therefore 1.15 must be something greater than the 5th root of 2. But the difference can be but small.

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We will therefore, in the next place, suppose the said 5th root to be $= 1.14$, and raise this number to the fifth power, in order to discover whether the said fifth power will be greater, or less, than 2, and consequently whether 1.14 will be greater, or less, than the fifth root of 2.

Now, if we multiply 1.14 four times successively into itself, we shall find that 1.14^5 is $= 1.925,414,582,4$; which is somewhat less than 2. Therefore the 5th root of 2 will be greater than 1.14, but less than 1.15; and therefore 1.14 will be a very convenient first near value of the 5th root of 2, and will be very fit to be made the basis of a further approximation to the true value of the said 5th root, by substituting it instead of a in either of the two expressions

$a + \frac{N - a^5}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, which have been derived from the foregoing Problem.

Art. 12. Now, if we suppose a to be $= 1.14$, we shall have $aa = 1.2996$, and $a^3 = 1.481,544$, and $a^5 = 1.925,414,582,4$, and consequently $N - a^5 (= 2.000,000,000,0 - 1.925,414,582,4) = 0.074,585,417,6$, and $N - a^5 \times a (= 0.074,585,417,6 \times 1.14) = 0.085,027,376,064$, and $2N (= 2 \times 2) = 4$, and $3a^5 (= 3 \times 1.925,414,582,4) = 5.776,243,747,2$, and $2N + 3a^5 (= 4 + 5.776,243,747,2) = 9.776,243,747,2$, and $\frac{N - a^5 \times a}{2N + 3a^5} (= \frac{0.085,027,376,064}{9.776,243,747,2}) = 0.008,697,34$. Therefore $a + \frac{N - a^5 \times a}{2N + 3a^5}$ will be $(= 1.14 + 0.008,697,34) = 1.148,697,34$; and consequently 1.148,697,34 will be very nearly equal to the 5th root of 2. Q. E. I.

This number 1.148,697,34 agrees with that found by *Vieta*, to wit, 1.148,697, in all its seven figures, but is carried to two more figures.

The other expression $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, may be computed as follows.

3 X

Since

Since a is $= 1.14$, we shall have $3a (= 3 \times 1.14) = 3.42$, and $\frac{3a}{4} (= \frac{3.42}{4}) = 0.855$, and $8N (= 8 \times 2) = 16$, and $3a^5 (= 3 \times 1.925,414,582,4) = 5.776,243,747,2$, and $8N - 3a^5 (= 16.000,000,000,0 - 5.776,243,747,2) = 10.223,756,252,8$, and $80a^3 (= 80 \times 1.481,544) = 118.523,520$, and $\frac{8N - 3a^5}{80a^3} (= \frac{10.223,756,252,8}{118.523,520}) = 0.086,259,303,240,40$, and $\sqrt{\frac{8N - 3a^5}{80a^3}} (= \sqrt{0.086259,303240,40}) = 0.293,699,34$, and consequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}} (= 0.855 + 0.293,699,34) = 1.148,699,34$. Therefore the fifth root of 2 will be very nearly equal to 1.148,699,34.

Q. E. I.

E X A M P L E III.

Art. 13. Let it be required to find the 5th root of the number 307,68282,11067,15625.

In order to find a first near value of the 5th root of this long number, we may begin by comparing it with the fifth powers of the numbers 10, 100, and 1000, and the following powers of 10. Now the fifth power of 10 is 100,000; which is very much less than the proposed number: and the fifth power of 100 is 10,000,000,000; which is also much less than the proposed number: and the fifth power of 1000 is 1000,000,000,000,000; which is also less than the proposed number: but the fifth power of 10,000 is 100,000,000,000,000,000,000; which is greater than the proposed number. We may therefore conclude that 1000 must be less, and that 10,000 must be greater than the fifth root

root of the proposed number. Therefore the fifth root of the said number must be of an intermediate magnitude between 1000 and 10,000.

Further, the proposed number 307,68282,11067,15625 is greater than 307,00000,00000,00000, or than 307×1000^5 . Therefore the fifth root of the said number will be greater than the product of the multiplication of the fifth root of 307 by 1000. We will therefore inquire what is the fifth root of 307.

Now the fifth power of 2 is 32, and the fifth power of 3 is 243, and the fifth power of 4 is 1024. Therefore the fifth power of 4 is much greater than 307, and the fifth power of 3 is a little less than 307. We may therefore conclude that the fifth root of 307 will be much less than 4, and a little greater than 3; and consequently the fifth root of the number 307,00000,00000,00000 will be much less than 4×1000 , or 4000, and a little greater than 3×1000 , or 3000. We may therefore reasonably conjecture that the fifth root of the proposed number 307,68282,11067,15625, will be nearly equal to 3100. And accordingly, if we raise this number 3100 to the fifth power, we shall find the said power of it to be = 286,29151,00000,00000, which is pretty nearly equal to, but somewhat less than, the proposed number 307,68282,11067,15625. Therefore 3100 will be a proper number to make the basis of a further approximation to the true value of the fifth root of the said proposed number, by means of either of the two ex-

pressions above-mentioned, to wit, $a + \frac{N - a^5 \times a}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 14. Here then we have $N = 307,68282,11067,15625$, and $a = 3100$, and consequently $a^5 = 286,29151,00000,00000$, and $3a^5 (= 3 \times 286,29151,00000,00000) = 858,87453,00000,00000$, and $2N (= 2 \times 307,68282,11067,15625) = 615,36564,22134,31250$, and $2N + 3a^5$

$$\begin{array}{r} 615,36564,22134,31250 \\ 858,87453,00000,00000 \\ \hline 1473,24017,22134,31250 \end{array}$$

(= 615,36564,22134,31250 + 858,87453,00000,00000)
 = 1474,24017,22134,31250, and $N - a^5$ (= 307,68282,
 11067,15625 - 286,29151,00000,00000) = 21,39131,
 11067,15625, and $N - a^5 \times a$ (= 21,39131,11067,15625
 $\times 3100$) = 66313,06443,08184,37500, and $\frac{N - a^5 \times a}{2N + 3a^3}$ (= $\frac{66313,06443,08184,37500}{1474,24017,22134,31250}$) = 44.98. Therefore $a + \frac{N - a^5 \times a}{2N + 3a^3}$
 will be (= 3100 + 44.98) = 3144.98, which will there-
 fore be nearly equal to the fifth root of the proposed num-
 ber 307,68282,11067,15625. Q. E. I.

The three first figures, 314, of this number 3144.98, or,
 rather, the five first figures of it, 3144.9, are exact, the
 error being only in the sixth figure 8, which ought to be
 a 9 instead of an 8. For the exact root of the proposed
 number 307,68282,11067,15625 is 3144.999,999, *ad infi-*
nitum, or the whole number 3145, as will appear by raising
 the said number 3145 to the fifth power.

Art. 15. The other, or irrational, expression for the se-
 cond near value of the fifth root of this number 307,68282,
 11067,15625, to wit, the expression $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, may
 be computed as follows.

Since a is = 3100, we shall have $3a$ (= 3×3100) =
 9300, and $\frac{3a}{4}$ (= $\frac{9300}{4}$) = 2325, and a^5 = 2,97910,00000,
 and a^5 = 286,29151,00000,00000, and $80a^3$ (= $80 \times$
 $2,97910,00000$) = 238,32800,00000, and $8N$ (= $8 \times$
 $307,68282,11067,15625$) = 2461,46256,88537,25000, and
 $3a^5$ = 858,87453,00000,00000, and $8N - 3a^5$ (= $2461,$
 $46256,88537,25000 - 858,87453,00000,00000$) = 1602,
 58803,88537,25000; and $\frac{8N - 3a^5}{80a^3}$ (= $\frac{1602,58803,88537,25000}{238,32800,00000}$)
 = 672.29.6091, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ (= $\sqrt{672.29.6091}$)

820.01,

820.01, and consequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^4}{80a^3}}$ ($= 2325 + 820.01$) $= 3145.01$. Therefore 3145.01 will be a second near value of the fifth root of the proposed number 307,68282,11067,15625. Q. E. I.

Art. 16. If we should chuse to find the fifth root of this number 307,68282,11067,15625, by means of Mr. Raphson's expression for its value, to wit, the expression $a +$

$\frac{N - a^5}{5a^4}$, or $a + \frac{N - a^5}{5a^4}$, (which certainly has the merit of being much simpler, and easier to be remembered, than either of Mr. de Lagny's expressions, and likewise much easier to compute,) the computation will be as follows.

Since a is $= 3100$, we shall have $a^4 = 9235,21000,00000$, and $a^5 = 286,29151,00000,00000$, and $5a^4 = 46176,05000,00000$, and $N - a^5 (= 307,68282,11067,15625 - 286,29151,00000,00000) = 21,39131,11067,15625$, and $\frac{N - a^5}{5a^4} (= \frac{21,39131,11067,15625}{46176,05000,00000}) = 46$. Therefore $a + \frac{N - a^5}{5a^4}$ will be $(= 3100 + 46) = 3146$; which therefore will be the second near value of the fifth root of the proposed number 307,68282,11067,15625, obtained by Mr. Raphson's approximation. Q. E. I.

A S C H O L I U M.

This last near value of the fifth root of the said proposed number, which has been obtained by Mr. Raphson's approximation, is greater than its true value, 3145, by only an unit, or the 3145th part of the said true value. So that this very simple method of approximating to the roots of numbers

bers may be justly considered as extremely useful as well as easy. And, if this process were to be repeated, by taking 3146 for the value of a , and supposing $a - z$ to be equal to $\sqrt[5]{N}$, (a , or 3146, being somewhat greater than the true value of the fifth root which we are in search of,) and by

computing the expression $a - \frac{N - a^5}{5a^4}$, that would result from

that supposition, this second process would double the number of figures that are exact in a , or 3146, or give us about four times the number of figures that were exact in 3100, or the former value of a ; which is more than is done by either of

the two expressions, $a + \frac{N - a^5}{2N + 3a^5} \times a$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$,

given us by Mr. de Lagny: So that two steps of Mr. Raphson's method of approximation are more than equivalent to one step of Mr. de Lagny's method. It may therefore be doubted, whether Mr. Raphson's method is not, upon the whole, to be preferred to Mr. de Lagny's, as Mr. Raphson himself always thought it to be. For he tells us in the Appendix to the second edition of his excellent Treatise, intitled, *Analysis Aequationum Universalis*, (which second edition was published in the year 1697, seven years after the first edition of it, and five or six years after the publication of Monsieur de Lagny's method,) that he himself had had thoughts

of resolving the quadratick equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 = N$, or $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times$

$a^{m-2} z^2 = N$, in the imperfect manner adopted by Mr. Lagny, in order to obtain his rational value of z , to wit,

$\frac{\sqrt{N - a^m} \times 2a}{m-1 \times N + m+1 \times a^m}$, to wit, by substituting in the term

$m \times \frac{m-1}{2} \times a^{m-2} z^2$ of the said quadratick equation, in-

stead of z , the value of z already obtained by the resolution

of the simple equation $a^m + m a^{m-1} z = N$, to wit, the fraction

fraction $\frac{N - a^m}{m a^{m-1}}$, (by which substitution the said quadratick equation would be converted into the following simple equation, $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z \times \sqrt{\frac{N - a^m}{m a^{m-1}}} = N$), and then resolving the simple equation thence resulting, to wit, the simple equation $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z \times \frac{N - a^m}{m a^{m-1}} = N$, or $m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} \times \sqrt{\frac{N - a^m}{m a^{m-1}}} \times z = N - a^m$, or $z \times \left[m a^{m-1} + m \times \frac{m-1}{2} \times a^{m-2} \times \sqrt{\frac{N - a^m}{m a^{m-1}}} \right] = N - a^m$, in the usual way, or by the single operation of Division, which would give us $z (=$

$$\begin{aligned} & \frac{N - a^m}{m a^{m-1} + m \times \frac{m-1}{2} \times a^{m-2} \times \sqrt{\frac{N - a^m}{m a^{m-1}}}} \\ &= \frac{N - a^m}{m a^{m-1} + \frac{m-1}{2} \times a^{m-2} \times \sqrt{\frac{N - a^m}{a^{m-1}}}} \\ &= \frac{2 \times N - a^m}{2 m a^{m-1} + (m-1) a^{m-2} \times \sqrt{\frac{N - a^m}{a^{m-1}}}} \\ &= \frac{2 \times N - a^m}{2 m a^{2m-2} + (m-1) \times a^{m-2} \times N - a^m} \\ & \quad a^{m-1} \end{aligned}$$

==

$$\begin{aligned}
 &= \frac{2 \times N - a^m}{2ma^{2m-2} + \overline{m-1} \times a^{m-2} \times N - ma^{2m-2} + a^{2m-2}} \\
 &= \frac{2 \times N - a^m}{ma^{2m-2} + a^{2m-2} + \overline{m-1} \times a^{m-2} \times N} \\
 &= \frac{2a \times N - a^m}{ma^{2m-1} + a^{2m-1} + \overline{m-1} \times a^{m-1} \times N} \\
 &= \frac{2a \times N - a^m}{ma^m + a^m + \overline{m-1} \times N} = \frac{2a \times N - a^m}{\overline{m+1} \times a^m + \overline{m-1} \times N} \\
 &= \frac{N - a^m \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}, \text{ and consequently } a + z = a + \frac{N - a^m}{\overline{m-1} \times N + \overline{m+1} \times a^m}
 \end{aligned}$$

which is Mr. de Lagny's rational expression for the second near value of the m th root of N . Mr. Raphson (I say,) tells us that he himself had had thoughts of resolving the quadratick equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z^2 = N$ in this manner, in order to obtain a value of z somewhat nearer to the truth than the fraction $\frac{N - a^m}{ma^{m-1}}$, which he had obtained by the resolution

of the simple equation $a^m + ma^{m-1}z = N$; but that he did not think proper to adopt this method, because he thought his own method of approximation, (which he published in the first edition of his *Analysis Aequationum Universalis*, in the year 1690, and which is derived from the simple equation $a^m + ma^{m-1}z = N$;) the easiest and clearest, and, upon the whole, the best, and fittest for practice, that could be followed. His words, in his Appendix, relating

relating to this subject, are as follows. *An Dominus de Lagny Librum meum unquam viderat, nec-ne, prorsus nescio. Quibuscumque modis non solum sua methodus, sed et etiam alia quam-plurima, eodem prorsus processu, et perpetuo inde derivata graduum scala, inveniri possint, hujusce Appendicis est ostendere; idque, quam possumus, brevissime.*

*Ipse equidem de gradatim inferendis (quas prius rejeceram in Thecramate Vietæo,) potestatibus olim cogitavi: Sed tamen non prosecutus fui; utpote qui methodum meam, harum omnium fundamentalem, veluti facillimam semper existimavi. Subsequentem processu earum omnium inventionem indagare cuilibet liceat. See Mr. Raphson's *Analysis Æquationum Universalis*, Edition 2d, 1697, page 49. And again, in page 55, he concludes his Appendix with these words: *Innumeras etiam alias methodos et abbreviationes (novarum quidem methodorum nomine insigniendas,) adinvenire liceat; quæ tamen omnia fundamentali hujus superiorum potestatum imprimis rejectionis methodo, posteaque gradatim retinendarum, innitantur. Nostram tamen simplicissimam fore et facillimam, cuivis pateat.**

EXAMPLE IV.

Art. 17. Let it be required to find the seventh root of the number 34,487,717,467,307,513,182,492,153,794,673; which, Mr. Bonycastle, of the Royal Military Academy at Woolwich, in his *Scholar's Guide to Arithmetick*, page 189, tells us, is 32017.

This number must, in the first place, be compared with the seventh powers of 10, 100, 1000, &c, to which it approaches nearest, in order to know between which two of those numbers 10, 100, 1000, &c, its seventh root will lie. Now the seventh power of 10 is 10,000,000, which is very much less than the said proposed number; and the seventh

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power

power of 100 is 100,000,000,000,000, which is likewise much less than the said proposed number; and the seventh power of 1000 is 1000,000,000,000,000,000,000, which is likewise less than the said proposed number; and the seventh power of 10,000 is 10,000,000,000,000,000,000,000,000,000, or 1 with twenty-eight cyphers annexed to it; which is likewise less than the said proposed number, which consists of 32 figures. But the seventh power of 100,000 is 100,000,000,000,000,000,000,000,000,000,000, or 1 with 35 cyphers annexed to it, and is therefore greater than the said proposed number, which consists of only 32 figures. Therefore 10,000 will be less, and 100,000 will be greater, than the seventh root of the said proposed number.

Further, the proposed number 34,487,717,467,307,513, 182,492,153,794,673, is greater than the number 34480,000,000,000,000,000,000,000,000,000,000, or than $3448 \times 10,000,000,000,000,000,000,000,000,000,000,000$, or than $3448 \times$ the seventh power of 10,000: but the difference between them is not great. Therefore the seventh root of the said proposed number will be greater than 10,000 times the seventh root of 3448: but the difference between them will not be great. And consequently, if we can find the seventh root of the number 3448 exact to two, or three, places of figures, we need only multiply the said seventh root by 10,000, in order to obtain the seventh root of 34480,000,000,000,000,000,000,000,000,000,000, exact to two or three places of figures. And, when we have obtained the said near value of the seventh root of the number 34480,000,000,000,000,000,000,000,000,000,000, the said near value will likewise be a near value of the seventh root of the proposed number 34487,717,467,307,513,182,492,153,794,673, and will be less than the true value of the seventh root of the said number, and therefore will serve as a convenient basis of a further approximation to the true value of the seventh root of the said proposed number, by means of one of *Monsieur de Lagny's* two expressions found above in the solution of the foregoing Problem. We must therefore now endeavour to find a near value of the seventh root of the number

number 3448. Now this may be done in the manner following.

Art. 18. The seventh power of the number 2 is 128, and the seventh power of 3 is 2187, and the seventh power of 4 is 16384. Therefore, since 3448 is greater than 2187, or the seventh power of 3, but is much less than 16384, or the seventh power of 4, it follows that the seventh root of 3448 must be greater than 3, but much less than 4. We may therefore reasonably conjecture, that it will be nearly equal to $3\frac{1}{7}$, or 3.2. And accordingly, upon trial, we shall find it to be so. For, if we raise 3.2 to its seventh power, we shall find the said power to be = 3435.973,836,8; which is less than 3448, but very nearly equal to it. Therefore 3.2 is a very near first value of the seventh root of the number 3448; and consequently $3.2 \times 10,000$, or 32000, will be a very near first value of the number 34480,000,000,000,000,000,000,000,000,000,000,000, and therefore will be also a pretty near first value of the proposed number $34487,717,467,307,513,182,492,153,794,673$. We will therefore suppose a to be = 32000, and proceed, upon that supposition, to compute the two expressions given in the solution of the foregoing Problem, for a second value of the seventh root of the said proposed number that shall approach nearer than a , or 32000, to its true value. These computations will be as follows.

Art. 19. The first, or rational, expression, given in the solution of the foregoing Problem, for the second near value of the m th root of any proposed number N , is $a +$

$$\frac{N - a^m \times 2a}{m-1 \times N + m+1 \times a^m}.$$

Now, when m is = 7, this expression will be $(= a + \frac{N - a^7 \times 2a}{7-1 \times N + 7+1 \times a^7} = a + \frac{N - a^7 \times 2a}{6N + 8a^7}) = a + \frac{N - a^7 \times a}{3N + 4a^7}.$

Now, since N is, in this case, = 34,487,717,467,307,
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513,

513,182,492,153,794,673, and a is = 32000, we shall have
 a^7 (= 32000⁷) = 34,359,738,368,000,000,000,000,000,000,
 000,000, and $N - a^7$

$$\begin{aligned}
 & (= 34,487,717,467,307,513,182,492,153,794,673 \\
 & \quad - 34,359,738,368,000,000,000,000,000,000) \\
 & = 127,979,099,307,513,182,492,153,794,673, \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 & N - a^7 \times a \quad (= 127,979,099,307,513,182,492,153,794,673 \\
 & \quad \times 32000) = 4,095,331,177,840,421,819,748,921,429, \\
 & 536,000, \text{ and } 3N \quad (= 3 \times 34,487,717,467,307,513,182, \\
 & 492,153,794,673) = 103,463,152,401,922,539,547,476, \\
 & 461,384,019, \text{ and } 4a^7
 \end{aligned}$$

$$\begin{aligned}
 & (= 4 \times 34,359,738,368,000,000,000,000,000,000,000) \\
 & = 137,438,945,472,000,000,000,000,000,000,000, \\
 & \text{and } 3N + 4a^7
 \end{aligned}$$

$$\begin{aligned}
 & (= 103,463,152,401,922,539,547,476,461,384,019 \\
 & \quad + 137,438,945,472,000,000,000,000,000,000,000) \\
 & = 240,902,097,873,922,539,547,476,461,384,019, \text{ and}
 \end{aligned}$$

$$\frac{N - a^7 \times a}{3N + 4a^7} \quad (= \frac{4,095,331,177,840,421,819,748,921,429,436,000}{240,902,097,873,922,539,547,476,461,384,019} =,$$

$$\text{nearly, } \frac{4,095,331}{240,902} = 16.99998. \text{ Therefore } a + \frac{N - a^7 \times a}{3N + 4a^7}$$

will be (= 32000 + 16.99998) = 32016.99998, and consequently the second near value of the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673 will be 32016.99998. Q. E. I.

This number is true in all the figures but the last, which ought to be a 9 instead of an 8, the true value of the seventh root of the said proposed number being 32016.999,999,999, 999, &c, *ad infinitum*, or the whole number 32017.

Art. 20. The second, or irrational, expression, given in the solution of the foregoing Problem, for the second near value of the m th root of any proposed number N , is $a -$

$$\frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} + \frac{2 \times N - a^m}{m \times \frac{a}{m-1} \times a^{m-2}}}; \text{ which, when } m \text{ is}$$

$$= 7,$$

= 7, (as is the case in the present example,) is (= $a -$

$$\frac{a}{7-1} + \sqrt{\frac{aa}{(7-1)^2} + \frac{2 \times N - a^7}{7 \times 7-1 \times a^5}} = a - \frac{a}{6} +$$

$$\sqrt{\frac{aa}{6^2} + \frac{2 \times N - a^7}{7 \times 6 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} + \frac{N - a^7}{7 \times 3 \times a^5}} = \frac{5a}{6}$$

$$+ \sqrt{\frac{aa}{36} + \frac{N - a^7}{21a^5}} = \frac{5a}{6} + \sqrt{\frac{21a^7}{21 \times 36a^5} + \frac{36N - 36a^7}{21 \times 36a^5}} = \frac{5a}{6} +$$

$$\sqrt{\frac{7a^7}{7 \times 36a^5} + \frac{12N - 12a^7}{7 \times 36a^5}} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{7 \times 36a^5}} = \frac{5a}{6} +$$

$$\sqrt{\frac{12N - 5a^7}{252a^5}}. \text{ This expression may be computed as follows.}$$

Since N is = 34,487,717,467,307,513,182,492,153,794,673, and a is = 32000, we shall have $5a$ (= 5×32000) = 160,000, and $\frac{5a}{6}$ (= $\frac{160,000}{6}$) = 26,666.666,666, &c,

and a^5 (= $\sqrt[5]{32000^5}$) = 33,554,432,000,000,000,000,000, and $252a^5$ (= $252 \times 33,554,432,000,000,000,000,000$) = 8,455,716,864,000,000,000,000,000, and a^7 (= $\sqrt[7]{32,000^7}$) = 34,359,738,368,000,000,000,000,000,000,000, and $5a^7$ (= $5 \times 34,359,738,368,000,000,000,000,000,000,000$) = 171,798,691,840,000,000,000,000,000,000,000, and $12N$ (= $12 \times 34,487,717,467,307,513,182,492,153,794,673$) = 413,852,609,607,690,158,189,905,845,536,076, and $12N - 5a^7$ (= $413,852,609,607,690,158,189,905,845,536,076 - 171,798,691,840,000,000,000,000,000,000,000$) = 242,053,917,767,690,158,189,905,845,536,076, and $\frac{12N - 5a^7}{252a^5}$ (= $\frac{242,053,917,767,690,158,189,905,845,536,076}{8,455,716,864,000,000,000,000,000}$) =

28,626,067.033799, and $\sqrt{\frac{12N - 5a^7}{252a^5}}$ (= $\sqrt{28,626,067.033799}$) = 5350.333,35.

Therefore $\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$ will be (= 26,666.666,66, &c, + 5350.333,35) = 32017.000,01; and consequently 32017.000,01 will be the second near

513,182,492,153,794,673, and a is = 32000, we shall have
 a^7 (= $\overline{32000^7}$) = 34,359,738,368,000,000,000,000,000,
 000,000, and $N - a^7$
 (= 34,487,717,467,307,513,182,492,153,794,673
 — 34,359,738,368,000,000,000,000,000,000)
 = 127,979,099,307,513,182,492,153,794,673, and
 $N - a^7 \times a$ (= 127,979,099,307,513,182,492,153,794,673
 $\times$ 32000) = 4,095,331,177,840,421,839,748,921,429,
 536,000, and $3N$ (= $3 \times 34,487,717,467,307,513,182,$
 $492,153,794,673$) = 103,463,152,401,922,539,547,470,
 461,384,019, and $4a^7$
 (= $4 \times 34,359,738,368,000,000,000,000,000,000$)
 = 137,438,945,472,000,000,000,000,000,000,
 and $3N + 4a^7$
 (= 103,463,152,401,922,539,547,476,461,384,019
 + 137,438,945,472,000,000,000,000,000,000)
 = 240,902,097,873,922,539,547,476,461,384,019, and
 $\frac{N - a^7 \times a}{3N + 4a^7}$ (= $\frac{4,095,331,177,840,421,839,748,921,429,536,000}{240,902,097,873,922,539,547,476,461,384,019}$ =,
 nearly, $\frac{4,095,331}{240,902}$) = 16.99998. Therefore $a + \frac{N - a^7 \times a}{3N + 4a^7}$
 will be (= 32000 + 16.99998) = 32016.99998, and conse-
 quently the second near value of the seventh root of the pro-
 posed number 34,487,717,467,307,513,182,492,153,794,
 673 will be 32016.99998. Q. E. I.

This number is true in all the figures but the last, which
 ought to be a 9 instead of an 8, the true value of the seventh
 root of the said proposed number being 32016.999,999,999,
 999, &c, *ad infinitum*, or the whole number 32017.

Art. 20. The second, or irrational, expression, given in
 the solution of the foregoing Problem, for the second near
 value of the m th root of any proposed number N , is $a -$

$$\frac{a}{m-1} + \sqrt{\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}}; \text{ which, when } m \text{ is}$$

$$= 7,$$

$= 7$, (as is the case in the present example,) is $(= a -$

$$\frac{a}{7-1} + \sqrt{\frac{aa}{(7-1)^2} + \frac{2 \times N - a^7}{7 \times 7-1 \times a^3}} = a - \frac{a}{6} +$$

$$\sqrt{\frac{aa}{6^2} + \frac{2 \times N - a^7}{7 \times 6 \times a^3}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} + \frac{N - a^7}{7 \times 3 \times a^3}} = \frac{5a}{6}$$

$$+ \sqrt{\frac{aa}{36} + \frac{N - a^7}{21a^3}} = \frac{5a}{6} + \sqrt{\frac{21a^7}{21 \times 36a^3} + \frac{36N - 36a^7}{21 \times 36a^3}} = \frac{5a}{6} +$$

$$\sqrt{\frac{7a^7}{7 \times 36a^3} + \frac{12N - 12a^7}{7 \times 36a^3}} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{7 \times 36a^3}} = \frac{5a}{6} +$$

$$\sqrt{\frac{12N - 5a^7}{252a^3}}. \text{ This expression may be computed as follows.}$$

Since N is $= 34,487,717,467,307,513,182,492,153,794,673$, and a is $= 32000$, we shall have $5a (= 5 \times 32000)$

$= 160,000$, and $\frac{5a}{6} (= \frac{160,000}{6}) = 26,666.666,666, \&c$,

and $a^3 (= 32000^3) = 33,554,432,000,000,000,000,000$,

and $252a^3 (= 252 \times 33,554,432,000,000,000,000,000) =$

$8,455,716,864,000,000,000,000,000$, and $a^7 (= 32,000^7)$

$= 34,359,738,368,000,000,000,000,000,000,000,000$, and $5a^7$

$(= 5 \times 34,359,738,368,000,000,000,000,000,000,000,000) =$

$171,798,691,840,000,000,000,000,000,000,000,000$, and $12N$

$(= 12 \times 34,487,717,467,307,513,182,492,153,794,673)$

$= 413,852,609,607,690,158,189,905,845,536,076$, and

$12N - 5a^7 (= 413,852,609,607,690,158,189,905,845,536,076$

$- 171,798,691,840,000,000,000,000,000,000,000,000)$

$= 242,053,917,767,690,158,189,905,845,536,076$, and

$\frac{12N - 5a^7}{252a^3} (= \frac{242,053,917,767,690,158,189,905,845,536,076}{8,455,716,864,000,000,000,000,000}) =$

$28,626,067.033799$, and $\sqrt{\frac{12N - 5a^7}{252a^3}} (= \sqrt{28,626,067.$

$033799}) = 5350.333,35$. Therefore $\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^3}}$

will be $(= 26,666.666,66, \&c, + 5350.333,35) = 32017.$

$000,01$; and consequently $32017.000,01$ will be the second near

513,182,492,153,794,673, and a is = 32000, we shall have a^7 ($= 32000^7$) = 34,359,738,368,000,000,000,000,000,000,000, and $N - a^7$

$$= 34,487,717,467,307,513,182,492,153,794,673 \\ - 34,359,738,368,000,000,000,000,000,000,000 \\ = 127,979,099,307,513,182,492,153,794,673, \text{ and}$$

$$N - a^7 \times a \text{ (} = 127,979,099,307,513,182,492,153,794,673 \\ \times 32000 \text{)} = 4,095,331,177,840,421,839,748,921,429, \\ 536,000, \text{ and } 3N \text{ (} = 3 \times 34,487,717,467,307,513,182, \\ 492,153,794,673 \text{)} = 103,463,152,401,922,539,547,476, \\ 461,384,019, \text{ and } 4a^7$$

$$\text{(} = 4 \times 34,359,738,368,000,000,000,000,000,000,000 \text{)} \\ = 137,438,945,472,000,000,000,000,000,000,000, \\ \text{and } 3N + 4a^7$$

$$\text{(} = 103,463,152,401,922,539,547,476,461,384,019 \\ + 137,438,945,472,000,000,000,000,000,000,000 \text{)} \\ = 240,902,097,873,922,539,547,476,461,384,019, \text{ and}$$

$$\frac{N - a^7 \times a}{3N + 4a^7} \text{ (} = \frac{4,095,331,177,840,421,839,748,921,429,436,000}{240,902,097,873,922,539,547,476,461,384,019} \text{)} = ,$$

$$\text{nearly, } \frac{4,095,331}{240,902} \text{)} = 16.99998. \text{ Therefore } a + \frac{N - a^7 \times a}{3N + 4a^7}$$

will be ($= 32000 + 16.9998$) $= 32016.9998$, and consequently the second near value of the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673 will be 32016.9998. Q. E. I.

This number is true in all the figures but the last, which ought to be a 9 instead of an 8, the true value of the seventh root of the said proposed number being 32016.999,999,999, 999, &c, *ad infinitum*, or the whole number 32017.

Art. 20. The second, or irrational, expression, given in the solution of the foregoing Problem, for the second near value of the m th root of any proposed number N , is $a -$

$$\frac{a}{m-1} + \sqrt{\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}}; \text{ which, when } m \text{ is}$$

$$= 7,$$

= 7, (as is the case in the present example,) is ($= a -$

$$\frac{a}{7-1} + \sqrt{\frac{aa}{(7-1)^2} + \frac{2 \times N - a^7}{7 \times 7-1 \times a^5}} = a - \frac{a}{6} +$$

$$\sqrt{\frac{aa}{6^2} + \frac{2 \times N - a^7}{7 \times 6 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} + \frac{N - a^7}{7 \times 3 \times a^5}} = \frac{5a}{6}$$

$$+ \sqrt{\frac{aa}{36} + \frac{N - a^7}{21a^5}} = \frac{5a}{6} + \sqrt{\frac{21a^7}{21 \times 36a^5} + \frac{36N - 36a^7}{21 \times 36a^5}} = \frac{5a}{6} +$$

$$\sqrt{\frac{7a^7}{7 \times 36a^5} + \frac{12N - 12a^7}{7 \times 36a^5}} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{7 \times 36a^5}} = \frac{5a}{6} +$$

$$\sqrt{\frac{12N - 5a^7}{252a^5}}. \text{ This expression may be computed as follows.}$$

Since N is = 34,487,717,467,307,513,182,492,153,794,673, and a is = 32000, we shall have $5a$ ($= 5 \times 32000$) = 160,000, and $\frac{5a}{6}$ ($= \frac{160,000}{6}$) = 26,666.666,666, &c,

and a^5 ($= \sqrt[5]{32000^5}$) = 33,554,432,000,000,000,000,000, and $252a^5$ ($= 252 \times 33,554,432,000,000,000,000,000$) = 8,455,716,864,000,000,000,000,000, and a^7 ($= \sqrt[7]{32,000^7}$) = 34,359,738,368,000,000,000,000,000,000,000, and $5a^7$ ($= 5 \times 34,359,738,368,000,000,000,000,000,000,000$) = 171,798,691,840,000,000,000,000,000,000,000, and $12N$ ($= 12 \times 34,487,717,467,307,513,182,492,153,794,673$) = 413,852,609,607,690,158,189,905,845,536,076, and $12N - 5a^7$ ($= 413,852,609,607,690,158,189,905,845,536,076 - 171,798,691,840,000,000,000,000,000,000,000$) = 242,053,917,767,690,158,189,905,845,536,076, and $\frac{12N - 5a^7}{252a^5}$ ($= \frac{242,053,917,767,690,158,189,905,845,536,076}{8,455,716,864,000,000,000,000,000,000}$) =

28,626,067.033799, and $\sqrt{\frac{12N - 5a^7}{252a^5}}$ ($= \sqrt{28,626,067.033799}$) = 5350.333,35.

Therefore $\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$ will be ($= 26,666.666,66, \&c. + 5350.333,35$) = 32017.000,01; and consequently 32017.000,01 will be the second near

near value of the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.000,01 is exact in the first nine figures, 32017.0000, and errs only in the 10th figure, 1, which ought to be a cypher, 0, instead of a 1, because the true value of this seventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 21. If we seek the value of this seventh root by Mr. Raphson's expression $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, the computation will be as follows.

Since a is = 32,000, we shall have a^6 ($= \overline{32,000^6}$) = 1,073,741,824,000,000,000,000,000,000, and $7a^6$ ($= 7 \times 1,073,741,824,000,000,000,000,000,000$) = 7,516,192,768,000,000,000,000,000,000, and a^7 ($= \overline{32,000^7}$) = 34,359,738,368,000,000,000,000,000,000,000, and $N - a^7$ ($= 34,487,717,467,307,513,182,492,153,794,673 - 34,359,738,368,000,000,000,000,000,000,000$) = 127,979,099,307,513,182,492,153,794,673, and $\frac{N - a^7}{7a^6}$ ($= \frac{127,979,099,307,513,182,492,153,794,673}{7,516,192,768,000,000,000,000,000,000}$) = 17.02.

Therefore $a + \frac{N - a^7}{7a^6}$ is ($= 32000 + 17.02$) = 32017.02; and consequently 32017.02 will be nearly equal to the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.02 is exact in the six first figures 32017.0, and errs only in the seventh figure 2, which ought to be a cypher instead of a 2, because the true value of this seventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 22. This expression $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, is

so

so much simpler than either of Mr. *de Lagny's* expressions above-mentioned, and so much less difficult to be computed, that I am inclined to agree with Mr. *Raphson* in thinking it, upon the whole, preferable to them. But, perhaps, when a , or the first value of the root sought, consists of only one figure, it may sometimes be adviseable to make use of one of Mr. *de Lagny's* expressions, in order to obtain a second near value of the root sought, and then to make use of Mr. *Raphson's* expression in order to obtain a third near value of it.

Art. 23. These four examples are, I presume, sufficient to illustrate *Monsieur de Lagny's* method of extracting the m th root of any proposed number denoted by the letter N , by

means of either of the two expressions $a + \frac{N - a^m}{m-1 \times N + m+1 \times a^m} \times 2a$

and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{m-1} + \frac{N - a^m}{m \times \frac{m-1}{2} \times a^{m-2}}}$, or $a -$

$\frac{a}{m-1} + \sqrt{\frac{aa}{m-1} + \frac{2 \times N - a^m}{m \times m-1 \times a^{m-2}}}$, when a , or the

first near value of $\sqrt[m]{N}$, which is supposed to be already known, is less than its true value; which is the case supposed in the foregoing Problem. I shall therefore now proceed to consider the other case, in which a , or the first near value of the m th root of the proposed number N , is greater than its true value, and to investigate similar expressions for a second near value of the said root that shall approach nearer than a to its true value. This may be done by a solution of the following Problem,

PROBLEM

P R O B L E M II.

Art. 24. Let N be any proposed number whatsoever, and m any proposed whole number whatsoever; and let a be a known number that is nearly equal to, but somewhat greater than, the m th root of the given number N . It is required to find a second near value of the said m th root of the given number N , that shall approach much nearer to it than a , or the former near value of it that is already known.

S O L U T I O N.

Let z be put for the unknown difference between a , the first near value of the m th root of the given number N , and the true value of the said root. Then, since a is supposed to be greater than $\sqrt[m]{N}$, and to exceed it by the difference z , it follows that $a - z$ will be $= \sqrt[m]{N}$, and consequently that $(a - z)^m$ will be $= N$.

But, by Sir Isaac Newton's residual theorem in the first and simplest case of it, to wit, the case of integral powers, $(a - z)^m$ will be $=$ the series $a^m - m a^{m-1} z + m \times \frac{m-1}{2}$

$$\times a^{m-2} z^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} z^3 + m \times \frac{m-1}{2}$$

$$\times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} z^4 - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$$

$$\times \frac{m-4}{5} \times a^{m-5} z^5 + \&c, \text{ continued to } m + 1 \text{ terms;}$$

or,

or, if, for the sake of brevity, we substitute the capital letters A, B, C, D, E, F, &c, instead of the several numeral coefficients 1, m , $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, respectively, $a - z$ will be = the series $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, continued to $m + 1$ terms. Therefore the said series $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, continued to $m + 1$ terms, will be = N. This is the original equation, by the resolution of which we are to find a near value of z , and consequently of $a - z$, or a second near value of $\sqrt[m]{N}$, which will approach nearer to it than a , or its former near value.

Art. 25. Now, since z is less, and usually much less, than a , being about a 10th, or a 100th, part of it, or some still lesser part of it, it is evident that all the terms in the afore-said series that involve zz , and z^3 , and z^4 , and the following powers of z , will be less, and usually much less, than the term $B a^{m-1} z$, which involves only the simple power of z . And therefore, if all the said terms of the series be neglected or omitted, and the two first terms alone, to wit, the terms $A a^m - B a^{m-1} z$, be retained, the said two terms alone will be nearly equal to the whole series, and consequently to the given number N; and therefore, if we add $B a^{m-1} z$ to both sides, we shall have $A a^m$, nearly, = $N + B a^{m-1} z$, and (subtracting N from both sides,) $A a^m - N$, nearly, = $B a^{m-1} z$, or $B a^{m-1} z$, nearly, = $A a^m - N$, and (dividing both sides by $B a^{m-1}$), we shall have

$3 Z :$ $z,$

z , nearly, $= \frac{A a^m - N}{B a^{m-1}}$, or (because A is $= 1$, and B is $= m$,) we shall have z , nearly, $= \frac{a^m - N}{m a^{m-1}}$; which fraction

may be derived from the known quantities N and a , by the common arithmetical operations of Multiplication, Subtraction, and Division. This therefore is an approximation to

the true value of z , and consequently $a - \sqrt[m]{\frac{a^m - N}{m a^{m-1}}}$ will

be an approximation to the true value of $a - z$, or of $\sqrt[m]{N}$, or will be a second near value of it that will approach nearer to it than a , or the first near value of it which was already known. And it will be still somewhat greater (as the former value a was,) than the true value of $\sqrt[m]{N}$; as may be demonstrated in the manner following.

The whole series $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, is $= N$. Therefore (adding $B a^{m-1} z$ to both sides,) we shall have the series $A a^m + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, $= N + B a^{m-1} z$, and (subtracting N from both sides,) we shall have the series $A a^m - N + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, $= B a^{m-1} z$. But, because $C a^{m-2} z^2$ is greater than $D a^{m-3} z^3$, and $E a^{m-4} z^4$ is greater than $F a^{m-5} z^5$, and, in like manner, every following term in the said series that is marked with the sign $+$, is greater than the term immediately following it, which is marked with the sign $-$, it follows that the series $A a^m - N + C a^{m-2} z^2 - D a^{m-3} z^3 +$
 $+$

+ $E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, will be greater than its two first terms $A a^m - N$. Therefore $B a^{m-1} z$ (which is equal to the said series,) will be greater than $A a^m - N$.

Therefore z will be greater than $\frac{A a^m - N}{B a^{m-1}}$, and conse-

quently $a - z$, or $\sqrt[m]{N}$, will be less than $a - \sqrt[m]{\frac{A a^m - N}{B a^{m-1}}}$,

or than $a - \sqrt[m]{\frac{a^m - N}{m a^{m-1}}}$, and therefore $a - \sqrt[m]{\frac{a^m - N}{m a^{m-1}}}$ will be greater than $a - z$; or the true value of the m th root of the proposed number N . Q. E. D.

This quantity, $a - \sqrt[m]{\frac{a^m - N}{m a^{m-1}}}$, is the expression given by

Mr. Raphson for the second near value of the m th root of the given number N . And it is a very useful approximation: for it usually gives us twice as many figures of the true value of $\sqrt[m]{N}$ exact as were exact in a , or the first near value of the said m th root. And it is evidently the most simple and easy approximation to the value of the said m th root that can well be imagined.

Art. 26. But Mr. de Lagny, being desirous of finding at once a still nearer value of the m th root of the number N , retains the third term $C a^{m-2} z^2$, as well as the two first terms $A a^m - B a^{m-1} z$, of the series $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 + \&c$, (which is equal to N), and thereby converts the original equation $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 +$

3 Z 2
E a

E $a^{m-4} z^4 - F a^{m-5} z^5 + \&c.$, = N into a quadratrick equation, to wit, the equation $A a^m - B a^{m-1} z + C a^{m-2} z^2 = N$, instead of converting it (as Mr. Raphson does,) into the simple equation $A a^m - B a^{m-1} z = N$. And this quadratrick equation he resolves in two different ways, to wit, first, imperfectly, or inaccurately, and then accurately. By the former, or inaccurate, resolution of it, he obtains a rational expression for the value of z , and consequently another rational expression for the value of $a - z$, or for a second near value of $\sqrt[m]{N}$, which is nearer to its true value than a , or the former near value of it, was; and by the accurate resolution of the same quadratrick equation he obtains a furd, or irrational, expression for the value of z , and consequently another furd, or irrational, expression for the value of $a - z$, or for the second near value of $\sqrt[m]{N}$, that approaches much nearer to its true value than its former near value, a , did. These resolutions of the said quadratrick equation $A a^m - B a^{m-1} z + C a^{m-2} z^2 = N$, may be performed in the following manner.

Art. 27. Since $A a^m - B a^{m-1} z + C a^{m-2} z^2 = N$, we shall have $A a^m + C a^{m-2} z^2 = N + B a^{m-1} z$, and $A a^m = N + B a^{m-1} z - C a^{m-2} z^2$, and $A a^m - N = B a^{m-1} z - C a^{m-2} z^2$, or $B a^{m-1} z - C a^{m-2} z^2 = A a^m - N$; that is, $z \times B a^{m-1} - z \times C a^{m-2} z$ will be $= A a^m - N$, or $z \times \sqrt{B a^{m-1} - C a^{m-2} z}$ will be $= A a^m - N$. Therefore (dividing both sides of the equation by the compound quantity $B a^{m-1} - C a^{m-2} z$), we shall have $z = \frac{A a^m - N}{B a^{m-1} - C a^{m-2} z}$.

Now

Now let $\frac{Aa^m - N}{Ba^{m-1}}$, or $\frac{a^m - N}{ma^{m-1}}$, (which has already been shewn to be nearly equal to z ,) be substituted instead of z in the second term, $Ca^{m-2}z$, of the denominator of the fraction last obtained, to wit, the fraction $\frac{Aa^m - N}{Ba^{m-1} - Ca^{m-2}z}$.

And we shall have $z = \frac{Aa^m - N}{Ba^{m-1} - Ca^{m-2} \times \sqrt{\frac{a^m - N}{ma^{m-1}}}}$, or

(because A is $= 1$, and B is $= m$,)

$$z = \frac{a^m - N}{ma^{m-1} - ca^{m-2} \times \sqrt{\frac{a^m - N}{ma^{m-1}}}}; \text{ which is}$$

$$= \frac{a^m - N}{\frac{m^2 a^{2m-2} - ca^{2m-2} \times a^m - N}{ma^{m-1}}}$$

$$= \frac{a^m - N}{\frac{m^2 a^{2m-2} - ca^{2m-2} + cNa^{m-2}}{ma^{m-1}}}$$

$$= a^m - N \times \frac{ma^{m-1}}{m^2 a^{2m-2} - ca^{2m-2} + cNa^{m-2}}$$

$$= a^m - N \times \frac{2aa \times ma^{m-1}}{2m^2 a^{2m} - 2ca^{2m} + 2cNa^m}$$

$$= a^m - N \times \frac{2a \times ma^m}{2m^2 a^{2m} - 2ca^{2m} + 2cNa^m}$$

$$= a^m - N \times \frac{2ma}{2m^2 a^m - 2ca^m + 2cN}$$

=

$$\begin{aligned}
&= \sqrt[m]{a^m - N} \times \frac{2ma}{2m^2a^m - 2 \times m \times \frac{m-1}{2} \times a^m + 2 \times m \times \frac{m-1}{2} \times N} \\
&= \sqrt[m]{a^m - N} \times \frac{2ma}{2m^2a^m - m \times \overline{m-1} \times a^m + m \times \overline{m-1} \times N} \\
&= \sqrt[m]{a^m - N} \times \frac{2a}{2ma^m - \overline{m-1} \times a^m + \overline{m-1} \times N} \\
&= \sqrt[m]{a^m - N} \times \frac{2a}{2ma^m - ma^m + a^m + \overline{m-1} \times N} \\
&= \sqrt[m]{a^m - N} \times \frac{2a}{ma^m + a^m + \overline{m-1} \times N} \\
&= \sqrt[m]{a^m - N} \times \frac{2a}{\overline{m+1} \times a^m + \overline{m-1} \times N} \\
&= \frac{\sqrt[m]{a^m - N} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}. \text{ Therefore } z \text{ will be } = \\
&\frac{\sqrt[m]{a^m - N} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}, \text{ or } \frac{2a \times \sqrt[m]{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}; \text{ and} \\
&\text{consequently } a - z \text{ will be } = \\
&a - \frac{2a \times \sqrt[m]{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}, \text{ or } a - \frac{2a \times \sqrt[m]{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}
\end{aligned}$$

will be a second near value of $\sqrt[m]{N}$, or of the m th root of the proposed number N . Q. E. I.

Art. 28. The accurate resolution of the quadratick equation $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 = N$, may be performed as follows.

Since $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2$ is $= N$, we shall have $Aa^m + Ca^{m-2}z^2 = N + Ba^{m-1}z$, and $Aa^m = N + Ba^{m-1}z - Ca^{m-2}z^2$, and $Aa^m - N$

$= B a^{m-1} z - C a^{m-2} z^2$, or $B a^{m-1} z - C a^{m-2} z^2 = A a^m - N$, or (because A is $= 1$, and B is $= m$, and C is $= m \times \frac{m-1}{2}$) $ma^{m-1} z - m \times \frac{m-1}{2} a^{m-2} z^2 = a^m - N$, and (multiplying both sides by 2,) $2ma^{m-1} z - m \times m-1 \times a^{m-2} z^2 = 2 \times a^m - N$, and, (dividing both sides of the equation by $m \times m-1 \times a^{m-2}$),

$$\frac{2ma^{m-1} \times z}{m \times m-1 \times a^{m-2}} - zz = \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}, \text{ or}$$

$$\frac{2a^{m-1} \times z}{m-1 \times a^{m-2}} - zz = \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}, \text{ or (because}$$

$$\frac{a^{m-1}}{a^{m-2}} \text{ is } = a,) \frac{2a}{m-1} \times z - zz = \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}.$$

Now $\frac{2a}{m-1} \times z - zz$ is $= z \times \left[\frac{2a}{m-1} - z \right]$. And, by Euclid's Elements, Book II, Prop. 5, the rectangle or product, under z and $\frac{2a}{m-1} - z$, must be less than the square of half the line $\frac{2a}{m-1}$. Therefore the compound quantity

$\frac{2az}{m-1} - zz$ will be less than the square of half $\frac{2a}{m-1}$, or than the square of $\frac{a}{m-1}$, or than $\frac{aa}{(m-1)^2}$. Therefore

$\frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}$, (which is equal to the compound quantity $\frac{2az}{m-1} - zz$), will also be less than $\frac{aa}{(m-1)^2}$. Therefore

both these quantities $\frac{2az}{m-1} - zz$ and $\frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}$ may
be

be subtracted from $\frac{aa}{m-1}^2$. Let them be so subtracted. And then we shall have $\frac{aa}{m-1}^2 - \frac{2az}{m-1} + zz = \frac{aa}{m-1}^2 - \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}$. Therefore the square-root of the trinomial quantity $\frac{aa}{m-1}^2 - \frac{2az}{m-1} + zz$ will be equal to the square root of the compound quantity $\frac{aa}{m-1}^2 - \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}$; or, if, for the sake of brevity, we denote the said compound quantity by the capital letter P, the square-root of the trinomial quantity $\frac{aa}{m-1}^2 - \frac{2az}{m-1} + zz$ will be $= \sqrt{P}$. But, whenever z is less than $\frac{a}{m-1}$, (as is commonly the case in these extractions of the roots of numbers,) $\frac{a}{m-1} - z$ will be the square-root of the trinomial quantity $\frac{aa}{m-1}^2 - \frac{2az}{m-1} + zz$. Therefore $\frac{a}{m-1} - z$ will $= \sqrt{P}$, and consequently $\frac{a}{m-1}$ will be $= \sqrt{P} + z$, and z will be $= \frac{a}{m-1} - \sqrt{P}$. Therefore $a - z$ will be $(= a - \sqrt{\frac{a}{m-1} - \sqrt{P}}) = a - \frac{a}{m-1} + \sqrt{P}$; and consequently $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{m-1}^2 - \frac{2 \times a^m - N}{m \times m-1 \times a^{m-2}}}$ will be a second near value of $\sqrt[m]{N}$, or of the m th root of the given number N.

Q. E. I.

Art. 29. When m is $= 3$, the former of these two expressions,

preffions, to wit, the rational expreffion $a -$

$$\frac{2a \times a^m - N}{(m-1) \times N + (m+1) \times a^m}, \text{ will be } (= a - \frac{2a \times \overline{a^3 - N}}{2N + 4a^3}) = a - \frac{a \times \overline{a^3 - N}}{N + 2a^3};$$

and the latter, or irrational, expreffion, $a -$

$$\frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} - \frac{2 \times \overline{a^m - N}}{m \times m-1 \times a^{m-2}}} \text{ will be } (= a -$$

$$\frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{2 \times \overline{a^3 - N}}{3 \times 2 \times a}} = \frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{(a^3 - N)}{3a}} = \frac{a}{2}$$

$$+ \sqrt{\frac{3a^3}{12a} - \frac{(4a^3 - 4N)}{12a}} = \frac{a}{2} + \sqrt{\frac{3a^3 - 4a^3 + 4N}{12a}} = \frac{a}{2} +$$

$$\sqrt{\frac{4N - a^3}{12a}}. \text{ Therefore the two expreffions for the second}$$

near value of the cube-root of a given number N , when a , or the former near value of it, is greater than its true value,

are $a - \frac{a \times \overline{a^3 - N}}{N + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4N - a^3}{12a}}.$

Art. 30. And, when m is $= 5$, the former of the two foregoing general expreffions, to wit, the rational expreffion

$$a - \frac{2a \times \overline{a^m - N}}{(m-1) \times N + (m+1) \times a^m}, \text{ will be } (= a - \frac{2a \times \overline{a^5 - N}}{4N + 6a^5})$$

$$= a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5};$$

and the latter, or irrational, expreffion,

$$a - \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} - \frac{2 \times \overline{a^m - N}}{m \times m-1 \times a^{m-2}}} \text{ will be } (=$$

$$a - \frac{a}{4} + \sqrt{\frac{aa}{16} - \frac{2 \times \overline{a^5 - N}}{5 \times 4 \times a^3}} = \frac{3a}{4} + \sqrt{\frac{aa}{16} - \frac{a^5 - N}{10a^3}}$$

$$= \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} - \frac{16a^5 - 16N}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5 - (16a^5 - 16N)}{160a^3}}$$

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$$= \frac{3a}{4} + \sqrt{\frac{10a^5 - 16a^3 + 16N}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{16N - 6a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}.$$

Therefore the two expressions for the second near value of the fifth root of a given number N , when a , or the former near value of the said root, is greater than its true value, are $a - \frac{a \times a^5 - N}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}.$

Art. 31. And, when m is $= 7$, the former of the two foregoing general expressions, to wit, the rational expression

$$a - \frac{2a \times a^m - N}{(m-1) \times N + (m+1) \times a^m}, \text{ will be } (= a - \frac{2a \times a^7 - N}{6N + 8a^7})$$

$$= a - \frac{a \times a^7 - N}{3N + 4a^7}; \text{ and the latter, or irrational, expression,}$$

$$a - \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} - \frac{2 \times a^m - N}{m \times (m-1) \times a^{m-2}}} \text{ will be}$$

$$(= a - \frac{a}{6} + \sqrt{\frac{aa}{36} - \frac{2 \times a^7 - N}{7 \times 6 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} - \frac{a^7 - N}{21a^5}})$$

$$= \frac{5a}{6} + \sqrt{\frac{21a^7}{36 \times 21a^5} - \frac{36a^7 - 36N}{36 \times 21a^5}} = \frac{5a}{6} + \sqrt{\frac{21a^7 - 36a^7 + 36N}{36 \times 21a^5}}$$

$$= \frac{5a}{6} + \sqrt{\frac{21a^7 - 36a^7 + 36N}{36 \times 21a^5}} = \frac{5a}{6} + \sqrt{\frac{36N - 15a^7}{36 \times 21a^5}} = \frac{5a}{6} +$$

$$\sqrt{\frac{12N - 5a^7}{36 \times 7a^5}} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}.$$

Therefore the two expressions for the second near value of the 7th root of a given number N , when a , or the former near value of the said

root, is greater than its true value, are $a - \frac{a \times a^7 - N}{3N + 4a^7}$ and

$$\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}},$$

Examples

Examples of the Extraction of the Roots of given Numbers by

means of the Two General Expressions $a - \frac{2a \times a^m - N}{(m-1) \times N + (m+1) \times a^m}$

and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} - \frac{2 \times a^m - N}{m \times (m-1) \times a^{m-2}}}$, which have been found by the Solution of Problem II, when a , or the First near Value of $\sqrt[m]{N}$, is greater than its true Value.

EXAMPLE I.

Art. 32. Let it be required to find the fifth root of 2, which has been already investigated by means of the two expressions investigated in Problem I, and found to be = 1.148,697,34. And let us suppose that we have already discovered that this root is greater than 1.14, but less than 1.15, and differs less from 1.15 than from 1.14; so that 1.15 may be taken for a , or its first near value. Then, by art. 30, the two expressions of the second near value of this root will be $a - \frac{a \times a^5 - N}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, in which 2 must be substituted instead of N , and 1.15 instead of a .

Now, since N is = 2, and a is = 1.15, we shall have $2N$ ($= 2 \times 2$) = 4, and a^3 = 1.520,875, and a^5 = 2.011,357,187,5, and $a^5 - N$ ($= 2.011,357,187,5 - 2.000,000,000,0$) = 0.011,357,187,5, and $a \times a^5 - N$ ($= 1.15 \times 0.011,357,187,5$) = 0.013,060,765,625, and $3a^5$ ($= 3 \times 2.011,357,187,5$) = 6.034,071,562,5, and $2N + 3a^5$ ($= 4 + 6.034,071,562,5$) = 10.034,071,562,5, and $\frac{a \times a^5 - N}{2N + 3a^5}$ ($= \frac{0.013,060,765,625}{10.034,071,562,5}$) = 0.001,301,641, and

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consequently $a = \frac{a \times \sqrt[5]{a^5 - N}}{2N + 3a^5}$ ($= 1.150,000,000 - 0.001,301,641$) $= 1.148,698,359$. Therefore 1.148,698,359 will be a second near value of the fifth root of the given number 2. Q. E. I.

And, since a is $= 1.15$, and a^3 is $= 1.520,875$, and a^4 is $= 2.011,357,187,5$, and $3a^5$ is $= 6.034,071,562,5$, we shall have $80a^3$ ($= 80 \times 1.520,875$) $= 121.670,000$, and $8N - 3a^5$ ($= 8 \times 2 - 6.034,071,562,5 = 16.000,000,000,0 - 6.034,071,562,5$) $= 9.965,928,437,5$, and $\frac{8N - 3a^5}{80a^3}$ ($= \frac{9.965,928,437,5}{121.670,000}$) $= 0.081909,496486$, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ ($= \sqrt{0.081909,496486}$) $= 0.286,198,351$, and $3a$ ($= 3 \times 1.15$) $= 3.45$, and $\frac{3a}{4}$ ($= \frac{3.45}{4}$) $= 0.8625$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ ($= 0.8625 + 0.286,198,351$) $= 1.148,698,351$. Therefore 1.148,698,351 will be a second near value of the fifth root of the given number 2.

Q. E. I.

As these two approximations to the fifth root of 2, to wit, 1.148,698,359 and 1.148,698,351, agree with each other in the first nine figures 1.148,698,35, we may reasonably conclude that those nine figures are exact, or are the first nine figures of a more accurate value of the said fifth root.

Art. 33. If we make use of Mr. Raphson's expression,

to wit, $a = \frac{a^m - N}{m a^{m-1}}$, or $a = \frac{a^5 - N}{5a^4}$, for the purpose of

obtaining a second near value of the fifth root of 2, after 1.15 has been taken for a , or, its first near value, the computation of it will be as follows.

Since a is $= 1.15$, we shall have $a^4 = 1.749,006,25$, and $a^5 = 2.011,357,187,5$, and $5a^4$ ($= 5 \times 1.749,006,25$) $=$

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$= 8.745,031,25$, and $a^5 - N (= 2.011,357,187,5 - 2)$
 $= 0.011,357,187,5$, and $\frac{a^5 - N}{5a^4} (= \frac{0.011,357,187,5}{8.745,031,25}) =$
 $0.001,298,7$, and $a - \sqrt[5]{\frac{a^5 - N}{5a^4}} (= 1.150,000,0 - 0.001,$
 $298,7) = 1.148,701,3$. Therefore $1.148,701,3$ will be the
 second near value of the fifth root of 2, resulting from
 Mr. Raphson's expression $a - \sqrt[5]{\frac{a^5 - N}{5a^4}}$. Q. E. I.

This number $1.148,701,3$, is greater than the true value
 of the 5th root of 2, to wit, the number $1.148,698,3$, but
 exceeds it by only the very small quantity $0.000,003,3$.

EXAMPLE II.

Art. 34. Let it be required to find the 5th root of the
 number $2,327,834,559,873$.

Now this number, which consists of thirteen figures, is
 greater than $100,000$, or the fifth power of 10 ; and it is
 likewise greater than $10,000,000,000$, or the fifth power of
 100 : but it is less than $1,000,000,000,000,000$, or the fifth
 power of 1000 . Therefore its fifth root must be greater
 than 100 , but less than 1000 .

Further, this number, $2,327,834,559,873$, is greater than
 $2,320,000,000,000$, or than $232 \times 10,000,000,000$, or
 than $232 \times$ the fifth power of 100 . Therefore the fifth
 root of $2,327,834,559,873$ will be greater than $100 \times$ the
 fifth root of 232 . But the difference will not be great; and
 consequently, if we can find a number that shall be nearly
 equal to the fifth root of 232 , we need only multiply the
 said number into 100 , and the product will be nearly equal
 to the fifth root of $2,320,000,000,000$, and therefore will
 likewise,

likewise be pretty nearly equal to the fifth root of the proposed number 2,327,834,559,873, so as to be a convenient first near value of the said fifth root, and a proper basis to found a further approximation upon to a second near value of the said fifth root, by either of the two foregoing expressions of Mr. de Lagny, which have been investigated above in the Solution of Problem II, or by Mr. Raphson's expression. We will therefore endeavour to find the fifth root of the number 232.

Art. 35. Now the fifth power of the number 2 is 32, which is much less than 232; and the fifth power of 3 is 243, which is a little greater than 232. Therefore the fifth root of 232 must be much greater than 2, and a little less than 3. We may therefore reasonably conjecture that it will be nearly equal to $2 + \frac{9}{10}$, or 2.9. We will therefore suppose it to be $\approx$ 2.9, and try what the result of that supposition will be.

Now the fifth power of 2.9 is 205.11149, which is less than 232. Therefore the fifth root of 232 will be greater than 2.9. And, as 232 differs much less from 243, or the fifth power of 3, than from 205.11149, or the fifth power of 2.9, we may reasonably suppose that the fifth root of 232 will differ much less from 3 than from 2.9; and therefore we will suppose that it is nearly $\approx$ 2.98, and will raise the said number 2.98 to its fifth power, in order to examine the truth of the said supposition.

Now $\overline{2.98^5}$ is $\approx$ 235.007,282,396,8; which is nearly equal to, but a little greater than, 232. Therefore 2.98 must be nearly equal to, but a little greater than, the fifth root of 232; and consequently 100×2.98 must be nearly equal to, but a little greater than, the fifth root of $232 \times 10,000,000,000$; or 298 must be nearly equal to, but a little greater than, the fifth root of 2,320,000,000,000. And, further, since 2.98 is the fifth root of the number 235.007,282,396,8, the number 100×2.98 , will be the fifth root of the number $235.007,282,396,8 \times 10,000,000,000$;

000; that is, the number 298 will be the fifth root of the number 2,350,072,823,968, which is greater than the proposed number 2,327,834,559,873. Therefore 298 will be greater than the fifth root of the said number 2,327,834,559,873. But it will be near enough to it to make it a very convenient basis of a further approximation to the true value of the fifth root of the said number, 2,327,834,559,873, by means of the two

$$\text{expressions } a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}, \text{ and } \frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}.$$

Art. 36. Here then we shall have $N = 2,327,834,559,873$, and $a = 298$. Therefore a^2 will be $= 26,463,592$, and a^4 will be $= 7,886,150,416$, and a^5 will be $= 2,350,072,823,968$; and consequently $a^5 - N$ will be $(= 2,350,072,823,968 - 2,327,834,559,873) = 22,238,264,095$, and $a^5 - N \times a$ will be $(= 22,238,264,095 \times 298) = 6,627,002,700,310$, and $2N$ will be $(= 2 \times 2,327,834,559,873) = 4,655,669,119,746$, and $3a^5$ will be $(= 3 \times 2,350,072,823,968) = 7,050,218,471,904$, and $2N + 3a^5$ will be $(= 4,655,669,119,746 + 7,050,218,471,904) = 11,705,887,591,650$, and $\frac{a^5 - N \times a}{2N + 3a^5}$ will be $(=$

$$\frac{6,627,002,700,310}{11,705,887,591,650}) = 0.566,125,605, \text{ and consequently}$$

$a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ will be $(= 298.000,000,000 - 0.566,125,605) = 297.433,874,395$. Therefore 297.433,874,395 will be a second near value of the fifth root of the proposed number 2,327,834,559,873. Q. E. I.

And $3a$ will be $(= 3 \times 298) = 894$, and $\frac{3a}{4}$ will be $(= \frac{894}{4}) = 223.5$, and a^3 will be $(= \overline{298^3}) = 26,463,592$, and $8N$ will be $(= 8 \times 2,327,834,559,873) = 18,622,676,478,984$, and a^5 will be $(= \overline{298^5}) = 2,350,072,823,968$, and $3a^5$ will be $(= 3 \times 2,350,072,823,968) = 7,050,218,471,904$, and $8N - 3a^5$ will be $(= 18,622,676,$

676,498,984 - 7,050,218,471,904 = 11,572,458,007,080,
 and $80a^3$ will be ($= 80 \times 26,463,502$) = 2,117,087,360,
 and $\frac{8N - 3a^3}{80a^3}$ will be ($= \frac{11,572,458,007,080}{2,117,087,360}$) = 5466.216,
 569,863,2, and $\sqrt{\frac{8N - 3a^3}{80a^3}}$ will be ($= \sqrt{5466.216569,$
 8632) = 72.933,866,190, and consequently $\frac{3a}{4} +$
 $\sqrt{\frac{8N - 3a^3}{80a^3}}$ will be ($= 223.5 + 72.933,866,190$) =
 297.433,866,190. Therefore 297.433,866,190 will be a
 second near value of the fifth root of the proposed number
 2,327,834,559,873. Q. E. I.

These two numbers 297.433,874,395, and 297.433,866,
 190, agree with each other in the first seven figures 297.
 433,8. Therefore we may conclude that these seven figures
 are exact, or are the first seven figures of a more accurate
 value of the fifth root of the proposed number 2,327,834,
 559,873.

Art. 37. If we make use of Mr. Raphson's expression,
 to wit, $a - \frac{a^m - N}{ma^{m-1}}$, or $a - \frac{a^5 - N}{5a^4}$, for the purpose of
 obtaining a second near value of the fifth root of the pro-
 posed number 2,327,834,559,873, after 298 has been taken
 for a , or its first near value, which is somewhat greater than
 the truth, the computation of it will be as follows.

Since a is = 298, we shall have $a^4 = 7,886,150,416$,
 and $a^5 = 2,350,072,823,968$, and $a^5 - N$ ($= 2,350,072,$
 823,968 - 2,327,834,559,873) = 22,238,264,095, and
 $5a^4$ ($= 5 \times 7,886,150,416$) = 39,430,752,080, and
 $\frac{a^5 - N}{5a^4}$ ($= \frac{22,238,264,095}{39,430,752,080}$) = 0.563,982, and $a - \frac{a^5 - N}{5a^4}$
 ($= 298.000,000 - 0.563,982$) = 297.436,018. There-
 fore 297.436,018 will be a second near value of the fifth
 root of the proposed number 2,327,834,559,873. Q. E. I.

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The first five figures, 297.43, of this number, 297.436,018, obtained by Mr. Raphson's expression, are exact.

And, if we make $a = 297.436$, and repeat the application of Mr. Raphson's expression, we shall obtain the value of the fifth root of the said proposed number 2,327,834,559,873, to a much greater degree of exactness. This may be done in the manner following.

If a is taken $= 297.436$, we shall have

$$a^4 = 7,826,617,827.880,165,417,216, \text{ and}$$

$$a^5 = 2,327,917,900,253.364,881,035,058,176, \text{ and}$$

$$a^5 - N (= 2,327,917,900,253.364,881,035,058,176$$

$$- 2,327,834,559,873.000,000,000,000,000)$$

$$= 83,340,380.364,881,035,058,176,$$

$$\text{and } 5a^4 (= 5 \times 7,826,617,827.880,165,417,216) =$$

$$39,133,089,139.400,827,086,080, \text{ and consequently } \frac{a^5 - N}{5a^4}$$

$$(\frac{83,340,380.364,881,035,058,176}{39,133,089,139.400,827,086,080}) = 0.002,129,662, \text{ and}$$

$$a - \frac{a^5 - N}{5a^4} (= 297.436,000,000 - 0.002,129,662) =$$

297 433,870,338. Therefore 297.433,870,338 will be the more accurate value of the fifth root of the proposed number 2,327,834,559,873. Q. E. I.

Of this number, 297.433,870,338, which we have now found for the last near value of the fifth root of 2,327,834,559,873, I believe the first ten figures 297.453,870,3 to be exact, if no mistakes have been made in the calculation. Yet Mr *Raphson* (from whose *Analysis Aequationum Universalis*, Problem IV, page 12, this example is taken,) makes this fifth root equal to 297.433,874,895. But I believe the four last figures, 4895, of this number to be erroneous; because in Mr. *Raphson*'s last process, (of which this number, 297.433,874,895, is the result,) the value of g (which answers to a in our notation,) was taken equal only to 297.46, which is exact only in the first four figures 297.4, whereas in the last process of the foregoing computation we

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took a equal to 297.436, which is exact in the first five figures, 297.43; and consequently the number resulting from this supposition ought to be more exact than that which results from the other less accurate supposition made by Mr. Raphson. But we may, at least, conclude that the first eight figures, 297.433,87, of those two numbers, which are the same in both, are exact, or are the first eight figures of a number approaching more nearly than either of them to the true value of the fifth root of the proposed number 2,327,834,559,873.

Art. 38. These two examples will, I presume, be sufficient to illustrate *Monsieur de Lagny's* method of extracting the m th root of any proposed number N , by means of

either of the two expressions, $a - \frac{2a \times a^m - N}{(m-1) \times N + (m+1) \times a^m}$

and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{(m-1)^2} - \frac{2 \times a^m - N}{m \times (m-1) \times a^{m-2}}}$, when a ,

or the first near value of $\sqrt[m]{N}$, which is supposed to be already known, is greater than its true value; which is the case supposed above in Problem II, by the solution of which those expressions were obtained. I have therefore nothing more to add concerning the explanation of *Monsieur de Lagny's* method aforesaid. But I will just make another observation, or two, concerning the said method of extracting the roots of numbers, as compared with other methods of performing the same thing.

Observations

*Observations on the several different Methods that may be taken
for Extracting the Roots of Numbers.*

Art. 39. In the 1st place, then, it is manifest that either of Mr. *de Lagny's* two expressions, the rational one and the irrational one, for obtaining a second near value of the root of a given number, when a former near value of it is already known, is greatly to be preferred to *the common*, or, rather, *the old*, method of extracting such root, by which, with a great deal of trouble, we obtain only one new figure of the root sought by every new process; except, perhaps, in extracting the square-root of a number, which is easy enough in the common way, (at least for the first three or four figures of the root sought,) to make it unnecessary to have recourse to other methods. But in extracting the cube-root, or the fifth root, or the seventh root, or any higher root, of a proposed number, the case is very different, and it will be found highly expedient to have recourse either to Mr. *de Lagny's* method of extracting them, or to Mr. *Raphson's*, or to some other method of performing the said extraction,

Secondly, if the m th root of any number is to be found only to four, or five, figures, it will be most adviseable to have recourse to a Table of Logarithms for this purpose. For, by the use of such a Table, we may always obtain any proposed root of a given number exact to four, or five, places of figures, with very great ease, and without making use of the proportional parts set down in those tables, and which are necessary to the obtaining the said roots exactly to six, or seven, or more, places of figures. Whenever therefore we want to find the m th root of a proposed number only to four, or five, places of figures, it seems best to have recourse at once to a Table of Logarithms for that purpose.

But, 3dly, if we wish to obtain the m th root of any number, exactly to nine, or ten, or more, places of figures, it

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will

will be convenient to have recourse to either Mr. *de Lagny's* or Mr. *Raphson's* methods of approximation for that purpose. And, if we wish to obtain the said root exact only to nine places of figures, I should think it would be expedient to make use of Mr. *Raphson's* expression for that purpose, in preference to either of Mr. *de Lagny's* expressions, as being simpler and easier to compute than the latter; but, if we wish to obtain the said root exact to fourteen, or fifteen, places of figures, I should think it would be most advisable to have recourse to one of Mr. *de Lagny's* expressions for that purpose, rather than to make use of Mr. *Raphson's* expression, and repeat the process a second time, as was done above in art. 37. And of Mr. *de Lagny's* two expressions, the latter, or irrational, expression will be found less troublesome to compute, and, usually, in a small degree more exact, than the rational expression. But it may often be prudent to compute them both, to be checks upon each other; and the number of figures in which the results of both expressions are found to agree, may be justly concluded to be exact.

And, 4thly, when we make use of either Mr. *Raphson's* or Mr. *de Lagny's* methods of extracting the m th root of a given number, I conceive it will be always advisable to make use of a Table of Logarithms first, in order to obtain the first near value of the root sought, from which we are afterwards to derive a second and more accurate near value of it, by means of the expressions invented by those ingenious Gentlemen. This, indeed, is not absolutely necessary, as it is always easy to find the proposed root exact to one, or two, figures, by some very simple reasonings and trials, as is shewn above in all the foregoing examples. But it will always be still easier to find these first figures by the help of a Table of Logarithms, and we may find them by that means not only to two places of figures, but to five.

End of the Treatise, intitled, Mr. de Lagny's General Method of Extracting the Roots of Numbers by Approximation.

OBSER-

OBSERVATIONS

ON

MR. RAPHSOON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

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OBSERVATIONS

ON

MR. RAPHSON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

Article 1. **I**N the foregoing Tract I have given a pretty full explanation of *Monsieur de Lagny's* Method of Extracting the Roots of Numbers by Approximation, and I have likewise mentioned Mr. *Raphson's* more simple and easy, though less exact, method of performing the same thing. But both these methods may be applied to the resolution of all sorts of equations, those which are called *affected* equations*, or in which the unknown quantity occurs in

* This expression of *affected equations* seems to require some further explanation. It was introduced by the celebrated Vieta, the great father and restorer of Algebra. He has many expressions peculiar to himself, and which have not been adopted by subsequent Algebräists. Amongst these are the following ones. He calls a set of quantities in continual geometrical proportion; (such as the quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c,) a set of *scalar* quantities, or *magnitudines scalares*; and, when there are several of these *scalar* quantities connected with each other by the signs + and -, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$;) he calls the highest quantity, or that which is farthest in the scale of quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c, (to wit, the quantity x^5 in the said compound quantity $x^5 + ax^4 - b^2x^3$;) *the power* of the fundamental quantity x , or of the second term in the said scale; and he calls the lower scalar quantities, which are involved in the second and third terms of the said compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities

** Vieta was an eminent French mathematician
born 1540 died AD. 1603.*

in more than one term, as well as those which are called *pure* equations, or in which the unknown quantity occurs in only one term, and which are resolved by the mere extraction of the roots of given numbers. And in all affected equations beyond biquadratics, or those of the fourth power, these methods of approximation are the only methods that can be taken for discovering their roots, or the values of the unknown quantities contained in them. And even in cubick and biquadratick equations, though particular methods have been invented by Mathematicians, for the accurate resolution of most of the cases of these equations, (to wit, the rules called *Cardan's rules* for the resolution of most cases of cubick equations, and the rules invented by *Lewis Ferrari* of *Bologna* in Italy; about the year 1545, and explained at large in *Bombelli's Algebra*, in the year 1579, and those afterwards invented by *Monsieur Des Cartes*, and published in his *Geometry* in the year 1637,

for

tica x^4 and x^3 , (or, in our present language, the inferior powers of x .) scalar quantities of a *parodic* degree to x^5 , or the power of the fundamental quantity x . This word *parodic* I take to be derived (though Vieta does not tell us so,) from the Greek words *παρά* and *ὁδός*, which signify *near* and *a way*, or *road*, because these inferior scalar quantities, x^3 and x^4 , lie in the way as you pass along in the scale of the aforesaid quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c, from 1 to x^5 , which he calls the power of x in the said compound quantity $x^5 + ax^4 - b^2x^3$. These inferior scalar quantities x^3 and x^4 are therefore *parodic*, or situated in the way to, or are leading to, the said power, or higher scalar quantity, x^5 . He then proceeds to define a *pure power* and an *affected power*, and tells us, that a *pure power* is a scalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* scalar quantity, and that an *affected power* is a scalar quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, scalar quantities, combined with co-efficients that raise them to the same dimension as the power itself, or make them *homogeneous* to it, and consequently capable of being added to it, or subtracted from it. Thus x^5 alone is a *pure power* of x , namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is an *affected power* of x , namely, its fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* scalar quantities, x^3 and x^4 , which are multiplied into bb and a , in order to make them *homogeneous* to, or of the same dimension with, x^5 itself, and consequently capable of being added to it, or subtracted from it. See Schooten's edition of Vieta's Works, published at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expressions a *pure power* and an *affected power*, the meaning of the corresponding expressions of a *pure equation*

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for the resolution of biquadratick equations, by the mediation of cubick equations,) it will be found that these methods of approximation will, for the most part, enable us to find the values of their roots to any proposed degree of exactness, with less trouble than the particular and accurate methods above-mentioned, which have been invented for that purpose. So that these methods of resolving equations by approximation ought to be considered as of the highest utility, and as being absolutely necessary to the completion of the Doctrine of the Resolution of Algebraick Equations, which is the most important branch of the Science of Algebra.

Art. 2. But it is not so easy to determine, which of these two methods of approximation, Mr. *Raphson's*, or Mr. *de Lagny's*, deserves to be preferred to the other on these occasions. Mr. *Raphson's* is certainly much simpler than the other, because it proceeds by considering the new, or transformed, equation, (resulting from the substitution of $a + x$,

tion and an affected equation follows from it of course: a pure equation signifying an equation in which a pure power of an unknown quantity is declared to be equal to some known quantity; such as the equation $x^3 = 79$; and an affected equation signifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, some inferior powers of the same unknown quantity, (multiplied into proper co-efficients in order to make them homogeneous to the said highest power of the said unknown quantity,) is declared to be equal to some known quantity; such as the equation $x^3 + ax^2 - b^2x^2 = 79$. This I take to be the original meaning of the expression an affected equation. But, as the language of *Vieta* has not been adopted by subsequent writers of Algebra, I should think it would be more convenient to call them by some other name. And, perhaps, those of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, because they would be equations in which a *binomial* quantity, or quantity consisting of two terms that involved the unknown quantity x , is declared to be equal to a known quantity; and, for a like reason, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^3x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I observe, in his excellent new Mathematical and Philosophical Dictionary, just now published, (Feb. 2, 1795,) calls them *compound* equations; which is likewise a very proper name for them, and less obscure than that of *affected* equations.

or $a - x$, inſtead of x , in the original equation,) as being only a ſimple equation, and reſolving it accordingly, or by the mere operation of Diviſion; whereas, in Mr. *de Lagny's* method, the ſaid new, or transformed, equation is conſidered as a quadratick equation, and reſolved accordingly; which, when a (or the firſt near value of the root, that is ſuppoſed to be already known,) is a number conſiſting of five, or fix, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. *Raphſon's* method in reſolving all affected equations, more eſpecially when the number a conſiſts of more than two figures: but it muſt be confeſſed that the celebrated Dr. *Halley* (who had much experience, and was an excellent judge of theſe matters,) was of a different opinion, and gave the preference to Mr. *de Lagny's* method, which he has therefore taken the pains to explain in a better manner than had been done by *Monsieur de Lagny* himſelf, and likewise to illuſtrate by examples, in his Tract in the Philoſophical Tranſactions, Number 210, intitled, “*A New, Exact, and Eaſy Method, of finding the Roots of any Equations Generally, and that without any previous Reduction,*” which was publiſhed in the year 1694. On the other hand we may obſerve, that Mr. *Raphſon* always continued to give his own method the preference, after the publication of the tracts of *Monsieur de Lagny* and Dr. *Halley* upon the ſubject, as well as before their publication, when he tells us he had himſelf had the thought of adopting the principle which was afterwards followed by Mr. *de Lagny* and Dr. *Halley*, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater eaſe and ſimplicity of the other method, in which the ſaid transformed equation is conſidered and treated as a ſimple equation. And Sir *Iſaac Newton* in his method of reſolving equations by approximation (which differs very little from Mr. *Raphſon's*,) ſeems alſo to prefer Mr. *Raphſon's* practice, of treating the transformed equation as a mere ſimple equation, to that of Mr. *de Lagny* and Dr. *Halley*, of treating the ſaid equation as a quadratick equation. I therefore cannot but recommend it to all young Algebräiſts to ſtudy Mr. *Raphſon's* excellent Treatiſe

tise on this subject, intituled, *Analysis Aëquationum Universalis*, with great attention, and to endeavour to make themselves masters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to say that they will thereby acquire more useful knowledge in Algebra, towards the business of resolving affected, or compound, or multinomial, equations, than by reading all that has been written by *Harriot* and *Des Cartes*, and his learned Commentator *Van Schooten*, and all his other Commentators, and their numerous followers, on the boasted doctrine of the Generation of Equations one from another, by supposing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and so on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and likewise all the abstruse and intricate matter that has been delivered by Sir *Isaac Newton*, and Mr. *Gravesende* and Mr. *Mac Laurin*, and other learned Algebräists of modern times, on the invention of Divisors, which is grounded on that doctrine of the Generation of Equations from each other.

Art. 3. Yet in reading this excellent Treatise of Mr. *Raphson*, which I so much recommend, there will now and then occur some difficulties which are not inherent in the subject itself, but which might have been avoided, if Mr. *Raphson* had not unfortunately adopted the perplexing doctrines of modern writers of Algebra, about negative quantities and negative roots of Equations. The quantities called *negative* are such as it is impossible to form any clear idea of, being defined, by Sir *Isaac Newton* and other Algebräists *,

* Quantitates vel *Affirmativæ* sunt, seu majores Nihilo, vel *Negativæ*, seu Nihilo minores.—*Newton's Arithmetica Universalis*, page 3.

When a greater quantity is taken from a lesser of the same kind, the remainder becomes of the opposite kind.—*Mac Laurin's Algebra*, page 5.

An affirmative quantity is a quantity greater than nothing, and is known by this sign, +; a negative quantity is a quantity less than nothing, and is known by this sign, —.—*Saundersen's Algebra*, Vol. I. page 50, article 2.

to be such quantities as are *less than nothing*, or as *arise from the subtraction of a greater quantity from a lesser*, which is an operation evidently impossible to be performed: and, as to the *negative roots* of an equation, they are in truth the real and positive roots of another equation consisting of the same terms as the first equation, but with different signs $+$ and $-$ prefixed to some of them; so that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and suppose the proposed, or first, equation to have not only its own proper roots (which they call its *affirmative*, or *positive*, roots,) but to have likewise the roots of a different equation, which they call its *negative* roots. Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, $+ 16$, and the negative root, $- 20$. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of absurd and fantastick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and considering them as if they were one and the same equation. Now this perplexing language is unfortunately used by Mr. Raphson in this valuable Treatise, and tends to throw an air of mystery and obscurity upon some of the Problems solved in it, from which they would otherwise have been intirely free. As a proof of the truth of this observation, I shall here insert one of the said Problems, the solution of which is by this means rendered so obscure, that I had a good deal of trouble to find out the meaning of it; though, if this language had been avoided, and the proper and natural language, belonging to the conditions of the Problem, had been used in its stead, there could not have been the least difficulty in understanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatim et litteratim*, as follows.

PROBLEMA

P R O B L E M A XXIV.

Æquationum Quintæ Potestatis Adfectarum Solutio.

Proponatur $\rightarrow aaaaa + 7aaaa - 20aaa + 155aa = 10,000.$

Hoc est, $\rightarrow aaaaa + baaaa - caaa + daa = f.$

$$\text{Theor. } x = f + \frac{gEEEE + cEEE - bEEEE - dgg}{4bEEE + 2dg - 5EEEE - 3cEE}$$

Sit $g = -5$

$$\begin{aligned} f + gEEEE + cEEE - bEEEE - dgg &= -3875 \\ 4bEEE + 2dg - 5EEEE - 3cEE &= -9675 \end{aligned} \quad (-3875,0 (+,4 = x$$

$$\begin{array}{r} -5, \\ +,4 \\ \hline \end{array}$$

$g = -4,6$

$$\begin{aligned} f + gEEEE + cEEE - bEEEE - dgg &= -420,36896 \\ 4bEEE + 2dg - 5EEEE - 3cEE &= -7659,736 \end{aligned} \quad (-420,36896 (+,055 = x$$

$$\begin{array}{r} -4,6 \\ +,055 \\ \hline \end{array}$$

$g = -4,545$

$$\begin{aligned} f + gEEEE + cEEE - bEEEE - dgg &= -5,960359465465625 \\ 4bEEE + 2dg - 5EEEE - 3cEE &= -7410,748 \end{aligned} \quad (-5,9603594 (+,00080428 = x$$

$$\begin{array}{r} -4,545 \\ +,000,804,28 \\ \hline \end{array}$$

$$x = -4,544,195,72$$

To this solution I have, in my copy of Mr. Raphson's Tract, subjoined the following Note.

Numerus 4.544,195,72 est radix æquationis $a^5 + 7a^4 + 20a^3 + 155a^2 = 10,000$; quod hic obscurè innuitur sub specie radices negativæ æquationis $\rightarrow a^5 + 7a^4 - 20a^3 + 155a^2 = 10,000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ discendâ et excolendâ deterrentur, ex hîsce radicibus negativis et alijs quantitativis negativis,

gativis, seu (ut hodierni Algebræ scriptores absurdè loquuntur,) nihilo minoribus, ortum habent.

In this Problem the letter a is used for the unknown quantity, or root of the equation, which is usually denoted by the letter x ; and the letter g is used for the first near value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter a ; and the letter x is used for the difference between g , the first near value of the root of the equation, and a , its true value, which difference has been denoted in the two foregoing Tracts by the letter z . So that, if we express the enunciation of the foregoing Problem in the notation that has been used in the two foregoing Tracts, it will be as follows.

$$\text{Proponatur } -xxxxx + 7xxx - 20xx + 155x = 10,000,$$

$$\text{Sive } -x^5 + 7x^4 - 20x^3 + 155x^2 = 10,000,$$

$$\text{Hoc est, } -xxxxx + bxxxx - cxxx + dxx = f,$$

$$\text{Sive } -x^5 + bx^4 - cx^3 + dx^2 = f.$$

$$\text{Theor. } z = \frac{f + aaaaa + caaa - baaaa - daa}{4baaa + 2da - 5aaaa - 3caa},$$

$$\text{or } z = \frac{f + a^5 + ca^3 - ba^4 - da^2}{4ba^3 + 2da - 5a^4 - 3ca^2}.$$

Art. 4. Here, then, the equation proposed by Mr. Raphson to be resolved, is said to be $-x^5 + 7x^4 - 20x^3 + 155x^2 = 10,000$, or $155x^2 - 20x^3 + 7x^4 - x^5 = 10,000$. But this is not the equation he resolves; and, indeed, it is not a possible equation, because the greatest possible magnitude of the compound quantity $155x^2 - 20x^3 + 7x^4 - x^5$ is that which it has when the infinitely small increment of the binomial quantity $20x^3 + x^5$ becomes equal to the contemporary increment of the binomial quantity $155x^2 + 7x^4$, that is, (if we put $\dot{x}$, or x with a point placed over it, for the infinitely small increment of x ;) when $20 \times 3x^2\dot{x} + 5x^4\dot{x}$ becomes equal to $155 \times 2x\dot{x} + 7 \times 4x^3\dot{x}$, or when $60x^3 + 5x^4$ is $= 310x + 28x^3$, or when $60x + 5x^3$ is $= 310 + 28xx$, or when $5x^3 - 28xx + 60x$ is $= 310$,
or

or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by substituting 5.5 instead of x in the terms of the said quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greatest possible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much less than 10,000, or the absolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, and consequently the said equation is impossible. But Mr. *Raphson*, though he sets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, as the equation that is to be resolved, yet really means to resolve a quite different equation, to wit, the equation that results from supposing x to be a negative quantity, or from substituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the said equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, instead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which substitution the said equation will be converted into the equation $155x + xx - 20x - x^3 + 7x + x^4 - 1x - x^5 = 10,000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, which is evidently a possible equation, and of which the root is 4,544, 195,72, or the same number which he obtains by his solution of the Problem, and which, with the sign $-$ prefixed to it, he calls the negative root of the proposed equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$. Now all this perplexity would have been avoided, if Mr. *Raphson* had proposed at first to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *positive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which equation is evidently possible, and can have only one root. And then all the steps of his solution would have been clear and easy, as will appear by resolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$ according to the principles of his method; which may be done in the manner following.

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The Resolution of the Affected Equation $x^5 + 7x^4 + 20x^3 + 155x = 10,000$, by Mr. Raphson's Method of Approximation.

Art. 5. In considering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, it is, in the 1st place, easy to see that x must be greater than 1. For, if we suppose x to be $= 1$, we shall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much less than the absolute term 10,000. Therefore 1 must be much less than x .

In the second place, if we suppose x to be $= 10$, we shall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10,000$, and $x^5 = 100,000$; so that x^4 alone will be equal to the absolute term 10,000, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ must be very much greater than the said absolute term; and consequently 10 must be much greater than x .

Thirdly, since x is less than 10 and greater than 1, let us suppose it to be equal to 5. Then we shall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx (= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875) = 13,875$; which is greater than the absolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

We will therefore, in the 4th place, suppose x to be $= 4$. And then we shall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx (= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480) = 6576$; which is less than the absolute term 10,000. Therefore 4 is less than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

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It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, is greater than 4, but less than 5. And either of these values might very well serve for a first near value of the said root, or for the basis of a further approximation to it. Mr. Raphson makes choice of 5, which is greater than the truth.

Art. 6. Let us then suppose a , or the first near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x . Then will x be $= a - z$, and consequently xx will be $(= \overline{a - z}^2) = aa - 2az + \&c$, and x^3 will be $(= \overline{a - z}^3) = a^3 - 3a^2z + \&c$, and x^4 will be $(= \overline{a - z}^4) = a^4 - 4a^3z + \&c$, and x^5 will be $(= \overline{a - z}^5) = a^5 - 5a^4z + \&c$. Therefore $x^5 + 7x^4 + 20x^3$

$$+ 155xx \text{ will be } = \left\{ \begin{array}{l} + 7 \times \frac{a^5 - 5a^4z + \&c}{a^4 - 4a^3z + \&c}, \\ + 20 \times \frac{a^3 - 3a^2z + \&c}{aa - 2az + \&c}, \\ + 155 \times \frac{aa - 2az + \&c}{aa - 2az + \&c}, \end{array} \right\}$$

$$= \left\{ \begin{array}{l} + a^5 - 5a^4z + \&c, \\ + 7a^4 - 28a^3z + \&c, \\ + 20a^3 - 60a^2z + \&c, \\ + 155aa - 310az + \&c. \end{array} \right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az + \&c$, will also be $= 10,000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) we shall have $a^5 + 7a^4 + 20a^3 + 155aa = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (because a is $= 5$, and consequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13,875$, as has been shewn in art. 5,) we shall have $13,875 = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, and consequently (subtracting 10,000 from both sides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times \sqrt{5a^4 + 28a^3 + 60a^2 + 310a}$.
4 D Therefore

Therefore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a} (=$
 $\frac{3875}{5 \times 5^4 + 28 \times 5^3 + 60 \times 5^2 + 310 \times 5} =$
 $\frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} = \frac{3875}{3125 + 3500 + 1500 + 1550}$
 $= \frac{3875}{9675}) = 0.4$. Therefore $a - z$, or x , will be $(= a$
 $- 0.4 = 5.0 - 0.4) = 4.6$; and 4.6 will be a second
 near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

We must next try whether this second near value of x is greater or less than its true value; and for this purpose we must substitute it, instead of x , in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we suppose x to be $= 4.6$, we shall have xx $(= \overline{4.6}^2) = 21.16$, and x^3 $(= \overline{4.6}^3) = 97.336$, and x^4 $(= \overline{4.6}^4) = 447.7456$, and x^5 $(= \overline{4.6}^5) = 2059.62976$, and $155xx$ $(= 155 \times 21.16) = 3279.80$, and $20x^3$ $(= 20 \times 97.336) = 1946.720$, and $7x^4$ $(= 7 \times 447.7456) = 3134.2192$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 2059.62976 + 3134.2192 + 1946.720 + 3279.80) = 10,420.36896$; which is greater than 10,000, or the absolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art. 7. To find a third near value of the root of this equation, let a be supposed to be $= 4.6$, and z be the difference by which a , or 4.6, exceeds the true value of the said root.

Then we shall have, as before, $x = a - z$, and consequently xx $(= \overline{a - z}^2) = aa - 2az + \&c$, and x^3 $(= \overline{a - z}^3) = a^3 - 3a^2z + \&c$, and x^4 $(= \overline{a - z}^4) = a^4 - 4a^3z + \&c$, and x^5 $(= \overline{a - z}^5) = a^5 - 5a^4z + \&c$,
 and

and $7x^4 (= 7 \times \overline{a^5 - 4a^3z + \&c.}) = 7a^5 - 28a^3z + \&c.$, and $20x^3 (= 20 \times \overline{a^5 - 3a^3z + \&c.}) = 20a^5 - 60a^3z + \&c.$, and $155xx (= 155 \times \overline{aa - 2az + \&c.}) = 155aa - 310az + \&c.$, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{ \begin{array}{l} a^5 - 5a^4z + \&c., \\ + 7a^5 - 28a^3z + \&c., \\ + 20a^5 - 60a^3z + \&c., \\ + 155aa - 310az + \&c. \end{array} \right\}.$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $\left\{ \begin{array}{l} a^5 - 5a^4z + \&c., \\ + 7a^5 - 28a^3z + \&c., \\ + 20a^5 - 60a^3z + \&c., \\ + 155aa - 310az + \&c., \end{array} \right\}$ will likewise

be $= 10,000$, and consequently (adding $5a^4z + 28a^3z + 60a^3z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^3z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.6^5} + 7 \times \overline{4.6^4} + 20 \times \overline{4.6^3} + 155 \times \overline{4.6^2}$, is $= 10,420.36896$.

Therefore $10,420.36896$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^3z + 310az$; and consequently (subtracting $10,000$ from both sides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^3z + 310az (= 5 \times \overline{4.6^4} \times z + 28 \times \overline{4.6^3} \times z + 60 \times \overline{4.6^2} \times z + 310 \times \overline{4.6} \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z) = 7659.7360 \times z$,

and consequently z will be $(= \frac{420.36896}{7659.7360}) = 0.0548$, or

nearly 0.055 . Therefore x , or $a - z$, or $4.6 - z$, will be nearly $(= 4.6 - 0.055) = 4.545$; and consequently this number 4.545 will be a third near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Q. E. I.

Now

Now let this number 4.545 be substituted instead of x in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to discover whether the result will be greater, or less, than 10,000, or the absolute term of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Now, if x be supposed to be $= 4.545$, we shall have xx ($= 4.545^2$) $= 20.657,025$, and x^3 ($= 4.545^3$) $= 93.886,178,625$, and x^4 ($= 4.545^4$) $= 426.712,681,850,625$, and x^5 ($= 4.545^5$) $= 1,939,409,139,011,090,625$, and consequently $7x^4$ ($= 7 \times 426.712,681,850,625$) $= 2,986.988,772,954,375$, and $20x^3$ ($= 20 \times 93.886,178,625$) $= 1,877.723,572,500$, and $155xx$ ($= 155 \times 20.657,025$) $= 3,201.838,875$, and $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 1,939,409,139,011,090,625 + 2,986.988,772,954,375 + 1,877.723,572,500 + 3,201.838,875$) $= 10,005.960,359,465,465,625$; which is greater than 10,000, or the absolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. 8. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, let a be supposed to be $= 4.545$, and z be supposed to be the difference by which a , or 4.545, exceeds the true value of the said root.

Then we shall, as before, have $x = a - z$, and consequently xx ($= a - z^2$) $= aa - 2az + \&c$, and x^3 ($= a - z^3$) $= a^3 - 3a^2z + \&c$, and x^4 ($= a - z^4$) $= a^4 - 4a^3z + \&c$, and x^5 ($= a - z^5$) $= a^5 - 5a^4z + \&c$, and $7x^4$ ($= 7 \times a^4 - 4a^3z + \&c$) $= 7a^4 - 28a^3z + \&c$, and $20x^3$ ($= 20 \times a^3 - 3a^2z + \&c$) $= 20a^3 - 60a^2z + \&c$, and $155xx$ ($= 155 \times aa - 2az + \&c$) $= 155aa - 310az + \&c$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx =$

a^5

$$\left\{ \begin{array}{l} a^5 - 5a^4z + 8c, \\ + 7a^4 - 28a^3z + 8c, \\ + 20a^3 - 60a^2z + 8c, \\ + 155a^2 - 310az + 8c. \end{array} \right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z + 8c, - 28a^3z + 8c, - 60a^2z + 8c, - 310az + 8c$, will likewise be $= 10,000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $4.545^5 + 7 \times 4.545^4 + 20 \times 4.545^3 + 155 \times 4.545^2$, is $= 10,005.960,359,465,465,625$.

Therefore $10,005.960,359,465,465,625$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting $10,000$ from both sides,) $5.960,359,465,465,625$ will be $= 5a^4z + 28a^3z + 60a^2z + 310az (= 5 \times 4.545^4 \times z + 28 \times 4.545^3 \times z + 60 \times 4.545^2 \times z + 310 \times 4.545 \times z = 5 \times 426.712,681,850,625 \times z + 28 \times 93,886,178,625 \times z + 60 \times 20,657,025 \times z + 310 \times 4.545 \times z = 2133.563,409,253,125 \times z + 2628.813,001,500 \times z + 1239.421,500 \times z + 1408.950 \times z) = 7410.747,910,753,125 \times z$. Therefore z will be $(= \frac{5.960,359,465,465,625}{7410.747,910,753,125}) = 0.000,804,28$, and x , or $a - z$, or $4.545 - z$, will be $(= 4.545,000,00 - 0.000,804,28) = 4.544,195,72$. Therefore $4.544,195,72$ will be a fourth near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

This number $4.544,195,72$, agrees with the number found by Mr. Raphson, in all its figures.

Art. 9. The foregoing resolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, has been performed at great

great length, in order to set forth, in as clear a manner as possible, the several reasonings upon which the arithmetical operations used in it are grounded, as well as the said operations themselves. And by so doing the subject is rendered so much easier than in Mr. Raphson's very concise and compressed way of treating it, (in which all the reasonings are dropped, and only the arithmetical operations are exhibited,) that, though the above resolution of the said equation is three, or four, times as long as Mr. Raphson's, yet I am fully persuaded that it may be read and understood in a third, or fourth, part of the time that is necessary to a thorough comprehension of Mr. Raphson's resolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10,000$. But that this may appear the more clearly, I will now repeat the foregoing resolution of this equation in the style and manner of Mr. Raphson, by omitting the several reasonings set forth in the foregoing articles, and making use of a Canon, or Theorem, for the purpose of computing the second, third, and fourth values of z , in the same manner as Mr. Raphson has done.

Art. 10. Since each of the three first successive near values of x , or the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are successively denoted by the letter a , is greater than the true value of x in the said equation, or than the root of the said equation, it follows that the second, and third, and fourth near values of x will, each of them, be successively denoted by the residual quantity $a - z$; and consequently, by applying the reasonings used in art. 6, in order to obtain the values of z , and of $a - z$, or x , we shall find that z will be, successively, nearly equal to the

value of the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$, and,

therefore, that $a - z$, or x , will be, successively, nearly equal to the value of the quantity $a -$ the fraction

$$a^5 +$$

$\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem, or Canon, by the application of which we are to compute the second, and third, and fourth, near values of $a - z$, or x , after taking 5 for the first near value of it, or for the first value of a .

Now, if a is = 5, we shall have $z =$ the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4$. Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the second near value of x , or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Secondly, if a be = 4.6, we shall have $z =$ $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360} = 0.0548$, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x , or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Thirdly, if a be = 4.545, we shall have $z =$ $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{5,960,359,465,465,625}{7410,747,910,753,125} = 0.000,804,28$. Therefore $a - z$ will be $(= 4.545 - 0.000,804,28) = 4.544,195,72$; which will therefore be the fourth near value of x , or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

Art. 11. Mr. Raphson's Canon, or Theorem, for the value of z , is expressed more concisely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uses the letters b, c, d , and f , for the co-efficients 7, 20, and 155, of the fourth, third, and second, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, and for 10,000, the absolute term of that equation, respectively; which produces the following Canon, or Theorem, for the value of z ,
to

to wit, $x = \frac{a^5 + 5a^4 + 10a^3 + 10a^2 - f}{5a^4 + 4ba^3 + 3ca^2 + 2da}$. But it appears to me

that, though we may seem to gain something in point of brevity by using this very general notation, we lose as much in the article of perspicuity, which is a matter of much greater importance. However, this latter resolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which is expressed in Mr. Raphson's concise style and manner, and the foregoing more explicit resolution of it in art. 5, 6, 7, and 8, (in which the reasonings, on which the several arithmetical operations are grounded, are distinctly set forth and repeated,) are, both of them, the same in substance, and are, as I believe, the very best method that can be taken for discovering the root of the said equation.

Art. 12. It has been observed above in art. 2, that Sir Isaac Newton's method of resolving numeral equations by approximation differed but little from Mr. Raphson's, both methods being founded on the same principle of considering the new, or transformed, equation, (resulting from the substitution of $a + z$, or $a - z$, instead of x , in the original equation,) as a mere simple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than its simple power; which reduces the resolution of all equations, of whatever orders, to the resolution of a simple equation, or, rather, to the resolution of several successive simple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Isaac Newton's method and Mr. Raphson's method perfectly agree; and, in finding the second near value of x , or in making the first approximation to the true value of x , after having obtained, by conjecture, or trial, or in some other manner, the value of what has been here called a , or a first near value of x , or the root sought, there is not the smallest difference between them. But in the investigation of the third, and fourth, and other following near values of x , there is a little difference in their manner of proceeding, which the reader may be glad to see examined. I shall therefore now compare

pare the two methods together, in the case of a very easy equation, by which Sir Isaac Newton himself has thought proper to illustrate his method.

A Comparison between Sir Isaac Newton's and Mr. Raphson's Methods of Resolving Numeral Equations by Approximation.

Art. 13. Sir Isaac Newton's Method of Resolving Numeral Equations by Approximation, is explained by himself in his curious little Tract, intituled, *Analysıs per Aequationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Isaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows.

Art. 14. Let it be required to resolve the cubick equation $x^3 - 2x = 5$.

Here, in the first place, it is easy to see that x is somewhat greater than 2, but much less than 3. For, if x is taken equal to 2, we shall have $2x = 4$, and $x^3 = 8$, and consequently $x^3 - 2x (= 8 - 4) = 4$; which is less than 5, or the true value of $x^3 - 2x$ in the proposed equation: and, if x is taken equal to 3, we shall have $2x = 6$, and $x^3 = 27$, and consequently $x^3 - 2x (= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the proposed equation. Therefore the true value of x in that equation must be much less than 3, and a little greater than 2. Let it therefore be supposed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be substituted, instead of x , in the proposed equation $x^3 - 2x = 5$. This may be done as follows.

Since x is $= 2 + z$, we shall have $x^3 (= 2^3 + 3 \times 2^2 \times z + 4E)$

$\times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times 2z^2 + z^3) = 8 + 12z + 6zz + z^3$, and $2x (= 2 \times 2 + z) = 4 + 2z$, and consequently $x^3 - 2x (= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will also be $= 5$, and consequently (subtracting 4 from both sides,) $10z + 6zz + z^3$ will be $= 1$; and, (subtracting $6zz + z^3$ from both sides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10} = \frac{1}{10} - \frac{6zz - z^3}{10} \pm 0.1 - \frac{6zz - z^3}{10}$, that is, z is less than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$.

Therefore x , or $2 + z$, is less than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the said quantity $\frac{6zz + z^3}{10}$; which, on account of the smallness of z , (which is less than $\frac{1}{10}$), will be a very small quantity in comparison of z , or of $\frac{1}{10}$, and, *à fortiori*, in comparison of 2, and consequently may be neglected. And therefore 2.1 will be a second near value of x , or the root of the proposed equation $x^3 - 2x = 5$, that will be a little greater than its true value, but nearer to it than any other number that consists of only two places of figures.

Q. E. I.

This is the first step of Sir Isaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the assumption of the number 2, by conjecture and trial, for its first near value. And in this first step of the approximation Sir Isaac Newton's and Mr. Raphson's methods exactly coincide.

Art. 15. But in the next step of the approximation to the value of x , in the said equation $x^3 - 2x = 5$, the two methods are somewhat different from each other, though the number of new figures of the true value of x , that are exact in the next near values of it resulting from both methods,

thods, is the same. The difference between the methods in this second stage of the approximation is as follows.

Mr. Raphson corrects the value of x , or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be somewhat greater than the truth,) by subtracting from it the unknown quantity by which it exceeds x , and which we may call v , and substituting 2.1 $- v$ instead of x in the said original equation, $x^3 - 2x = 5$, whereby it is transformed into another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by resolving the said transformed equation as if it were only a simple equation, or by neglecting the terms which involve the square and cube of v , on account of their smallness, just as we before neglected the terms $6xz$ and z^3 in the foregoing transformed equation $10z + 6xz + z^3 = 1$ for the same reason. But Sir Isaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole process of his approximation; but, instead of the said original equation, he considers the former transformed equation, $10z + 6xz + z^3 = 1$, which was derived from it, and investigates the value of its root, z , to a greater degree of exactness than that to which it was before obtained. And this he does in the manner following.

Since it has been seen that z is less than 0.1, let the quantity by which 0.1 exceeds it be called v , so that z shall be $= 0.1 - v$; and let 0.1 $- v$ be substituted, instead of z , in the transformed equation $10z + 6xz + z^3 = 1$, This may be done as follows,

Since z is $= 0.1 - v$, we shall have

$$2x (= 0.1 - v)^2 = 0.01 - 0.2v + vv,$$

$$\text{and } z^3 (= 0.1 - v)^3 = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv - v^3 = 0.001 - 0.03v + 0.3vv - v^3,$$

$$\text{and } 10z (= 10 \times 0.1 - v) = 1 - 10v,$$

$$\text{and } 6xz = 0.06 - 1.2v + 6vv,$$

$$4 E 2$$

$$\text{and}$$

and consequently

$$\left\{ \begin{array}{l} + 10z \\ + 6xz \\ + z^3 \end{array} \right\} = \left\{ \begin{array}{l} 1.00 - 10v \\ + 0.06 - 1.2v + 6vv \\ + 0.001 - 0.03v + 0.3vv - v^3 \end{array} \right\} \\ = 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6xz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewise be $= 1$. And consequently (adding $11.23v$ to both sides,) we shall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (subtracting 1 from both sides,) we shall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconsiderable in comparison of 0.061 and $11.23v$) we shall have $0.061 = 11.23v$, or $11.23v = 0.061$; and consequently (dividing both sides by 11.23 ;) we shall have $v (= \frac{0.061}{11.23}) = 0.0054$. Therefore z , or $0.1 - v$, will be $(= 0.1 - 0.0054) = 0.0946$, and consequently x , or $2 + z$, will be $(= 2 + 0.0946) = 2.0946$. Q. E. 1.

In this manner Sir Isaac Newton finds the root of the proposed equation $x^3 - 2x = 5$ to be equal to 2.0946 , which is as near the truth as five figures can express it.

Art. 16. He then carries the investigation one step further, by which he obtains the value of x exact to nine places of figures; and for this purpose he proceeds in the manner following.

The last transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, because $6.3vv$ is greater than v^3 . Since, therefore, v is greater than 0.0054 , let us suppose it to be $= 0.0054 + w$; and let this binomial quantity be substituted, instead of v , in the last transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation

equation $11.23v - 6.3vv + v^3 = 0.061$, consisting of the same terms as the former, but in which the terms involving the unknown quantity v are all brought to the same side of the equation, and ranged according to the powers of v , beginning from its lowest power, or the simple power of v . This may be done in the manner following.

Since v is $= 0.0054 + w$, we shall have

$$vv (= \overbrace{0.0054 + w}^2) = 0.0054^2 + 2 \times 0.0054 \times w + w^2 \\ = 0.000,029,16 + 0.0108 \times w + w^2,$$

$$\text{and } v^3 (= \overbrace{0.0054 + w}^3) = 0.0054^3 + 3 \times 0.0054^2 \times w \\ + 3 \times 0.0054 \times w^2 + w^3 \\ = 0.000,000,157,464 + 3 \times 0.000,029,16 \times w \\ + 0.0162 \times w^2 + w^3 \\ = 0.000,000,157,464 + 0.000,087,48 \times w + \\ 0.0162 \times w^2 + w^3,$$

$$\text{and } 11.23v (= 11.23 \times \overbrace{0.0054 + w}) = 0.060,642 + \\ 11.23 \times w,$$

$$\text{and } 6.3vv (= 6.3 \times \overbrace{0.000,029,16 + 0.0108 \times w + w^2}) \\ = 0.000,183,708 + 0.068,04 \times w + 6.3ww;$$

and consequently $11.23v - 6.3vv + v^3$ will be =

$$\left\{ \begin{array}{l} 0.060,642 + 11.23 \times w \\ - 0.000,183,708 - 0.068,04 \times w - 6.3ww \\ + 0.000,000,157,464 + 0.000,087,48 \times w + 0.0162w^2 + w^3 \end{array} \right\} \\ = \left\{ \begin{array}{l} 0.060,642,157,464 + 11.230,087,48 \times w + 0.0162w^2 + w^3 \\ - 0.000,183,708 \quad - 0.068,04 \times w \quad - 6.3 \times w^2 \end{array} \right\} \\ = 0.060,458,449,464 + 11.162,047,48w - 6.2838w^2 + w^3,$$

But $11.23v - 6.3vv + v^3$ is $= 0.061$.

Therefore $0.060,458,449,464 + 11.162,047,48 \times w - 6.2838 \times ww + w^3$ will likewise be $= 0.061$; and consequently (subtracting $0.060,458,449,464$ from both sides,) $11.162,047,48 \times w - 6.2838ww + w^3$ will be $(= 0.061,000,000,000 - 0.060,458,449,464) = 0.000,541,550,536$; and (neglecting the terms $6.2838ww$ and w^3 , as inconsiderable in comparison of $11.162,047,48 \times w$), we shall have $11.162,047,48 \times w = 0.000,541,550,536$, and conse-

consequently $w (= \frac{0.000,541,550,536}{11.162,047,48} = 0.000,048,52,$

Therefore v , or $0.0054 + w$, will be $(= 0.0054 + 0.000,048,52) = 0.005,448,52$, and z , or $0.1 - v$, will be $(= 0.100,000,00 - 0.005,448,52) = 0.094,551,48$, and x , or $2 + z$, will be $(= 2 + 0.094,551,48) = 2.094,551,48$; that is, the root of the proposed equation $x^3 - 2x = 5$ will be $= 2.094,551,48$. Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be shewn in a subsequent article.

Art. 17. Having thus set forth Sir Isaac Newton's method of investigating the root of the proposed equation $x^3 - 2x = 5$ to nine places of figures, we must now perform the same thing by Mr. Raphson's method, in order to make a comparison between the necessary operations of the two methods.

Now Mr. Raphson's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6xx - x^3}{10}$, or to be somewhat less

than 2.1, is to put v for the unknown quantity by which it falls short of 2.1, and then to substitute the residual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the said equation will be transformed into another cubick equation, in which v will be the only unknown quantity: and then he determines the value of v by resolving the said transformed equation as if it was a mere simple equation, or by neglecting the terms in which the square or the cube of v occur. This may be done in the manner following.

Since x is $= 2.1 - v$, we shall have $xx (= \overline{2.1 - v}^2 = \overline{2.1}^2 - 2 \times 2.1 \times v + \&c) = 4.41 - 4.2v + \&c$, and $x^3 (= \overline{2.1 - v}^3 = \overline{2.1}^3 - 3 \times \overline{2.1}^2 \times v + \&c = 9.261 - 3 \times 4.41 \times v + \&c) = 9.261 - 13.23 \times v + \&c$.

&c, and $2x (= 2 \times 2.1 - v) = 4.2 - 2v$, and consequently $x^3 - 2x (= 9.261 - 13.23 \times v + \&c - 4.2 + 2v) = 5.061 - 11.23 \times v \&c$.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v \&c$, will likewise be $= 5$, and consequently (adding $11.23 \times v$ to both sides,) we shall have $5.061 = 5 + 11.23 \times v$, and (subtracting 5 from both sides,) we shall have $11.23 \times v = 0.061$, and consequently $v (= \frac{0.061}{11.23}) = 0.0054$. Therefore x , or $2.1 - v$, will be $(= 2.1 - 0.0054) = 2.0946$; or 2.0946 will be a third near value of the root of the proposed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very same with the third near value of it obtained above, in art. 15, by Sir Isaac Newton's method.

Art. 18. In this step of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the proposed equation $x^3 - 2x = 5$, the principal difference between the two methods seems to consist in this, to wit, that by Mr. Raphson's method we are obliged to raise the two first terms of the powers of the compound quantity $2.1 - v$, and consequently to raise the powers of the number 2.1, which consists of two figures; whereas in Sir Isaac Newton's method of proceeding, we had occasion only to raise the powers of the compound quantity $0.1 - v$, and consequently to raise the powers of the number 0.1, which consists of only one figure; which is somewhat easier than to raise the powers of 2.1. But both operations are so easy, that the difference of the labour of performing them is hardly worth considering. And, with respect to the simplicity of conception in the two methods, Mr. Raphson's method seems to be preferable to Sir Isaac Newton's; because the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has
more

more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. 19. But in the next step of the approximation by Mr. Raphson's method, we shall find the labour of raising the powers of the value of x already found, to wit, the powers of 2.0946, to be considerably greater than that of raising the powers of the last preceeding supplement of it according to Sir Isaac Newton's method, that supplement being only the decimal fraction 0.0054, in which there are only two significant figures. This will appear by performing this step of the approximation by Mr. Raphson's method; which may be done as follows.

Art. 20. The last near value we found for x , or the root of the equation $x^3 - 2x = 5$, by Mr. Raphson's method, was 2.0946. Now this near value of x is greater than its true value. For, if we suppose x to be $= 2.0946$, we shall have $x^3 (= 2.0946^3) = 9.189,741,550,536$, and $2x (= 2 \times 2.0946) = 4.1892$, and consequently $x^3 - 2x (= 9.189,741,550,536 - 4.1892) = 5.000,541,550,536$; which is greater than 5, or the absolute term of the equation $x^3 - 2x = 5$: and consequently 2.0946 must be greater than the true value of the root of the said equation.

We will therefore suppose x to be $= 2.0946 - w$, and substitute this residual quantity instead of x in the terms of the equation $x^3 - 2x = 5$.

Now, since x is $= 2.0946 - w$, we shall have $xx (= 2.0946 - w)^2 = 2.0946^2 - 2 \times 2.0946 \times w + &c) = 4.387,349,16 - 4.1892 \times w + &c$, and $x^3 (= 2.0946 - w)^3 = 2.0946^3 - 3 \times 2.0946^2 \times w + &c = 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + &c) = 9.189,741,550,536 - 13.162,047,48 \times w + &c$, and $2x (= 2 \times 2.0946 - w) = 4.1892 - 2w$, and consequently $x^3 - 2x =$

$$\left\{ \begin{array}{l} 9.189,741,550,536 - 13.162,047,48 \times w + &c \\ - 4.189,2 \quad \quad \quad + 2.000,000,00 \times w \end{array} \right\}$$

$$= 5.000,541,550,536 - 11.162,047,48 \times w + &c.$$

But

But $x^3 - 2x$ is = 5.

Therefore $5,000,541,550,536 - 11.162,047,48 \times w +$
 $\&c.$, will be = 5; and consequently (adding $11.162,047,48$
 $\times w$ to both sides,) we shall have $5,000,541,550,536 = 5$
 $+ 11.162,047,48 \times w$, and (subtracting 5 from both
sides,) $0.000,541,550,536 = 11.162,047,48 \times w$, or $11.162,$
 $047,48 \times w = 0.000,541,550,536$. Therefore w will be
 $(= \frac{0.000,541,550,536}{11.162,047,48}) = 0.000,048,52$; and consequently x ,
or $2.0946 - w$, will be $(= 2.094,600,00 - 0.000,048,52)$
 $= 2.094,551,48$. Therefore $2.094,551,48$ will be a fourth
near value of x , or the root of the proposed equation $x^3 -$
 $2x = 5$. Q. E. I.

This fourth near value of x is the very same with the
fourth near value of it obtained above, in art. 16, by Sir
Isaac Newton's method.

Art. 21. In this last stage of Mr. Raphson's approxima-
tion to the root of the proposed equation $x^3 - 2x = 5$, we
have been obliged to raise the powers of the number 2.0946 ,
which consists of five places of figures; whereas in Sir Isaac
Newton's way of proceeding we only raised the powers of
the decimal fraction 0.0054 , which contains only two signi-
ficant figures. But then in that way of proceeding we were
obliged to multiply v , or $0.0054 + w$, into 11.23 , and vv ,
or $0.000,029,16 + 0.0108 \times w + w^2$, into 6.3 ; whereas
in Mr. Raphson's way of proceeding we have only to mul-
tiply x , or $2.0946 - w$, into the very simple co-efficient 2.
So that, upon the whole, the difference of the labour of
computation in the two methods is not very considerable,
though it is rather less in Sir Isaac Newton's method than
in Mr. Raphson's. But in point of simplicity of conception
Mr. Raphson's method seems much superiour to Sir Isaac's,
because it never loses sight of the original equation $x^3 - 2x$
 $= 5$, which is to be resolved.

And, further, we may observe, in favour of Mr. Raph-
son's method; that it never requires us to raise any more
4 F than

than the two first terms of the binomial and residual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are substituted instead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Isaac Newton's method it is necessary to raise the other terms of the binomial and residual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increases the number and intricacy of the operations of the investigation. And therefore, upon the whole, I consider Mr. Raphson's method of approximating to the values of the roots of such equations as preferable to Sir Isaac Newton's.

A Proof of the Exactness of the Number 2.094,551,48, that has been found by the foregoing Methods of Approximation for the Root of the Equation $x^3 - 2x = 5$.

Art. 22. It remains that we prove the work to have been rightly performed, or that we shew that the last number 2.094,551,48, obtained by both these methods, is a very near value of the root x of the proposed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plainest and best method of doing this is to substitute the number 2.094,551,48, instead of x , in the compound quantity $x^3 - 2x$, in order to discover whether the quantity resulting from this substitution will be greater, or less, than 5, or the absolute term of the proposed equation $x^3 - 2x = 5$: and, if it shall appear that the said result is greater than 5, we may conclude that the said number 2.094,551,48 is greater than the true value of x in the said equation; and, if it shall appear that the said result is less than 5, we may conclude that the said number is less than the true value of x . And, when this has been thus disco-

discovered, we must, in the next place, endeavour to determine to how many figures this number 2.094,551,48 coincides with the more accurate value of x : and, for this purpose, we must, if this number be less than x , increase it by the addition of an unit in the last place of figures; and, if it be greater than x , we must diminish it by the same small quantity, and then substitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, instead of x , in the compound quantity $x^3 - 2x$. And, if it shall appear that the value of that compound quantity resulting from that substitution is greater, or less, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater, or less, than the true value of x , in the equation $x^3 - 2x = 5$, and consequently that the said true value is of an intermediate magnitude between 2.094,551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094,551,48$, we shall have

$$xx = 4.387,145,902,370,190,42$$

$$\text{and } x^3 = 9.189,102,942,785,417,810,201,792,$$

$$\text{and } 2x = 4.189,102,96,$$

and consequently

$$x^3 - 2x = 4.999,999,982,785,417,810,201,792; \text{ which number is somewhat less than 5, or the absolute term of the proposed equation } x^3 - 2x = 5. \text{ Therefore 2.094,551,48 must be somewhat less than the true value of } x \text{ in the said equation,}$$

Secondly, since x is greater than 2.094,551,48, we must now compare it with 2.094,551,49, by substituting that number instead of it in the compound quantity $x^3 - 2x$.

$$\begin{aligned} \text{Now, if } x \text{ is taken} &= 2.094,551,49, \text{ or } 2.094,551,48 + \\ &0.000,000,01, \text{ we shall have } x^3 (= 2.094,551,48)^3 + 3 \times \\ &2.094,551,48)^2 \times 0.000,000,01 + 3 \times 2.094,551,48 \times \\ &0.000,000,01 + 0.000,000,01)^3 = 9.189,102,942, \&c., + \\ &4 \times 2.094,551,48 \times 0.000,000,01 + 0.000,000,01 \end{aligned}$$

$3 \times 4.387,145,902, \&c \times 0.000,000,01 + 0.000,000,000, \&c + 0.000,000,000, \&c = 9.189,102,942, \&c + 13.161,437,706, \&c \times 0.000,000,01 + 0.000,000,000, \&c + 0.000,000,000, \&c = 9.189,102,942, \&c + 0.000,000,131, \&c + 0.000,000,000, \&c + 0.000,000,000, \&c) = 9.189,103,073, \&c$; and $2x (= 2 \times 2.094,551,49) = 4.189,102,98$; and consequently $x^3 - 2x (= 9.189,103,07, \&c - 4.189,102,98) = 5.000,000,09, \&c$; which is greater than 5. Therefore 2.094,551,49 must be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been shewn that 2.094,551,48 is less than the said true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and consequently all the figures of the number 2.094,551,48, which we found by the foregoing processes of Sir Isaac Newton's and Mr. Raphson's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of the Difficulty of finding a , or the First near Value of the Root of an Affected Equation, in certain Cases.

Art. 23. There is another difficulty that occurs sometimes in resolving high equations by approximation, whether by Sir Isaac Newton's method or by Mr. Raphson's; which indeed are substantially the same. The difficulty I mean, is that of finding the first near value of the root sought (which we have called a in this discourse,) to one, or two places of figures, in order to make it the basis of a further approximation to the true value of the root by either of these methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for

(for all other roots are not worth considering,) this difficulty will not be great; because it will always be easy to find a tolerably near value of the root by conjectures and trials, and particularly by supposing x , or the root of the proposed equation, first, to be equal to 1, and 2dly, to be $= 10$, and 3dly, to be equal to some short intermediate number consisting of only one figure, or, if the root appears to be greater than 10, by supposing it to be equal to 100, or 1000, and afterwards supposing it to be equal to some short intermediate number consisting of two figures; as was done above in art. 5, in finding the first near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. But, when the equation consists of terms connected together partly by the sign $+$, and partly by the sign $-$, and consequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforesaid method of conjectures and trials (though by no means useless,) is less expeditious and satisfactory in assisting us to find the first near value of one of the roots than in the former case; and we are often puzzled to know which of the roots it would be most expedient to begin to investigate. Now, in most of these cases, I believe, it will be advisable to begin by investigating the least root, and for that purpose to expunge from the equation all the terms that have the sign $-$ prefixed to them, and to find, to about two places, or, at most, to three places, of figures, the root of the remaining equation. For this root will always be less than the least root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be less than the only root of the original equation, if (notwithstanding the appearances to the contrary,) it really has but one root. When the root of this second, or curtailed, equation, has been discovered, it may be called a , and made the ground-work of an approximation to the least root of the original equation, and the binomial quantity $a + x$ may be substituted in the original equation instead of x , and the transformed equation thence arising may be resolved as if it was a mere simple equation, agreeably to Mr. Raphson's

son's method of approximation; and the value of z thereby obtained, being added to a , will give us a known value of $a + z$, or a second near value of the least, or the only, root of the proposed equation: after which we may proceed to find the said least, or only, root of the proposed equation by a further prosecution of Mr. Raphson's method of approximation above-described. This method of finding a first near value, a , of the least root of a proposed equation that seems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Discourse on the Reversion of Infinite Serieses published in that Volume; to which I refer the reader. See the said 3d Volume, pages 724, 725, 726, 727, &c, - - - to page 761. And, with this improvement of it in the case of equations that have, or seem to have, more than one real and positive root, I believe it may safely be affirmed that Mr. Raphson's Method of Resolving Affected Equations is the best General Method of effecting that purpose in all equations above quadratics that has hitherto been discovered.

*End of the Observations on Mr. Raphson's Method of
Resolving Affected Equations by Approximation.*

A TABLE

A
T A B L E

OF THE

SQUARE AND CUBE ROOTS OF THE NATURAL
NUMBERS 1, 2, 3, 4, 5, &c, to 180;

*Being Table XIX. of Mr. James Dodson's valuable Tables of
Computation, intitled The Calculator, that were
published in the Year 1747.*

No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
1	1.000,000	1.000,000	31	5.567,764	3.141,381	61	7.810,250	3.936,497
2	1.414,214	1.259,921	32	5.656,854	3.174,808	62	7.874,008	3.957,898
3	1.732,051	1.442,250	33	5.744,563	3.207,534	63	7.937,254	3.979,057
4	2.000,000	1.587,401	34	5.830,952	3.239,612	64	8.000,000	4.000,000
5	2.236,068	1.709,976	35	5.916,080	3.271,066	65	8.062,258	4.020,726
6	2.449,490	1.817,121	36	6.000,000	3.301,927	66	8.124,038	4.041,240
7	2.645,751	1.912,933	37	6.082,763	3.332,222	67	8.185,353	4.061,548
8	2.828,427	2.000,000	38	6.164,414	3.361,975	68	8.246,211	4.081,656
9	3.000,000	2.080,084	39	6.244,998	3.391,211	69	8.306,624	4.101,566
10	3.162,278	2.154,435	40	6.324,555	3.419,952	70	8.366,600	4.121,285
11	3.316,625	2.223,980	41	6.403,124	3.448,217	71	8.426,150	4.140,818
12	3.464,102	2.289,428	42	6.480,741	3.476,027	72	8.485,281	4.160,168
13	3.605,551	2.351,335	43	6.557,439	3.503,398	73	8.544,004	4.179,339
14	3.741,657	2.410,142	44	6.633,250	3.530,348	74	8.602,325	4.198,336
15	3.872,983	2.466,212	45	6.708,204	3.556,893	75	8.660,254	4.217,163
16	4.000,000	2.519,842	46	6.782,330	3.583,048	76	8.717,798	4.235,824
17	4.123,106	2.571,282	47	6.855,655	3.608,826	77	8.774,964	4.254,321
18	4.242,641	2.620,741	48	6.928,203	3.634,241	78	8.831,761	4.272,659
19	4.358,899	2.668,402	49	7.000,000	3.659,305	79	8.888,194	4.290,841
20	4.472,136	2.714,418	50	7.071,068	3.684,031	80	8.944,272	4.308,870
21	4.582,576	2.758,923	51	7.141,428	3.708,430	81	9.000,000	4.326,749
22	4.690,416	2.802,039	52	7.211,103	3.732,511	82	9.055,385	4.344,481
23	4.795,832	2.843,867	53	7.280,110	3.756,286	83	9.110,434	4.362,071
24	4.898,979	2.884,499	54	7.348,469	3.779,763	84	9.165,151	4.379,519
25	5.000,000	2.924,018	55	7.416,198	3.802,953	85	9.219,544	4.396,830
26	5.099,020	2.962,496	56	7.483,315	3.825,862	86	9.273,518	4.414,005
27	5.196,152	3.000,000	57	7.549,834	3.848,501	87	9.327,379	4.431,047
28	5.291,503	3.036,589	58	7.615,773	3.870,877	88	9.380,832	4.447,960
29	5.385,165	3.072,317	59	7.681,146	3.892,996	89	9.433,981	4.464,745
30	5.477,226	3.107,232	60	7.745,967	3.914,867	90	9.486,833	4.481,405
No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.

No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
91	9.539,392	4.497,942	121	11.000,00	4.946,088	151	12.288,21	5.325,074
92	9.591,663	4.514,357	122	11.045,36	4.959,675	152	12.328,83	5.336,803
93	9.643,651	4.530,655	123	11.090,54	4.973,190	153	12.369,32	5.348,481
94	9.695,360	4.546,836	124	11.135,53	4.986,631	154	12.409,67	5.360,108
95	9.746,794	4.562,903	125	11.180,34	5.000,000	155	12.449,90	5.371,685
96	9.797,959	4.578,857	126	11.224,97	5.013,208	156	12.490,00	5.383,213
97	9.848,858	4.594,701	127	11.269,43	5.026,526	157	12.529,96	5.394,690
98	9.899,495	4.610,436	128	11.313,71	5.039,684	158	12.569,81	5.406,120
99	9.949,874	4.626,065	129	11.357,82	5.052,774	159	12.609,52	5.417,501
100	10.000,00	4.641,589	130	11.401,75	5.065,797	160	12.649,11	5.428,835
101	10.049,88	4.657,010	131	11.445,52	5.078,753	161	12.688,58	5.440,122
102	10.099,50	4.672,339	132	11.489,12	5.091,643	162	12.727,92	5.451,362
103	10.148,89	4.687,548	133	11.532,56	5.104,469	163	12.767,15	5.462,556
104	10.198,04	4.702,669	134	11.575,84	5.117,230	164	12.806,25	5.473,703
105	10.246,95	4.717,694	135	11.618,95	5.129,928	165	12.845,23	5.484,806
106	10.295,63	4.732,624	136	11.661,90	5.142,563	166	12.884,10	5.495,865
107	10.344,08	4.747,459	137	11.704,70	5.155,137	167	12.922,85	5.506,879
108	10.392,30	4.762,203	138	11.747,34	5.167,649	168	12.961,48	5.517,848
109	10.440,31	4.776,856	139	11.789,83	5.180,101	169	13.000,00	5.528,775
110	10.488,09	4.791,420	140	11.832,16	5.192,494	170	13.038,40	5.539,658
111	10.535,65	4.805,896	141	11.874,34	5.204,828	171	13.076,70	5.550,499
112	10.583,01	4.820,284	142	11.916,38	5.217,103	172	13.114,88	5.561,298
113	10.630,15	4.834,588	143	11.958,26	5.229,321	173	13.152,95	5.572,054
114	10.677,08	4.848,808	144	12.000,00	5.241,482	174	13.190,91	5.582,770
115	10.723,81	4.862,944	145	12.041,59	5.253,588	175	13.228,76	5.593,445
116	10.770,33	4.876,999	146	12.083,06	5.265,637	176	13.266,50	5.604,079
117	10.816,65	4.890,973	147	12.124,36	5.277,632	177	13.304,13	5.614,673
118	10.862,75	4.904,868	148	12.165,53	5.289,572	178	13.341,66	5.625,226
119	10.908,71	4.918,685	149	12.206,56	5.301,459	179	13.379,09	5.635,741
120	10.954,45	4.932,424	150	12.247,45	5.313,293	180	13.416,41	5.646,216
No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.

A TABLE

A
T A B L E
OF THE
SQUARE-ROOTS AND RECIPROCAL
OF ALL NUMBERS,

From 1 to 1000.

*Computed by Dr. CHARLES HUTTON, Professor of Mathematicks
at the Royal Military Academy at Woolwich in Kent.*

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
1	1.0	1.000,000,000,0	51	0.019,607,8	7.141,428,428,5
2	0.5	1.414,213,562,4	52	0.019,230,8	7.211,102,550,9
3	0.333,333,3	1.733,050,807,6	53	0.018,867,9	7.280,109,889,3
4	0.25	2.000,000,000,0	54	0.018,518,5	7.348,469,228,3
5	0.2	2.236,067,977,5	55	0.018,181,8	7.416,198,487,1
6	0.166,666,6	2.449,489,742,8	56	0.017,857,1	7.483,314,773,5
7	0.142,857,1	2.645,751,311,1	57	0.017,543,9	7.549,834,435,3
8	0.125	2.828,427,124,7	58	0.017,241,4	7.615,773,105,9
9	0.111,111,1	3.000,000,000,0	59	0.016,949,0	7.681,145,747,9
10	0.1	3.162,277,660,2	60	0.016,666,6	7.745,976,692,4
11	0.090,909,0	3.316,624,790,4	61	0.016,393,4	7.810,249,675,9
12	0.083,333,3	3.464,101,615,1	62	0.016,129,0	7.874,007,874,0
13	0.076,923,0	3.605,551,275,5	63	0.015,873,0	7.937,253,933,2
14	0.071,428,5	3.741,657,386,8	64	0.015,625,1	8.000,000,000,0
15	0.066,666,6	3.872,983,346,2	65	0.015,384,6	8.062,257,748,3
16	0.062,5	4.000,000,000,0	66	0.015,151,5	8.124,038,404,6
17	0.058,823,5	4.123,105,625,6	67	0.014,925,4	8.185,352,771,9
18	0.055,555,5	4.242,642,687,1	68	0.014,705,9	8.246,211,251,2
19	0.052,631,6	4.358,898,943,5	69	0.014,492,8	8.306,623,862,9
20	0.05	4.472,135,955,0	70	0.014,285,7	8.366,600,265,3
21	0.047,619,0	4.582,575,695,8	71	0.014,084,5	8.426,149,773,2
22	0.045,454,5	4.690,415,759,8	72	0.013,888,8	8.485,281,374,2
23	0.043,478,3	4.795,831,523,3	73	0.013,698,6	8.544,003,745,3
24	0.041,666,6	4.898,979,485,6	74	0.013,513,5	8.602,325,267,0
25	0.04	5.000,000,000,0	75	0.013,333,3	8.660,254,037,8
26	0.038,461,5	5.099,019,513,6	76	0.013,157,9	8.717,797,887,1
27	0.037,037,0	5.196,152,422,7	77	0.012,987,0	8.774,964,387,4
28	0.035,714,3	5.291,502,622,1	78	0.012,820,5	8.831,760,865,3
29	0.034,482,8	5.385,164,807,1	79	0.012,658,2	8.888,194,417,3
30	0.033,333,3	5.477,225,575,1	80	0.012,5	8.944,271,910,0
31	0.032,258,1	5.567,764,362,8	81	0.012,345,7	9.000,000,000,0
32	0.031,25	5.656,854,240,5	82	0.012,195,0	9.055,385,138,1
33	0.030,303,0	5.744,562,646,5	83	0.012,048,2	9.110,433,579,1
34	0.029,411,8	5.832,951,894,8	84	0.011,904,8	9.165,251,389,5
35	0.028,571,4	5.916,079,783,1	85	0.011,764,7	9.219,544,457,3
36	0.027,777,7	6.000,000,000,0	86	0.011,627,9	9.273,618,495,5
37	0.027,027,0	6.082,762,530,3	87	0.011,494,3	9.327,379,053,1
38	0.026,315,8	6.164,414,003,0	88	0.011,363,6	9.380,831,519,6
39	0.025,641,0	6.244,997,998,4	89	0.012,236,0	9.433,981,132,1
40	0.025	6.324,555,320,3	90	0.011,111,1	9.486,832,980,5
41	0.024,390,2	6.403,124,237,4	91	0.010,939,0	9.539,392,014,2
42	0.023,809,5	6.480,740,698,4	92	0.010,869,6	9.591,663,046,6
43	0.023,255,8	6.557,438,524,3	93	0.010,752,7	9.643,650,761,0
44	0.022,727,2	6.633,249,580,7	94	0.010,638,3	9.695,859,714,8
45	0.022,222,2	6.708,203,932,5	95	0.010,526,3	9.746,794,344,8
46	0.021,739,1	6.782,329,983,1	96	0.010,416,6	9.797,958,971,1
47	0.021,276,6	6.855,654,600,4	97	0.010,309,3	9.848,857,801,8
48	0.020,833,3	6.928,203,230,3	98	0.010,204,1	9.899,494,936,6
49	0.020,408,2	7.000,000,000,0	99	0.010,101,0	9.949,874,371,1
50	0.02	7.071,067,811,9	100	0.01	10.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
101	0.009,900,9	10.049,875,621,1	151	0.006,622,5	12.288,205,727,4
102	0.009,803,9	10.099,504,938,4	152	0.006,578,9	12.328,828,005,9
103	0.009,708,7	10.148,891,565,1	153	0.006,535,9	12.369,316,876,9
104	0.009,615,4	10.198,039,027,2	154	0.006,493,5	12.409,673,646,0
105	0.009,523,8	10.246,950,766,0	155	0.006,451,6	12.449,899,598,3
106	0.009,434,0	10.295,630,141,0	156	0.006,410,3	12.489,995,996,8
107	0.009,345,8	10.344,080,432,8	157	0.006,369,4	12.529,964,086,1
108	0.009,259,2	10.393,304,845,4	158	0.006,329,1	12.569,805,1090,0
109	0.009,174,3	10.440,306,508,9	159	0.006,289,3	12.609,520,212,9
110	0.009,090,9	10.488,088,481,7	160	0.006,25	12.649,110,640,7
111	0.009,009,0	10.535,653,752,9	161	0.006,211,2	12.688,577,540,4
112	0.008,928,6	10.583,005,244,3	162	0.006,172,8	12.727,922,061,4
113	0.008,849,6	10.630,145,812,7	163	0.006,135,0	12.767,145,334,8
114	0.008,771,9	10.677,078,252,0	164	0.006,097,5	12.806,248,474,9
115	0.008,695,7	10.723,805,244,8	165	0.006,060,6	12.845,232,578,7
116	0.008,620,7	10.770,329,614,3	166	0.006,024,1	12.884,098,726,7
117	0.008,547,0	10.816,653,826,4	167	0.005,988,0	12.922,847,983,3
118	0.008,474,5	10.862,780,491,2	168	0.005,952,4	12.961,448,1396,8
119	0.008,403,4	10.908,712,114,6	169	0.005,917,2	13.000,000,000,0
120	0.008,333,3	10.954,451,150,8	170	0.005,882,4	13.038,404,810,4
121	0.008,264,5	11.000,000,000,0	171	0.005,848,0	13.076,696,830,6
122	0.008,196,7	11.045,361,017,2	172	0.005,814,0	13.114,877,048,6
123	0.008,130,0	11.092,536,506,4	173	0.005,780,3	13.152,946,438,0
124	0.008,064,5	11.135,528,725,7	174	0.005,747,1	13.190,905,995,3
125	0.008,	11.180,339,887,5	175	0.005,714,3	13.228,756,555,3
126	0.007,995,5	11.224,972,160,3	176	0.005,681,8	13.266,499,161,4
127	0.007,927,0	11.269,427,669,6	177	0.005,649,7	13.304,134,695,7
128	0.007,862,5	11.313,708,499,0	178	0.005,618,0	13.341,664,064,1
129	0.007,795,9	11.357,816,691,6	179	0.005,586,6	13.379,088,160,3
130	0.007,732,3	11.401,754,251,0	180	0.005,555,5	13.416,407,866,0
131	0.007,663,6	11.445,523,142,3	181	0.005,524,9	13.453,624,047,1
132	0.007,595,7	11.489,125,293,1	182	0.005,494,5	13.490,737,563,2
133	0.007,528,8	11.532,562,594,7	183	0.005,464,5	13.527,749,258,5
134	0.007,462,7	11.575,836,902,8	184	0.005,434,8	13.564,659,966,3
135	0.007,407,4	11.618,950,238,6	185	0.005,405,4	13.601,470,508,7
136	0.007,352,9	11.661,903,789,7	186	0.005,376,3	13.638,181,697,0
137	0.007,299,3	11.704,699,911,1	187	0.005,347,6	13.674,794,331,2
138	0.007,246,4	11.747,344,380,8	188	0.005,319,1	13.711,309,200,9
139	0.007,194,2	11.789,826,122,6	189	0.005,291,0	13.747,727,084,9
140	0.007,142,9	11.832,159,566,2	190	0.005,263,2	13.784,048,752,1
141	0.007,092,2	11.874,342,087,0	191	0.005,235,6	13.820,274,961,1
142	0.007,042,3	11.916,375,287,8	192	0.005,208,3	13.856,406,460,6
143	0.006,993,0	11.958,260,743,1	193	0.005,181,3	13.892,443,989,4
144	0.006,944,4	12.000,000,000,0	194	0.005,154,6	13.928,388,277,2
145	0.006,896,6	12.041,594,578,8	195	0.005,128,2	13.964,240,043,8
146	0.006,849,3	12.083,045,973,6	196	0.005,102,0	14.000,000,000,0
147	0.006,802,7	12.124,355,653,0	197	0.005,076,1	14.035,668,844,1
148	0.006,756,7	12.165,525,060,6	198	0.005,050,5	14.071,247,279,5
149	0.006,711,4	12.206,555,615,3	199	0.005,025,1	14.106,735,979,7
150	0.006,666,6	12.247,448,713,9	200	0.005,	14.142,135,623,2

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
201	0.004,975,1	14.17,446,878,8	251	0.003,984,1	15.842,979,517,8
202	0.004,950,4	14.212,670,403,6	252	0.003,968,3	15.874,507,866,4
203	0.004,926,1	14.247,806,848,8	253	0.003,952,6	15.905,973,720,6
204	0.004,902,0	14.282,856,857,1	254	0.003,937,0	15.937,377,450,5
205	0.004,875,0	14.317,821,063,3	255	0.003,921,6	15.968,719,422,7
206	0.004,854,4	14.352,700,094,4	256	0.003,906,3	16.000,000,000,0
207	0.004,830,9	14.387,494,569,9	257	0.003,891,1	16.031,219,541,9
208	0.004,807,7	14.422,205,101,9	258	0.003,876,0	16.062,378,404,2
209	0.004,784,7	14.456,832,294,8	259	0.003,861,0	16.093,476,939,4
210	0.004,761,9	14.491,376,746,2	260	0.003,846,2	16.124,515,496,6
211	0.004,739,3	14.525,839,046,3	261	0.003,831,4	16.155,494,421,4
212	0.004,717,0	14.560,219,778,6	262	0.003,816,8	16.186,414,056,2
213	0.004,694,8	14.594,519,519,3	263	0.003,802,3	16.217,274,740,2
214	0.004,672,9	14.628,738,838,3	264	0.003,787,8	16.248,076,809,2
215	0.004,651,2	14.662,878,298,6	265	0.003,773,6	16.278,820,596,1
216	0.004,629,6	14.696,938,456,7	266	0.003,759,4	16.309,506,430,3
217	0.004,608,3	14.730,919,862,7	267	0.003,745,3	16.340,134,638,4
218	0.004,587,2	14.764,823,060,2	268	0.003,731,3	16.370,705,543,7
219	0.004,566,2	14.798,648,586,9	269	0.003,717,5	16.401,219,466,9
220	0.004,545,4	14.832,396,974,2	270	0.003,703,7	16.431,676,725,2
221	0.004,524,9	14.866,068,747,3	271	0.003,690,0	16.462,077,633,2
222	0.004,504,5	14.899,664,425,8	272	0.003,676,5	16.492,422,502,5
223	0.004,484,3	14.933,184,523,1	273	0.003,663,0	16.522,711,641,9
224	0.004,464,3	14.966,629,547,1	274	0.003,649,6	16.552,915,356,9
225	0.004,444,4	15.000,000,000,0	275	0.003,636,3	16.583,123,951,8
226	0.004,424,8	15.033,296,378,4	276	0.003,623,2	16.613,247,725,8
227	0.004,405,3	15.066,519,173,3	277	0.003,610,1	16.643,316,977,1
228	0.004,386,0	15.099,668,870,5	278	0.003,597,1	16.673,332,000,5
229	0.004,366,8	15.132,745,950,4	279	0.003,584,2	16.703,293,088,5
230	0.004,347,8	15.165,750,888,1	280	0.003,571,4	16.733,200,530,7
231	0.004,329,0	15.198,684,153,6	281	0.003,558,7	16.763,054,614,2
232	0.004,310,3	15.231,546,211,7	282	0.003,546,1	16.792,855,623,7
233	0.004,291,8	15.264,337,522,5	283	0.003,533,6	16.822,603,841,3
234	0.004,273,5	15.297,058,540,8	284	0.003,521,1	16.852,299,546,4
235	0.004,255,3	15.329,709,716,8	285	0.003,508,8	16.881,943,016,1
236	0.004,237,3	15.362,291,495,7	286	0.003,496,5	16.911,534,525,3
237	0.004,219,4	15.394,804,318,3	287	0.003,484,3	16.941,074,346,1
238	0.004,201,7	15.427,248,620,9	288	0.003,472,2	16.970,562,748,5
239	0.004,184,1	15.459,624,833,7	289	0.003,460,2	17.000,000,000,0
240	0.004,166,6	15.491,933,384,8	290	0.003,448,3	17.029,386,365,9
241	0.004,149,4	15.524,174,696,3	291	0.003,436,4	17.058,722,109,2
242	0.004,132,2	15.556,349,186,1	292	0.003,424,6	17.088,007,490,6
243	0.004,115,2	15.588,457,268,1	293	0.003,413,0	17.117,242,768,6
244	0.004,098,4	15.620,499,351,8	294	0.003,401,4	17.146,428,199,5
245	0.004,081,6	15.652,475,842,5	295	0.003,389,8	17.175,564,037,3
246	0.004,065	15.684,387,141,4	296	0.003,378,3	17.204,650,531,1
247	0.004,048,6	15.716,233,645,5	297	0.003,367,0	17.233,687,939,6
248	0.004,032,3	15.748,015,748,0	298	0.003,355,7	17.262,676,501,6
249	0.004,016,1	15.779,733,838,1	299	0.003,344,5	17.291,616,465,8
250	0.004,001	15.811,288,200,8	300	0.003,333,3	17.320,508,075,7

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
301	0.003,322,3	17.349,351,572,9	351	0.002,849,0	18.734,993,995,2
302	0.003,311,3	17.378,147,196,9	352	0.002,840,9	18.761,663,039,3
303	0.003,300,3	17.406,895,185,5	353	0.002,832,9	18.788,204,228,1
304	0.003,289,5	17.435,595,774,2	354	0.002,824,8	18.814,887,722,2
305	0.003,278,7	17.464,249,196,6	355	0.002,816,9	18.841,443,681,2
306	0.003,268,0	17.492,855,684,5	356	0.002,809,0	18.867,962,264,1
307	0.003,257,3	17.521,415,467,9	357	0.002,801,1	18.894,443,627,7
308	0.003,246,8	17.549,928,774,8	358	0.002,793,3	18.920,887,928,4
309	0.003,236,2	17.578,395,831,2	359	0.002,785,5	18.947,295,321,5
310	0.003,225,8	17.606,816,861,7	360	0.002,777,7	18.973,665,961,0
311	0.003,215,4	17.635,192,088,5	361	0.002,770,1	19.000,000,000,0
312	0.003,205,1	17.663,521,732,7	362	0.002,762,4	19.026,297,590,4
313	0.003,194,9	17.691,806,013,0	363	0.002,754,8	19.052,558,885,3
314	0.003,184,7	17.720,045,146,7	364	0.002,747,3	19.078,784,028,3
315	0.003,174,6	17.748,230,349,3	365	0.002,739,7	19.104,973,174,5
316	0.003,164,6	17.776,388,834,6	366	0.002,732,2	19.131,126,469,7
317	0.003,154,6	17.804,493,814,8	367	0.002,724,8	19.157,244,060,7
318	0.003,144,7	17.832,554,500,1	368	0.002,717,4	19.183,326,093,8
319	0.003,134,8	17.860,571,099,5	369	0.002,710,0	19.209,372,712,3
320	0.003,125,3	17.888,543,820,0	370	0.002,702,7	19.235,384,061,7
321	0.003,115,3	17.916,472,867,2	371	0.002,695,4	19.261,360,284,3
322	0.003,105,6	17.944,358,444,9	372	0.002,688,2	19.287,301,522,0
323	0.003,096,0	17.972,200,755,6	373	0.002,681,0	19.313,207,915,9
324	0.003,086,4	18.000,000,000,0	374	0.002,673,8	19.339,005,3751,4
325	0.003,076,9	18.027,756,377,3	375	0.002,666,6	19.364,916,731,0
326	0.003,067,5	18.055,470,085,3	376	0.002,659,6	19.390,719,429,7
327	0.003,058,1	18.083,141,320,0	377	0.002,652,2	19.416,487,838,9
328	0.003,048,8	18.110,770,276,3	378	0.002,645,5	19.442,222,095,2
329	0.003,039,5	18.138,357,147,2	379	0.002,638,5	19.467,922,333,0
330	0.003,030,3	18.165,902,124,6	380	0.002,631,6	19.493,588,689,6
331	0.003,021,1	18.193,405,398,7	381	0.002,624,7	19.519,221,295,0
332	0.003,012,0	18.220,867,158,3	382	0.002,617,8	19.544,820,285,7
333	0.003,003,0	18.248,287,590,9	383	0.002,611,0	19.570,385,790,8
334	0.002,994,0	18.275,666,882,5	384	0.002,604,2	19.595,917,942,3
335	0.002,985,1	18.303,005,217,7	385	0.002,597,4	19.621,416,870,5
336	0.002,976,2	18.330,302,779,8	386	0.002,590,7	19.646,882,704,2
337	0.002,967,4	18.357,559,750,7	387	0.002,584,0	19.672,315,572,9
338	0.002,958,5	18.384,776,310,9	388	0.002,577,3	19.697,715,603,6
339	0.002,949,9	18.411,952,639,5	389	0.002,570,7	19.723,082,923,1
340	0.002,941,2	18.439,088,914,6	390	0.002,564,1	19.748,417,658,1
341	0.002,932,6	18.466,185,312,6	391	0.002,557,5	19.773,719,933,3
342	0.002,924,0	18.493,242,008,9	392	0.002,551,0	19.798,680,873,2
343	0.002,915,5	18.520,259,177,5	393	0.002,544,5	19.824,227,601,6
344	0.002,907,0	18.547,236,991,0	394	0.002,538,1	19.849,433,241,5
345	0.002,898,6	18.574,175,621,0	395	0.002,531,6	19.874,606,914,4
346	0.002,890,2	18.601,075,237,7	396	0.002,525,2	19.899,748,742,1
347	0.002,881,8	18.627,936,010,2	397	0.002,518,9	19.924,858,845,2
348	0.002,873,6	18.654,758,106,2	398	0.002,512,6	19.949,937,343,3
349	0.002,865,3	18.681,541,692,3	399	0.002,506,3	19.974,984,355,4
350	0.002,857,1	18.708,286,933,9	400	0.002,5	20.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
401	0.002,493,8	20.024,984,394,5	451	0.002,217,3	21.236,760,581,6
402	0.002,487,6	20.049,937,655,8	452	0.002,212,4	21.260,291,625,5
403	0.002,481,4	20.074,859,899,9	453	0.002,207,5	21.283,796,653,8
404	0.002,475,2	20.099,751,242,2	454	0.002,202,6	21.307,275,752,7
405	0.002,469,1	20.124,611,797,5	455	0.002,197,8	21.330,729,007,7
406	0.002,463,1	20.149,441,579,6	456	0.002,193,0	21.354,156,504,1
407	0.002,457,0	20.174,241,001,8	457	0.002,188,2	21.377,558,326,4
408	0.002,451,0	20.199,009,876,7	458	0.002,183,4	21.400,934,559,0
409	0.002,445,0	20.223,748,416,2	459	0.002,178,6	21.424,285,285,6
410	0.002,439,0	20.248,456,731,3	460	0.002,173,9	21.447,610,589,5
411	0.002,423,1	20.273,134,932,7	461	0.002,169,2	21.470,910,553,6
412	0.002,427,2	20.297,783,130,2	462	0.002,164,5	21.494,185,257,9
413	0.002,421,3	20.322,401,432,9	463	0.002,159,8	21.517,434,791,4
414	0.002,415,5	20.346,989,949,4	464	0.002,155,2	21.540,659,228,5
415	0.002,409,6	20.371,548,787,5	465	0.002,150,5	21.563,858,652,8
416	0.002,403,8	20.396,078,054,4	466	0.002,145,9	21.587,033,144,9
417	0.002,398,1	20.420,577,856,7	467	0.002,141,3	21.610,182,785,0
418	0.002,392,3	20.445,048,300,3	468	0.002,136,8	21.633,307,652,8
419	0.002,386,6	20.469,489,490,5	469	0.002,132,2	21.656,407,827,7
420	0.002,381,0	20.493,901,531,9	470	0.002,127,7	21.679,483,388,7
421	0.002,375,3	20.518,284,528,7	471	0.002,123,1	21.702,534,414,2
422	0.002,369,7	20.542,638,584,2	472	0.002,118,7	21.725,560,982,4
423	0.002,364,1	20.566,963,801,2	473	0.002,114,2	21.748,563,170,9
424	0.002,358,5	20.591,260,282,0	474	0.002,109,7	21.771,541,057,1
425	0.002,352,9	20.615,528,128,1	475	0.002,105,3	21.794,494,717,7
426	0.002,347,4	20.639,767,440,6	476	0.002,100,8	21.817,424,229,3
427	0.002,341,9	20.663,978,319,8	477	0.002,096,4	21.840,329,667,8
428	0.002,336,4	20.688,160,865,6	478	0.002,092,1	21.863,211,109,1
429	0.002,331,0	20.712,315,177,2	479	0.002,087,7	21.886,068,628,2
430	0.002,325,6	20.736,441,353,3	480	0.002,083,3	21.908,902,300,2
431	0.002,320,2	20.760,539,492,0	481	0.002,079,0	21.931,712,199,5
432	0.002,314,8	20.784,609,690,8	482	0.002,074,7	21.954,498,402,4
433	0.002,309,5	20.808,652,046,7	483	0.002,070,4	21.977,260,975,8
434	0.002,304,2	20.832,666,656,0	484	0.002,066,1	22.000,000,000,0
435	0.002,298,9	20.856,653,614,6	485	0.002,061,9	22.022,715,545,5
436	0.002,293,6	20.880,613,017,8	486	0.002,057,6	22.045,407,685,0
437	0.002,288,3	20.904,544,960,4	487	0.002,053,4	22.068,076,490,7
438	0.002,283,1	20.928,449,536,5	488	0.002,049,2	22.090,722,034,4
439	0.002,277,9	20.952,326,839,8	489	0.002,045,0	22.113,344,387,5
440	0.002,272,7	20.976,176,963,4	490	0.002,040,8	22.135,943,621,2
441	0.002,267,6	21.000,000,000,0	491	0.002,036,7	22.158,519,806,2
442	0.002,262,4	21.023,796,041,6	492	0.002,032,5	22.181,073,012,8
443	0.002,257,3	21.047,565,179,8	493	0.002,028,4	22.203,603,311,2
444	0.002,252,2	21.071,307,505,7	494	0.002,024,3	22.226,110,770,9
445	0.002,247,2	21.095,023,109,7	495	0.002,020,2	22.248,595,461,3
446	0.002,242,2	21.118,712,031,9	496	0.002,016,2	22.271,057,451,3
447	0.002,237,1	21.142,374,511,9	497	0.002,012,1	22.293,497,809,6
448	0.002,232,1	21.166,010,488,5	498	0.002,008,0	22.315,913,604,4
449	0.002,227,2	21.189,620,100,4	499	0.002,004,0	22.338,507,903,9
450	0.002,222,2	21.213,203,435,6	500	0.002	22.360,679,775,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
501	0.001,996,0	22.383,029,285,6	551	0.001,814,9	23.473,589,188,6
502	0.001,992,0	22.405,356,502,4	552	0.001,811,6	23.494,680,248,9
503	0.001,988,1	22.427,661,492,0	553	0.001,808,3	23.515,952,032,6
504	0.001,984,1	22.449,944,320,6	554	0.001,805,1	23.537,204,591,9
505	0.001,980,3	22.472,205,054,2	555	0.001,801,8	23.558,437,978,8
506	0.001,976,3	22.494,443,758,4	556	0.001,798,6	23.579,652,245,1
507	0.001,972,4	22.516,660,498,4	557	0.001,795,3	23.600,847,442,4
508	0.001,968,5	22.538,855,339,2	558	0.001,792,1	23.622,023,622,0
509	0.001,964,6	22.561,028,345,4	559	0.001,788,9	23.643,180,835,1
510	0.001,960,8	22.583,179,581,3	560	0.001,785,7	23.664,319,132,4
511	0.001,956,9	22.605,309,110,9	561	0.001,782,5	23.685,438,564,7
512	0.001,953,1	22.627,416,998,0	562	0.001,779,4	23.706,539,182,3
513	0.001,949,3	22.649,503,305,8	563	0.001,776,2	23.727,621,035,4
514	0.001,945,5	22.671,568,097,5	564	0.001,773,0	23.748,684,174,1
515	0.001,941,7	22.693,611,435,8	565	0.001,769,9	23.769,728,648,0
516	0.001,938,0	22.715,633,383,2	566	0.001,766,8	23.790,754,506,5
517	0.001,934,2	22.737,634,001,8	567	0.001,763,7	23.811,761,799,4
518	0.001,930,5	22.759,613,353,5	568	0.001,760,6	23.832,750,575,6
519	0.001,926,8	22.781,571,499,8	569	0.001,757,5	23.853,720,683,8
520	0.001,923,1	22.803,508,502,0	570	0.001,754,4	23.874,672,772,6
521	0.001,919,4	22.825,424,421,0	571	0.001,751,3	23.895,606,290,7
522	0.001,915,7	22.847,319,317,6	572	0.001,748,2	23.916,521,486,2
523	0.001,912,0	22.869,193,252,1	573	0.001,745,2	23.937,418,407,2
524	0.001,908,4	22.891,046,284,5	574	0.001,742,2	23.958,297,101,4
525	0.001,904,8	22.912,878,474,8	575	0.001,739,1	23.979,157,416,6
526	0.001,901,1	22.934,689,882,4	576	0.001,736,1	24.000,000,000,0
527	0.001,897,5	22.956,480,566,5	577	0.001,733,0	24.020,824,298,9
528	0.001,893,9	22.978,250,586,2	578	0.001,730,1	24.041,630,560,3
529	0.001,890,4	23.000,000,000,0	579	0.001,727,1	24.062,418,831,0
530	0.001,886,8	23.021,728,866,4	580	0.001,724,1	24.083,168,306,2
531	0.001,883,2	23.043,437,243,6	581	0.001,721,2	24.103,941,586,4
532	0.001,879,7	23.065,125,189,3	582	0.001,718,2	24.124,676,163,6
533	0.001,876,2	23.086,792,761,2	583	0.001,715,3	24.145,392,935,3
534	0.001,872,7	23.108,440,016,6	584	0.001,712,3	24.166,091,947,2
535	0.001,869,2	23.130,067,012,4	585	0.001,709,4	24.186,773,244,9
536	0.001,865,7	23.151,673,805,6	586	0.001,706,5	24.207,436,873,6
537	0.001,862,2	23.173,260,452,5	587	0.001,703,6	24.228,082,879,2
538	0.001,858,7	23.194,827,009,5	588	0.001,700,7	24.248,711,306,0
539	0.001,855,3	23.216,373,532,5	589	0.001,697,8	24.269,322,199,0
540	0.001,851,8	23.237,900,077,2	590	0.001,694,9	24.289,915,603,0
541	0.001,848,4	23.259,406,699,2	591	0.001,692,0	24.310,491,562,3
542	0.001,845,0	23.280,893,453,6	592	0.001,689,1	24.331,050,121,2
543	0.001,841,6	23.302,360,395,5	593	0.001,686,5	24.351,591,323,8
544	0.001,838,2	23.323,807,579,4	594	0.001,683,5	24.372,115,213,9
545	0.001,834,9	23.345,235,059,9	595	0.001,680,7	24.392,621,835,3
546	0.001,831,5	23.366,642,891,1	596	0.001,677,9	24.413,111,231,5
547	0.001,828,2	23.388,031,127,1	597	0.001,675,0	24.433,583,445,7
548	0.001,824,8	23.409,399,821,4	598	0.001,672,2	24.454,038,521,3
549	0.001,821,5	23.430,749,027,7	599	0.001,669,4	24.474,476,501,0
550	0.001,818,1	23.452,078,799,1	600	0.001,666,6	24.494,897,427,8

TABLE of Square-Roots and Reciprocals.

600

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
601	0.001,663,9	24.515,301,344,3	651	0.001,536,1	25.514,701,644,3
602	0.001,661,1	24.533,688,292,8	652	0.001,533,7	25.534,290,669,6
603	0.001,658,4	24.556,058,315,6	653	0.001,531,4	25.553,864,678,4
604	0.001,655,6	24.576,411,454,9	654	0.001,529,1	25.573,423,705,1
605	0.001,652,9	24.596,747,752,5	655	0.001,526,7	25.592,967,784,1
606	0.001,650,1	24.617,067,250,2	656	0.001,524,4	25.612,496,949,7
607	0.001,647,4	24.637,369,989,5	657	0.001,522,1	25.632,011,236,0
608	0.001,644,7	24.657,656,011,9	658	0.001,519,8	25.651,510,676,8
609	0.001,642,0	24.677,925,358,5	659	0.001,517,5	25.670,995,306,0
610	0.001,639,3	24.698,178,070,5	660	0.001,515,1	25.690,465,157,3
611	0.001,636,7	24.718,414,188,6	661	0.001,512,9	25.709,920,264,4
612	0.001,634,0	24.738,633,753,7	662	0.001,510,6	25.729,360,660,5
613	0.001,631,3	24.758,836,806,3	663	0.001,508,3	25.748,786,379,2
614	0.001,628,7	24.779,023,386,7	664	0.001,506,0	25.768,197,453,5
615	0.001,626,0	24.799,193,535,3	665	0.001,503,8	25.787,593,916,5
616	0.001,623,4	24.819,347,292,0	666	0.001,501,5	25.806,975,801,1
617	0.001,620,7	24.839,484,696,7	667	0.001,499,3	25.826,343,140,3
618	0.001,618,1	24.859,605,789,3	668	0.001,497,0	25.845,695,966,6
619	0.001,615,5	24.879,710,609,2	669	0.001,494,8	25.865,034,312,8
620	0.001,612,9	24.899,799,196,0	670	0.001,492,5	25.884,358,211,1
621	0.001,610,3	24.919,871,588,8	671	0.001,490,3	25.903,667,694,0
622	0.001,607,7	24.939,927,826,7	672	0.001,488,1	25.922,962,793,6
623	0.001,605,1	24.959,967,948,7	673	0.001,485,9	25.942,243,542,1
624	0.001,602,6	24.979,991,993,6	674	0.001,483,7	25.961,509,971,5
625	0.001,6	25.000,000,000,0	675	0.001,481,4	25.980,762,113,5
626	0.001,597,4	25.019,992,006,4	676	0.001,479,3	26.000,000,000,0
627	0.001,594,9	25.039,968,051,1	677	0.001,477,1	26.019,223,662,5
628	0.001,592,4	25.059,928,172,3	678	0.001,474,9	26.038,433,132,6
629	0.001,589,8	25.079,872,408,0	679	0.001,472,8	26.057,628,441,6
630	0.001,587,3	25.099,800,796,0	680	0.001,470,6	26.076,809,620,8
631	0.001,584,8	25.119,713,374,2	681	0.001,468,4	26.095,976,701,4
632	0.001,582,3	25.139,610,180,0	682	0.001,466,3	26.115,129,714,4
633	0.001,579,8	25.159,491,250,8	683	0.001,464,1	26.134,268,690,7
634	0.001,577,3	25.179,356,620,1	684	0.001,462,0	26.153,393,661,2
635	0.001,574,8	25.199,206,336,7	685	0.001,459,9	26.172,504,656,6
636	0.001,572,3	25.219,040,425,8	686	0.001,457,7	26.191,601,707,4
637	0.001,569,9	25.238,858,928,2	687	0.001,455,6	26.210,684,844,2
638	0.001,567,4	25.258,661,880,6	688	0.001,453,5	26.229,754,097,2
639	0.001,564,9	25.278,449,319,5	689	0.001,451,4	26.248,809,496,8
640	0.001,562,5	25.298,221,281,3	690	0.001,449,3	26.267,851,073,1
641	0.001,560,1	25.317,977,802,3	691	0.001,447,2	26.286,878,856,2
642	0.001,557,6	25.337,718,918,6	692	0.001,445,1	26.305,892,875,9
643	0.001,555,2	25.357,444,666,2	693	0.001,443,0	26.324,893,162,2
644	0.001,552,8	25.377,155,080,9	694	0.001,440,9	26.343,879,744,6
645	0.001,550,4	25.396,850,198,4	695	0.001,438,8	26.362,852,652,9
646	0.001,548,0	25.416,530,051,3	696	0.001,436,8	26.381,811,916,5
647	0.001,545,6	25.436,194,684,0	697	0.001,434,7	26.400,757,564,9
648	0.001,543,2	25.455,844,122,7	698	0.001,432,7	26.419,689,627,2
649	0.001,540,8	25.475,478,405,7	699	0.001,430,6	26.438,608,132,8
650	0.001,538,5	25.495,097,568,0	700	0.001,428,6	26.457,513,110,6

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
701	0.001,426,5	26.476,404,589,7	751	0.001,331,6	27.404,379,212,1
702	0.001,424,5	26.495,282,599,0	752	0.001,329,8	27.422,618,401,6
703	0.001,422,5	26.514,147,167,1	753	0.001,328,0	27.440,845,468,0
704	0.001,420,5	26.532,998,322,8	754	0.001,326,3	27.459,060,435,5
705	0.001,418,4	26.551,836,094,7	755	0.001,324,5	27.477,263,328,1
706	0.001,416,4	26.570,660,511,2	756	0.001,322,8	27.495,454,169,7
707	0.001,414,4	26.589,471,600,6	757	0.001,321,0	27.513,632,984,4
708	0.001,412,4	26.608,269,391,3	758	0.001,319,3	27.531,799,795,9
709	0.001,410,4	26.627,053,911,4	759	0.001,317,5	27.549,954,627,9
710	0.001,408,5	26.645,825,188,9	760	0.001,315,8	27.568,097,504,2
711	0.001,406,5	26.664,583,251,9	761	0.001,314,1	27.586,228,448,3
712	0.001,404,5	26.683,328,128,3	762	0.001,312,3	27.604,347,483,7
713	0.001,402,5	26.702,059,845,6	763	0.001,310,6	27.622,454,633,9
714	0.001,400,6	26.720,776,431,8	764	0.001,308,9	27.640,549,922,2
715	0.001,398,6	26.739,483,914,2	765	0.001,307,2	27.658,633,371,9
716	0.001,396,6	26.758,176,320,5	766	0.001,305,5	27.676,705,006,2
717	0.001,394,7	26.776,855,678,0	767	0.001,303,8	27.694,764,848,3
718	0.001,392,8	26.795,522,013,9	768	0.001,302,1	27.712,812,921,1
719	0.001,390,8	26.814,175,355,6	769	0.001,300,4	27.730,849,247,7
720	0.001,388,8	26.832,815,730,0	770	0.001,298,7	27.748,873,851,0
721	0.001,387,0	26.851,443,164,2	771	0.001,297,0	27.766,886,753,8
722	0.001,385,0	26.870,057,685,1	772	0.001,295,3	27.784,887,978,9
723	0.001,383,1	26.888,659,319,5	773	0.001,293,7	27.802,877,548,9
724	0.001,381,2	26.907,248,094,1	774	0.001,292,0	27.820,855,548,5
725	0.001,379,3	26.925,824,035,7	775	0.001,290,3	27.838,821,814,2
726	0.001,377,4	26.944,387,170,6	776	0.001,288,7	27.856,776,554,4
727	0.001,375,5	26.962,937,525,4	777	0.001,287,0	27.874,719,729,5
728	0.001,373,6	26.981,475,126,5	778	0.001,285,3	27.892,651,362,0
729	0.001,371,7	27.000,000,000,0	779	0.001,283,7	27.910,571,473,9
730	0.001,369,9	27.018,512,172,2	780	0.001,282,1	27.928,480,087,5
731	0.001,368,0	27.037,011,669,2	781	0.001,280,4	27.946,377,225,0
732	0.001,366,1	27.055,498,516,9	782	0.001,278,8	27.964,262,908,2
733	0.001,364,3	27.073,972,741,4	783	0.001,277,1	27.982,137,159,3
734	0.001,362,4	27.092,434,368,3	784	0.001,275,5	28.000,000,000,0
735	0.001,360,5	27.110,883,423,5	785	0.001,273,9	28.017,851,452,2
736	0.001,358,7	27.129,319,932,5	786	0.001,272,3	28.035,691,537,8
737	0.001,356,9	27.147,743,921,0	787	0.001,270,6	28.053,520,278,2
738	0.001,355,0	27.166,155,414,4	788	0.001,269,0	28.071,337,688,1
739	0.001,353,2	27.184,554,438,1	789	0.001,267,4	28.089,143,810,4
740	0.001,351,3	27.202,941,017,5	790	0.001,265,8	28.106,938,645,1
741	0.001,349,5	27.221,315,177,6	791	0.001,264,2	28.124,722,220,9
742	0.001,347,7	27.239,676,943,8	792	0.001,262,6	28.142,494,558,9
743	0.001,345,9	27.258,026,340,9	793	0.001,261,0	28.150,255,680,7
744	0.001,344,1	27.276,363,394,0	794	0.001,259,4	28.178,005,607,2
745	0.001,342,3	27.294,688,127,9	795	0.001,257,9	28.195,744,359,7
746	0.001,340,5	27.313,000,567,5	796	0.001,256,3	28.213,471,959,3
747	0.001,338,7	27.331,300,737,4	797	0.001,254,7	28.231,186,427,0
748	0.001,336,9	27.349,588,662,4	798	0.001,253,1	28.248,893,783,7
749	0.001,335,1	27.367,864,366,8	799	0.001,251,6	28.266,588,050,2
750	0.001,333,3	27.386,127,875,3	800	0.001,25	28.284,271,247,5

TABLE of Square-Roots and Reciprocals.

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No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
801	0.001,248,4	28.301,943,396,2	851	0.001,175,1	29.171,904,291,6
802	0.001,246,9	28.319,604,517,0	852	0.001,173,7	29.189,039,038,7
803	0.001,245,3	28.337,254,630,6	853	0.001,172,3	29.206,163,733,0
804	0.001,243,8	28.354,893,757,5	854	0.001,171,0	29.223,278,392,4
805	0.001,242,2	28.372,521,918,2	855	0.001,169,6	29.240,383,034,4
806	0.001,240,7	28.390,139,133,2	856	0.001,168,2	29.257,477,676,7
807	0.001,239,2	28.407,745,422,7	857	0.001,166,9	29.274,562,336,6
808	0.001,237,6	28.425,340,807,1	858	0.001,165,5	29.291,637,031,8
809	0.001,236,1	28.442,925,306,7	859	0.001,164,1	29.308,701,779,5
810	0.001,234,6	28.460,498,941,5	860	0.001,162,8	29.325,756,597,2
811	0.001,233,0	28.478,061,731,8	861	0.001,161,4	29.342,801,502,2
812	0.001,231,5	28.495,613,697,6	862	0.001,160,1	29.359,836,511,8
813	0.001,230,0	28.513,154,858,8	863	0.001,158,7	29.376,861,643,1
814	0.001,228,5	28.530,685,235,4	864	0.001,157,4	29.393,876,913,4
815	0.001,227,0	28.548,204,847,2	865	0.001,156,1	29.410,882,339,7
816	0.001,225,5	28.565,713,714,2	866	0.001,154,7	29.427,877,939,1
817	0.001,224,0	28.583,211,855,9	867	0.001,153,4	29.444,863,728,7
818	0.001,222,5	28.600,699,292,2	868	0.001,152,1	29.461,839,725,3
819	0.001,221,0	28.618,176,042,5	869	0.001,150,7	29.478,805,946,0
820	0.001,219,5	28.635,642,126,6	870	0.001,149,4	29.495,762,407,5
821	0.001,218,0	28.653,097,563,8	871	0.001,148,1	29.512,709,126,7
822	0.001,216,5	28.670,542,373,7	872	0.001,146,8	29.529,646,120,5
823	0.001,215,1	28.687,976,575,6	873	0.001,145,5	29.546,573,405,4
824	0.001,213,6	28.705,400,188,8	874	0.001,144,2	29.563,490,998,2
825	0.001,212,1	28.722,813,232,7	875	0.001,142,9	29.580,398,915,5
826	0.001,210,6	28.740,215,726,4	876	0.001,141,6	29.597,297,173,9
827	0.001,209,2	28.757,607,689,1	877	0.001,140,3	29.614,185,789,9
828	0.001,207,7	28.774,989,139,9	878	0.001,139,0	29.631,064,780,1
829	0.001,206,3	28.792,360,097,8	879	0.001,137,7	29.647,932,474,3
830	0.001,204,8	28.809,720,581,8	880	0.001,136,3	29.664,793,948,4
831	0.001,203,4	28.827,070,610,8	881	0.001,135,1	29.681,644,159,3
832	0.001,201,9	28.844,410,203,7	882	0.001,133,8	29.698,484,809,8
833	0.001,200,5	28.861,739,379,3	883	0.001,132,5	29.715,315,916,2
834	0.001,199,0	28.879,058,156,4	884	0.001,131,2	29.732,137,494,6
835	0.001,197,6	28.896,366,553,6	885	0.001,129,9	29.748,949,561,3
836	0.001,196,2	28.913,664,589,6	886	0.001,128,7	29.765,752,132,3
837	0.001,194,7	28.930,952,283,0	887	0.001,127,4	29.782,545,223,7
838	0.001,193,3	28.948,229,652,3	888	0.001,126,1	29.799,328,851,5
839	0.001,191,9	28.965,496,715,9	889	0.001,124,9	29.816,103,031,8
840	0.001,190,5	28.982,753,492,4	890	0.001,123,6	29.832,867,780,4
841	0.001,189,1	29.000,000,000,0	891	0.001,122,3	29.849,623,113,2
842	0.001,187,6	29.017,236,257,1	892	0.001,121,1	29.866,369,046,1
843	0.001,186,2	29.034,462,281,9	893	0.001,119,8	29.883,105,595,0
844	0.001,184,8	29.051,678,092,7	894	0.001,118,6	29.899,832,775,5
845	0.001,183,4	29.068,883,707,5	895	0.001,117,3	29.916,550,603,3
846	0.001,182,0	29.086,079,144,5	896	0.001,116,1	29.933,259,094,2
847	0.001,180,6	29.103,264,421,7	897	0.001,114,8	29.949,958,263,7
848	0.001,179,2	29.120,439,557,1	898	0.001,113,6	29.966,648,127,5
849	0.001,177,9	29.137,604,568,7	899	0.001,112,3	29.983,328,701,1
850	0.001,176,5	29.154,759,474,2	900	0.001,111,1	30.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
901	0.001,109,9	30.016,662,039,6	951	0.001,051,5	30.838,287,890,2
902	0.001,108,6	30.033,314,835,4	952	0.001,050,4	30.854,449,741,7
903	0.001,107,4	30.049,958,402,6	953	0.001,049,3	30.870,608,080,9
904	0.001,106,2	30.066,592,756,7	954	0.001,048,2	30.886,871,422,9
905	0.001,105,0	30.083,217,913,0	955	0.001,047,1	30.903,071,881,7
906	0.001,103,8	30.099,833,886,6	956	0.001,046,0	30.919,246,666,5
907	0.001,102,5	30.116,440,692,8	957	0.001,044,9	30.935,416,559,5
908	0.001,101,3	30.133,038,346,6	958	0.001,043,8	30.951,557,508,1
909	0.001,100,1	30.149,626,863,4	959	0.001,042,8	30.967,725,134,4
910	0.001,098,9	30.166,206,258,0	960	0.001,041,6	30.983,866,769,7
911	0.001,097,7	30.182,776,545,6	961	0.001,040,6	31.000,000,000,0
912	0.001,096,5	30.199,337,741,1	962	0.001,039,5	31.016,124,838,5
913	0.001,095,3	30.215,889,859,5	963	0.001,038,4	31.032,241,298,4
914	0.001,094,1	30.232,432,915,7	964	0.001,037,3	31.048,349,392,5
915	0.001,092,9	30.248,966,924,5	965	0.001,036,3	31.064,449,134,0
916	0.001,091,7	30.265,491,900,8	966	0.001,035,2	31.080,540,535,8
917	0.001,090,5	30.282,007,859,5	967	0.001,034,1	31.096,623,610,9
918	0.001,089,3	30.298,514,815,1	968	0.001,033,1	31.112,698,372,2
919	0.001,088,1	30.315,012,782,4	969	0.001,032,0	31.128,764,832,5
920	0.001,087,0	30.331,501,776,2	970	0.001,030,9	31.144,823,004,8
921	0.001,085,8	30.347,981,811,0	971	0.001,029,9	31.160,872,901,8
922	0.001,084,6	30.364,452,901,4	972	0.001,028,8	31.176,914,536,2
923	0.001,083,4	30.380,915,061,9	973	0.001,027,7	31.192,947,921,0
924	0.001,082,2	30.397,368,307,1	974	0.001,026,7	31.208,973,068,7
925	0.001,081,0	30.413,812,651,5	975	0.001,025,6	31.224,989,992,0
926	0.001,079,9	30.430,248,109,4	976	0.001,024,6	31.240,998,703,6
927	0.001,078,7	30.446,674,695,3	977	0.001,023,5	31.256,999,216,2
928	0.001,077,6	30.463,092,423,5	978	0.001,022,5	31.272,991,542,2
929	0.001,076,4	30.479,501,308,3	979	0.001,021,5	31.288,975,694,3
930	0.001,075,3	30.495,901,364,0	980	0.001,020,4	31.304,951,685,0
931	0.001,074,1	30.512,292,604,8	981	0.001,019,4	31.320,919,526,7
932	0.001,073,0	30.528,675,044,9	982	0.001,018,3	31.336,879,232,0
933	0.001,071,8	30.545,048,698,6	983	0.001,017,3	31.352,830,813,2
934	0.001,070,7	30.561,413,579,9	984	0.001,016,3	31.368,774,282,7
935	0.001,069,5	30.577,769,702,8	985	0.001,015,2	31.384,709,653,0
936	0.001,068,4	30.594,117,081,6	986	0.001,014,2	31.400,636,936,2
937	0.001,067,2	30.610,455,730,0	987	0.001,013,2	31.416,556,144,8
938	0.001,066,1	30.626,785,662,2	988	0.001,012,1	31.432,467,201,0
939	0.001,065,0	30.643,106,892,1	989	0.001,011,1	31.448,370,387,0
940	0.001,063,8	30.659,419,433,5	990	0.001,010,1	31.464,265,445,1
941	0.001,062,7	30.675,723,300,4	991	0.001,009,1	31.480,152,477,4
942	0.001,061,6	30.692,018,506,4	992	0.001,008,1	31.496,031,496,0
943	0.001,060,4	30.708,305,065,6	993	0.001,007,0	31.511,902,513,2
944	0.001,059,3	30.724,582,991,5	994	0.001,006,0	31.527,765,540,9
945	0.001,058,2	30.740,852,297,9	995	0.001,005,5	31.543,620,591,2
946	0.001,057,1	30.757,112,998,5	996	0.001,004,0	31.559,467,676,1
947	0.001,056,0	30.773,365,106,9	997	0.001,003,0	31.575,306,807,7
948	0.001,054,9	30.789,608,636,7	998	0.001,002,0	31.591,137,997,9
949	0.001,053,7	30.805,843,601,5	999	0.001,001,0	31.606,961,258,6
950	0.001,052,6	30.822,070,014,8	1000	0.001,000,0	31.622,776,601,5

. HUTTON's *Account of the foregoing Table of the Reciprocals and Square-Roots of the Natural Numbers 1, 2, 3, 4, 5, 6, &c. to 1,000, given at the end of the Fourth Volume of the Collection of Mathematical Problems and Traits, intituled Miscellanea Mathematica, published in Four little Volumes, Duodecimo, in the Year 1775.*

OF the preceeding Table, the use is evidently to shorten arithmetical calculations, and will appear eminently great to those mathematicians and others who are frequently concerned in such kinds of computations. The structure of the table is evident; the first column contains the natural series of numbers from 1 to 1,000, the 2d the reciprocals, and the 3d the square-roots of the same numbers, very accurately calculated and printed. These reciprocals and roots are the results preserved of many years occasional and accidental calculations in various subjects: in frequently making such computations, I found that I had often to make divisions by, and to extract the roots of, the same numbers; and as it seemed probable that this might be the case with me for many years longer, I formed the resolution of preserving all such roots and reciprocals as I should occasionally produce in my calculations, that I might have them always ready on any future occasion; which I did, by entering them always in a little book, ruled for the purpose, till I have at last collected to the number of 1,000, as above; and I now publish them here in this cheap and easy manner, that they may be of like use to other persons as to myself. In the numerical calculations of such kinds of problems as have appeared in this Miscellany and the Diary, the use of this table will be found to be very great, because of the frequent divisions and extractions of roots which are to be made: and the manner and cases of applying these numbers are

